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ON A STAFFANS TYPE INEQUALITY FOR VECTOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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1. Introduction. In a recent paper Staffans [4] studies a nonlinear Volterra integro-differential equation

$$\dot{x}(t)+\int_0^t g\big(x(s)\big)\,d\mu(t-s)=f(t), \qquad t\geqslant 0, \tag{1}$$

in the real scalar case. His main result is the inequality

$$\int_{G(x(0))}^{G(x(t))}\frac{dy}{u(y)}\leqslant\int_0^t |f(s)|\,ds, \tag{2}$$

where x is any solution of (1), G is an integral of g, and u is the function defined by

$$u(y)=\sup\{|g(z)|:\ G(z)\leqslant y\}.$$

From (2) he is able to obtain bounds for the solutions of (1) provided that some additional conditions on g are assumed.

To prove (2) Staffans, among others, assumes that the function G satisfies

$$\inf_{x\in\boldsymbol{R}} G(x)>-\infty, \tag{3}$$

which, however, adds a restriction on the function g. Inequality (3) is not satisfied, if, e.g., we get $g(x)=-3x^2$, $x\in\boldsymbol{R}$. Furthermore, the function u by its definition might not get finite values so that inequality (2) does not give any information about the solutions of equation (1).

It is the purpose of this note to improve the Staffans results, namely, to examine the above-mentioned cases, and we will see how one can get a nontrivial inequality like (2), even for the n-dimensional nonautonomous equation

$$\dot{x}(t)+\int_0^t d\mu(t-s)g(s, x(s)) = f(t, x(t)), \quad t \geqslant 0, \tag{4}$$

where a condition similar to (3) is not used. The key idea is to move along solutions, and so for any finite interval of time an inequality, like (3), is satisfied. The other interesting point where u may take no finite values is faced by multiplying $|g(z)|$ in the definition of u by a factor $b(z)$ which, of course, will appear in the final inequality. In this way we can obtain "a priori" bounds for the solutions of (1). Such bounds in the n-dimensional case for Volterra integral equations were also obtained by Levin [2].

2. Main results. A basic assumption in [4] for (1) is the positiveness of the measure μ as was defined in Halanay's fundamental work [1]. Extending this notion to matrix-valued kernels we give the following definition:

Definition. Let μ be defined on $\boldsymbol{R}^+$ with values in the set of $(n \times n)$-matrices with real elements. We say that μ is *positive definite* if for each $\varphi \in C(\boldsymbol{R}^+, \boldsymbol{R}^n)$ and $r > 0$ it satisfies

$$\int_0^r \langle \varphi(t), \int_0^t d\mu(t-s)\varphi(s)\rangle dt \geqslant 0, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\boldsymbol{R}^n$.

Following the lines of Staffans in [3] (p. 207) one could give an equivalent definition for positiveness of μ by using its distribution Fourier transform, but it is not our purpose here to discuss this notion further.

Our main theorem is the following

THEOREM. *Consider equation* (4) *where μ is positive definite as in the Definition. Assume also that the functions g and f satisfy the conditions*:

(g) *g: $\boldsymbol{R}^+ \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and such that, for each fixed $t \in \boldsymbol{R}^+$, $g(t, \cdot)$ is the gradient of a function $G(t, \cdot)$: $\boldsymbol{R}^n \to \boldsymbol{R}$ for which assume that $G(\cdot, \cdot)$ is continuous and, for each (t, x), $(\partial G/\partial t)(t, x)$ is nonpositive.*

(f) *f maps $\boldsymbol{R}^+ \times \boldsymbol{R}^n$ to $\boldsymbol{R}^n$ and there are functions b: $\boldsymbol{R}^+ \times \boldsymbol{R}^n \to (0, \infty)$ and $m \in L^1_{\mathrm{loc}}(\boldsymbol{R}^+)$ such that*

$$|f(t, x)| \leqslant m(t)b(t, x) \tag{6}$$

for all $(t, x) \in \boldsymbol{R}^+ \times \boldsymbol{R}^n$.

Let u_b be the function defined by

$$u_b(y) = \sup\{b(t, x)|g(t, x)|: G(t, x) \leqslant y\}$$

for all those y's for which this takes real values.

Then any solution $x(t)$, $t \in [0, T)$, *of* (4) *satisfies the inequality*

$$\int_{G(0,x(0))}^{G(t,x(t))} \frac{dy}{u_b(y)} \leqslant \int_0^t m(s)\,ds \tag{7}$$

for all $t \in [0, T)$.

Proof. Fix a point $r \in (0, T)$. Since G is continuous, we have

$$\gamma \equiv \inf\{G(t, x(t)) \colon t \in [0, r]\} > -\infty.$$

Replacing G, if necessary, by $1-\gamma+G$, we can assume that $\gamma = 1$ since, if G satisfies (g), then so does $G+c$ for any constant $c \in \boldsymbol{R}$. Define the function

$$v_b(y) = \sup\{b(t, x(t))\,|g(t, x(t))|\colon t \in [0, r] \text{ and } G(t, x(t)) \leqslant y\}$$

for $y \geqslant 1$.

Now, multiple (4) by $g(t, x(t))$ from the left and integrate over $[0, \tau]$ $(0 \leqslant \tau \leqslant r)$; we then obtain

$$\int_0^\tau \langle g(t, x(t)), \dot{x}(t)\rangle dt + \int_0^\tau \langle g(t, x(t)), \int_0^t g(s, x(s))\,d\mu(t-s)\rangle dt = \int_0^\tau \langle g(t, x(t)), f(t, x(t))\rangle dt. \tag{8}$$

Because of (g) we have

$$G(\tau, x(\tau)) - G(0, x(0)) \leqslant \int_0^\tau \langle g(t, x(t)), \dot{x}(t)\rangle dt,$$

and so from (5) and (8) we obtain

$$G(\tau, x(\tau)) \leqslant G(0, x(0)) + \int_0^\tau \langle g(t, x(t)), f(t, x(t))\rangle dt$$

for all $\tau \in [0, r]$. Taking into account (6) and the definition of v_b we finally get

$$G(t, x(t)) \leqslant G(0, x(0)) + \int_0^t v_b(G(s, x(s)))\,m(s)\,ds, \qquad t \in [0, r]. \tag{9}$$

Assuming for a moment that v_b is continuous on $[1, \infty)$ we extend it continuously on $[0, \infty)$ in such a way that $v_b(0) \geqslant 0$ and v_b be nondecreasing. Then Bihari's inequality applies to (9) and gives

$$\int_{G(0,x(0))}^{G(t,x(t))} \frac{dy}{u_b(y)} \leqslant \int_0^t m(s)\,ds, \qquad t \in [0, r]. \tag{10}$$

Assume that v_b is discontinuous and fix a $t \in [0, r]$. Let v_n be a sequence of continuous nondecreasing functions on $[0, \infty)$ satisfying $v_n \to v_b$ pointwise and such that

$$v_b(y) \leqslant v_n(y), \quad y \in [1, \sup_{s \in [0,\bar{t}]} G(s, x(s))],$$

where now $v_b(y)$ is defined on $[0, 1]$ as before.

Since for each index n the function v_n satisfies (9) as well, extending it linearly on $[0, 1]$ as v_b above we obtain again

$$\int\limits_{G(0,x(0))}^{G(\bar{t},x(\bar{t}))} \frac{dy}{v_n(y)} \leqslant \int\limits_0^{\bar{t}} m(s)\,ds, \quad n = 1, 2, \ldots \tag{11}$$

The Lebesgue Dominated Theorem applies to (11) and gives (10), since $\bar{t}$ is arbitrary.

Finally, fix a $t \in [0, r]$. If $G(0, x(0)) \geqslant G(t, x(t))$, then clearly (7) holds. If $G(0, x(0)) < G(t, x(t))$, from (10) and due to the fact that $v_b(y) \leqslant u_b(y)$ we obtain (7) since r is arbitrary.

Example a. Consider the n-dimensional equation

$$\dot{x}(t) + 2\int\limits_0^t d\mu(t-s)\frac{x(s)}{1+s} = \sin|x(t)|, \tag{12}$$

where μ is n-dimensional positive definite.

Choose $G(t, x) = |x|^2/(1+t)$ and $b(t, x) = |x| + \sqrt{1+t}$. Then we obtain

$$u_b(y) = 2(y + \sqrt{y}), \quad y \geqslant 0,$$

and so (7) gives

$$|x(t)| \leqslant \sqrt{1+t}\left(-1 + (1+|x(0)|)\exp\{2(\sqrt{1+t}-1)\}\right)$$

for any solution x of (12) and any t in the interval of definition of x.

Example b. Let μ be positive definite and consider the real scalar integro-differential equation

$$\dot{x}(t) + 4\int\limits_0^t x^3(s)e^{-s}\,d\mu(t-s) = \frac{t}{1+|x(t)|}.$$

In this case, set $G(t, x) = x^4 e^{-t}$ and $b(t, x) = e^{t/4}$. Then (7) applies with $u_b(y) = 4y^{3/4}$, $y \geqslant 0$, and gives

$$|x(t)| \leqslant e^{t/4}\left(|x(0)| + \frac{1}{16} - \frac{1}{16}e^{-t/4} - \frac{1}{4}te^{-t/4}\right)$$

for all t in the interval of definition of the solution x.

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