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REALIZATION OF SMALL CONCRETE CATEGORIES BY ALGEBRAS AND INJECTIVE HOMOMORPHISMS

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0. Jónsson and Płonka (see [2]-[4]) characterized permutation groups which are automorphism groups of a universal algebra. The purpose of this paper is to extend their results to the case where — roughly speaking — not one, but a family of permutation groups and bijections between their underlying sets is given; more precisely, a small concrete category K with only injective morphisms is given and we find (in Theorem 2) some necessary and sufficient conditions for K to be realizable by algebras and injective homomorphisms. Theorem 2 is an easy application of Theorem 1 in which necessary and sufficient conditions are found for K to be realizable by algebras and arbitrary homomorphisms. We suppose that K has only injective and constant morphisms. At present, it seems hopeless to prove a similar result for K arbitrary, since no characterization is known as yet even if K has only one object. On the other hand, a characterization of small concrete categories which are realizable by partial algebras and homomorphisms is given in [5] and [1]. Theorems 3 and 4 extend Płonka's results on automorphism groups of algebras with bounded arities of operations.

1. If A is an ordered pair, then its first member is denoted by $\dot{A}$.

By a *concrete category* we mean a category K satisfying the following three conditions:

(1) every K-object is an ordered pair, its first member being a non-empty set;

(2) if a is a K-morphism, $a\colon A\to B$, then $a = \langle f, A, B\rangle$ for some mapping f of $\dot{A}$ into $\dot{B}$;

(3) the mapping assigning to every K-object A the set $\dot{A}$ and to every K-morphism $\langle f, A, B\rangle$ the triple $\langle f, \dot{A}, \dot{B}\rangle$ is a covariant functor of K into the category of non-empty sets.

The functor defined in (3) is called the *forgetful functor of K*.

Evidently, a category K is concrete if and only if the following four conditions are satisfied:

(1′) as (1);

(2′) every K-morphism is an ordered triple $\langle f, A, B\rangle$, where A and B are K-objects and f is a mapping of $\dot{A}$ into $\dot{B}$;

(3′) for every K-object A, the triple $\langle 1_{\dot{A}}, A, A\rangle$ is a K-morphism;

(4′) if $\langle f, A, B\rangle$ and $\langle g, B, C\rangle$ are K-morphisms, then $\langle g \circ f, A, C\rangle$ is a K-morphism.

Let a type Δ be given, i.e., $\Delta = (n_i)_{i\in I}$, where I is a set and every n_i is a non-negative integer. An *algebra of type* Δ is an ordered pair $A = \langle X, (f_i)_{i\in I}\rangle$, where X is a non-empty set and f_i is an n_i-ary operation in X for every $i \in I$. We denote f_i by i_A.

Let A and B be two algebras of type Δ. By a *homomorphism of A into B* we mean a mapping f of $\dot{A}$ into $\dot{B}$ such that

$$f\big(i_A(a_1, \ldots, a_{n_i})\big) = i_B\big(f(a_1), \ldots, f(a_{n_i})\big)$$
$$\text{for all } i \in I \text{ and } a_1, \ldots, a_{n_i} \in \dot{A}.$$

With every type Δ we associate two concrete categories H_Δ and M_Δ: objects of both categories are algebras of type Δ; H_Δ-morphisms are triples $\langle f, A, B\rangle$ such that A and B are algebras of type Δ and f is a homomorphism of A into B; M_Δ-morphisms are H_Δ-morphisms $\langle f, A, B\rangle$ such that f is injective.

Let two concrete categories K_1 and K_2 be given. A one-to-one functor F of K_1 onto a full subcategory of K_2 is said to be a *realization of K_1 in K_2* if the following two conditions are satisfied:

(I) if $F(A) = B$ for some K_1-object A, then $\dot{A} = \dot{B}$;

(II) if $F(\langle f, A, B\rangle) = \langle g, C, D\rangle$, then $f = g$.

Evidently, a realization F is defined if and only if $F(A)$ is defined for all K_1-objects A and the following there conditions are satisfied:

(I′) as (I);

(II′) $F(A) = F(B)$ implies $A = B$ for all K_1-objects A and B;

(III′) for every two K_1-objects A and B, $\langle f, A, B\rangle$ is a K_1-morphism if and only if $\langle f, F(A), F(B)\rangle$ is a K_2-morphism.

We call K_1 *realizable in* K_2 if there exists a realization of K_1 in K_2.

Let K be a concrete category. By a *path in* K we mean a finite sequence

$$a = \langle A_0, f_1, A_1, f_2, A_2, \ldots, f_n, A_n\rangle$$

such that n is an even positive integer, $A_0, A_1, A_2, \ldots, A_n$ are K-objects, $f_1, f_2, \ldots, f_n$ are injective mappings, $\langle f_k, A_{k-1}, A_k\rangle$ is a K-morphism for all odd numbers $k \in \{1, \ldots, n\}$, and $\langle f_k, A_k, A_{k-1}\rangle$ is a K-morphism for all even $k \in \{1, \ldots, n\}$. We call α a *path from* A_0 *to* A_n. Write

$$\hat{\alpha} = f_n^{-1} \circ f_{n-1} \circ \ldots \circ f_2^{-1} \circ f_1,$$

so that $\hat{\alpha}$ is an injective mapping of a subset $D(\hat{\alpha})$ of $\dot{A}_0$ into $\dot{A}_n$.

Let A be a K-object and $X \subseteq \dot{A}$. We denote by $C_A^K(X)$ the set of all $x \in \dot{A}$ such that

(i) if α is a path in K from A to some B and $X \subseteq D(\hat{\alpha})$, then $x \in D(\hat{\alpha})$;

(ii) if α and β are two paths in K from A to B and $\hat{\alpha}|_X = \hat{\beta}|_X$ (so that $X \subseteq D(\hat{\alpha}) \cap D(\hat{\beta})$), then $\hat{\alpha}(x) = \hat{\beta}(x)$.

Evidently, $X \subseteq C_A^K(X)$.

If there are given a set A and an element a, then the (unique) mapping of A into $\{a\}$ is denoted by O_a^A.

An n-ary operation g is called *quasi-trivial* if

$$g(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\} \quad \text{for all } a_1, \ldots, a_n.$$

2. THEOREM 1. *Let K be a small concrete category such that if $\langle f, A, B\rangle$ is a K-morphism, then the mapping f is either injective or constant. K is realizable in some H_Δ if and only if the following conditions are satisfied:*

(U) *if A and B are K-objects such that $\dot{A} = \dot{B}$ and $\langle 1_{\dot{A}}, A, B\rangle$ and $\langle 1_{\dot{A}}, B, A\rangle$ are K-morphisms, then $A = B$;*

(α) *if A and B are K-objects and f is an injective mapping of $\dot{A}$ into $\dot{B}$ such that, for every finite $X \subseteq \dot{A}$, there exists a path α in K from A to B satisfying $\hat{\alpha}|_X = f|_X$, then $\langle f, A, B\rangle$ is a K-morphism;*

(β) *if A, B are K-objects and $a \in \dot{A}$, then $\langle O_a^{\dot{A}}, A, A\rangle$ is a K-morphism if and only if $\langle O_a^{\dot{B}}, B, A\rangle$ is a K-morphism;*

(γ) *if A, B are K-objects, $a \in \dot{A}$, f is an injective mapping of $\dot{A}$ into $\dot{B}$, and $\langle f, A, B\rangle$ and $\langle O_{f(a)}^{\dot{B}}, B, B\rangle$ are K-morphisms, then $\langle O_a^{\dot{A}}, A, A\rangle$ is a K-morphism;*

(δ) *if A is a K-object, $a \in \dot{A}$, and $\langle O_a^{\dot{A}}, A, A\rangle$ is not a K-morphism, then $C_A^K(\{a\})$ has at least two elements.*

If these conditions are satisfied, then there exists a realization F such that every $F(A)$ is an algebra with one unary and some quasi-trivial operations.

Proof. Let F be a realization of K in H_Δ. (U) is evident and (α) is easy. To prove (β) and (γ) it is sufficient to notice that $O_a^{\dot{B}}$ is a homomorphism of $F(B)$ into $F(A)$ if and only if a is an idempotent of $F(A)$, i.e., $i_{F(A)}(a, \ldots, a) = a$ for all $i \in I$. The set $C_A^K(\{a\})$ contains a and all $i_{F(A)}(a, \ldots, a)$; if a is not an idempotent, some $i_{F(A)}(a, \ldots, a)$ is different from a and we get (δ).

Let (U), (α), (β), (γ) and (δ) be satisfied. Denote by V the set of all $\langle A, a\rangle$ such that A is a K-object, $a \in \dot{A}$, and $\langle O_a^{\dot{A}}, A, A\rangle$ is not a K-morphism.

Define a binary relation $\sim$ on V as follows: $\langle A_1, a_1\rangle \sim \langle A_2, a_2\rangle$ if and only if there exists a path α in K from A_1 to A_2 such that $a_1 \in D(\hat{\alpha})$ and $\hat{\alpha}(a_1) = a_2$. Evidently, $\sim$ is an equivalence relation. There exists a subset Z of V such that every element of V is equivalent to exactly one element of Z.

Define a mapping h with domain Z as follows: if $\langle A, a\rangle \in Z$, then $h(\langle A, a\rangle)$ is an element of $C_A^K(\{a\}) - \{a\}$ (this set is non-empty by (δ)).

Denote by I_0 the set of all $\langle e_1, \ldots, e_m, E\rangle$ such that E is a K-object, $m \geqslant 2$ and $e_1, \ldots, e_m$ are pairwise different elements of $\dot{E}$. Put $I = \{\lambda\} \cup I_0$, where λ is an element not belonging to I_0. Define the type $\Delta = (n_i)_{i \in I}$ by

$$n_\lambda = 1, \quad n_{\langle e_1, \ldots, e_m, E\rangle} = m.$$

For every K-object A, define an algebra $F(A)$ of type Δ: the first member of $F(A)$ is $\dot{A}$; if $a \in \dot{A}$ and $\langle A, a\rangle \notin V$, then $\lambda_{F(A)}(a) = a$; if $\langle A, a\rangle \in V$, we denote by $\langle D, d\rangle$ the element of Z equivalent to $\langle A, a\rangle$, choose a path α from D to A satisfying $\hat{\alpha}(d) = a$ and put $\lambda_{F(A)}(a) = \hat{\alpha}\big(h(\langle D, d\rangle)\big)$; if

$$i = \langle e_1, \ldots, e_m, E\rangle \in I_0 \quad \text{and} \quad a_1, \ldots, a_m \in \dot{A},$$

then

$$i_{F(A)}(a_1, \ldots, a_m) = a_2$$

if there exists a path α from E to A such that $\hat{\alpha}(e_1) = a_1, \ldots, \hat{\alpha}(e_m) = a_m$, while

$$i_{F(A)}(a_1, \ldots, a_m) = a_1$$

if such a path does not exist. (It is easy to see that the definition of $\lambda_{F(A)}$ is correct.)

We shall prove that F is a realization of K in H_Δ. (I) is evident. By (U), (II′) will be proved if we prove (III′).

Let $\langle f, A, B\rangle$ be a K-morphism. We shall prove that f is a homomorphism of $F(A)$ into $F(B)$. If f is constant, $f = O_b^{\dot{A}}$ for some $b \in \dot{B}$; then $\langle B, b\rangle \notin V$ by (β), so that $\lambda_{F(B)}(b) = b$; of course, we have

$$i_{F(B)}(b, \ldots, b) = b \quad \text{for all } i \in I_0$$

and, consequently, f is a homomorphism of $F(A)$ into $F(B)$. The case of f injective remains. If $\langle A, a\rangle \notin V$, then $\langle B, f(a)\rangle \notin V$, so that

$$f\big(\lambda_{F(A)}(a)\big) = f(a) = \lambda_{F(B)}\big(f(a)\big).$$

If $\langle A, a\rangle \in V$, then $\langle B, f(a)\rangle \in V$ by (γ). Denote by $\langle D, d\rangle$ the element of Z equivalent to $\langle A, a\rangle$ and by

$$\alpha = \langle D, f_1, A_1, \ldots, f_n, A\rangle$$

a path from D to A such that $\hat{\alpha}(d) = a$. Evidently,

$$\beta = \langle D, f_1, A_1, \ldots, f_n, A, f, B, 1_{\dot{B}}, B\rangle$$

is a path from D to B such that $\hat{\beta}(d) = f(a)$. We get

$$f\big(\lambda_{F(A)}(a)\big) = f\big(\hat{\alpha}(h(\langle D, d\rangle))\big) = \hat{\beta}\big(h(\langle D, d\rangle)\big) = \lambda_{F(B)}\big(f(a)\big).$$

Let $i = \langle e_1, \ldots, e_m, E\rangle \in I_0$ and $a_1, \ldots, a_m \in \dot{A}$. If there exists a path

$$\alpha = \langle E, f_1, A_1, \ldots, f_n, A\rangle$$

from E to A such that $\hat{\alpha}(e_1) = a_1, \ldots, \hat{\alpha}(e_m) = a_m$, then

$$\beta = \langle E, f_1, A_1, \ldots, f_n, A, f, B, 1_{\dot{B}}, B\rangle$$

is a path from E to B such that $\hat{\beta}(e_1) = f(a_1), \ldots, \hat{\beta}(e_m) = f(a_m)$, so that

$$f\big(i_{F(A)}(a_1, \ldots, a_m)\big) = f(a_2) = i_{F(B)}\big(f(a_1), \ldots, f(a_m)\big).$$

Assume that α does not exist. If

$$\beta = \langle E, g_1, B_1, \ldots, g_n, B\rangle$$

were a path such that

$$\hat{\beta}(e_1) = f(a_1), \ldots, \hat{\beta}(e_m) = f(a_m),$$

then $\langle E, g_1, B_1, \ldots, g_n, B, 1_{\dot{B}}, B, f, A\rangle$ would have the properties of α. Hence, β does not exist, and we have

$$f\big(i_{F(A)}(a_1, \ldots, a_m)\big) = f(a_1) = i_{F(B)}\big(f(a_1), \ldots, f(a_m)\big).$$

We have proved that f is a homomorphism of $F(A)$ into $F(B)$.

Let A, B be two K-objects and f a homomorphism of $F(A)$ into $F(B)$. We shall prove that $\langle f, A, B\rangle$ is a K-morphism. Suppose that there exist three different elements a, b, c of $\dot{A}$ such that $f(a) \neq f(b) = f(c)$. Put $i = \langle a, b, c, A\rangle$, so that $i \in I_0$. There does not exist any path α from A to B such that $\hat{\alpha}(a) = f(a)$, $\hat{\alpha}(b) = f(b)$ and $\hat{\alpha}(c) = f(c)$, as $\hat{\alpha}$ is injective. On the other hand, $\langle A, 1_{\dot{A}}, A, 1_{\dot{A}}, A\rangle$ is a path and we get

$$f\big(i_{F(A)}(a, b, c)\big) = f(b) \neq f(a) = i_{F(B)}\big(f(a), f(b), f(c)\big),$$

a contradiction. This shows that f is either constant or injective.

Let f be constant, $f = O_b^{\dot{A}}$ for some $b \in \dot{B}$. Evidently, $\lambda_{F(B)}(b) = b$, so that $\langle B, b\rangle \notin V$. Hence, $\langle f, A, B\rangle$ is a K-morphism.

Let f be not constant, and thus injective. Suppose that $\langle f, A, B\rangle$ is not a K-morphism. By (α), there exists a finite $X \subseteq \dot{A}$ such that $\hat{a}|_X = f|_X$ for no path a from A to B. Put $m = \operatorname{Card} X$. Since f is not constant, we have $\operatorname{Card}\dot{A} \geqslant 2$ and we may suppose $m \geqslant 2$. Denote by $a_1, \ldots, a_m$ the elements of X and put $i = \langle a_1, \ldots, a_m, A\rangle$, so that $i \in I_0$. We have

$$f\big(i_{F(A)}(a_1, \ldots, a_m)\big) = f(a_2) \neq f(a_1) = i_{F(B)}\big(f(a_1), \ldots, f(a_m)\big),$$

a contradiction.

THEOREM 2. *Let K be a small concrete category such that if $\langle f, A, B\rangle$ is a K-morphism, then the mapping f is injective. K is realizable in some M_Δ if and only if conditions* (U) *and* (α) *are satisfied and*

(δ') *if A, B are K-objects, $a \in \dot{A}$, $\operatorname{Card}\dot{B} = 1$, and $\langle O_a^{\dot{B}}, B, A\rangle$ is not a K-morphism, then $C_A^K(\{a\})$ has at least two elements.*

Proof. The direct implication is easy (as in Theorem 1). Let (U), (α) and (δ') be fulfilled. Construct a new concrete category L as follows: L has the same class of objects as K; if $\operatorname{Card}\dot{D} \geqslant 2$ for all K-objects D, then $\langle f, A, B\rangle$ is an L-morphism if and only if it is a K-morphism or else f is constant; if $\operatorname{Card}\dot{D} = 1$ for some D, then $\langle f, A, B\rangle$ is an L-morphism if and only if it is a K-morphism or else $f = O_b^{\dot{A}}$ for some $b \in \dot{B}$ and $\langle O_b^{\dot{D}}, D, B\rangle$ is a K-morphism. It is easy to see that L satisfies conditions (U), (α), (β), (γ) and (δ), so that, by Theorem 1, there exists a type Δ and a realization F of L in H_Δ. Evidently, F is a realization of K in M_Δ.

3. COROLLARY 1. *Let K be a small concrete category such that $\operatorname{Card}\dot{A} \geqslant 2$ for all K-objects A. If $\langle f, A, B\rangle$ is a K-morphism, then f is injective. K is realizable in some M_Δ if and only if it satisfies* (U) *and* (α). *The constructed realization F is such that, for every K-object A, all subsets of $\dot{A}$ are subuniverses of $F(A)$.*

COROLLARY 2. *Let H be a set of injective mappings of a set A into A, closed under composition and containing 1_A. H is the set of all injective endomorphisms of some algebra if and only if H contains any f such that, for every finite $X \subseteq A$, there exists a finite sequence $f_1, \ldots, f_{2n}$ of elements of H such that*

$$f_{2n}^{-1} \circ f_{2n-1} \circ \ldots \circ f_2^{-1} \circ f_1$$

is defined and coincides with f on X.

Denote by L_Δ the concrete category whose objects are algebras of type Δ and morphisms are triples $\langle f, A, B\rangle$ such that f is either an injective or constant homomorphism of A into B.

THEOREM 3. *Let a natural number $n \geqslant 2$ be given and let K be a small concrete category such that if $\langle f, A, B\rangle$ is a K-morphism, then f is either* *injective or constant. Then K is realizable in some L_Δ with $\Delta = (n_i)_{i\in I}$ satisfying $n_i \leqslant n$ for all $i \in I$ if and only if conditions* (U), (β), (γ) *and* (δ) *are satisfied and*

(α_n) *if A, B are K-objects and f is an injective mapping of $\dot{A}$ into $\dot{B}$ such that, for every $X \subseteq \dot{A}$ of cardinality not greater than n, there exists a path a in K from A to B satisfying $\hat{a}|_{C_A^K(X)} = f|_{C_A^K(X)}$, then $\langle f, A, B\rangle$ is a K-morphism.*

If $n \geqslant 3$, we may replace L_Δ by H_Δ.

Proof. The direct implication is easy. For the converse, define $\lambda_{F(A)}$ as in the proof of Theorem 1; define I_0 as the set of all $\langle e_1, \ldots, e_m, e, E\rangle$ such that E is a K-object, $2 \leqslant m \leqslant n$, $e_1, \ldots, e_m$ are pairwise different elements of $\dot{E}$, $e \in C_E^K(\{e_1, \ldots, e_m\})$; put $I = \{\lambda\} \cup I_0$; if

$$i = \langle e_1, \ldots, e_m, e, E\rangle \in I_0,$$

put $n_i = m$ and define $i_{F(A)}$ as follows: if there exists a path a from E to A such that

$$\hat{a}(e_1) = a_1, \ldots, \hat{a}(e_m) = a_m,$$

then

$$i_{F(A)}(a_1, \ldots, a_m) = \hat{a}(e);$$

if a does not exist, then

$$i_{F(A)}(a_1, \ldots, a_m) = a_1.$$

Let us prove only that if f is an injective but not constant homomorphism of $F(A)$ into $F(B)$, then $\langle f, A, B\rangle$ is a K-morphism. Suppose that $\langle f, A, B\rangle$ is not a K-morphism. By (α_n), there exists a set $X \subseteq \dot{A}$ of cardinality $m \leqslant n$ with the prescribed properties. We may suppose $m \geqslant 2$. Denote by $a_1, \ldots, a_m$ the elements of X. If there exists a path a from A to B such that $\hat{a}|_X = f|_X$, choose an $a \in C_A^K(X) - X$ such that $\hat{a}(a) \neq f(a)$, and put

$$i = \langle a_1, \ldots, a_m, a, A\rangle;$$

we have

$$f\big(i_{F(A)}(a_1, \ldots, a_m)\big) = f(a) \neq \hat{a}(a) = i_{F(B)}\big(f(a_1), \ldots, f(a_m)\big),$$

a contradiction. If a does not exist, put

$$i = \langle a_1, \ldots, a_m, a_2, A\rangle;$$

we have

$$f\big(i_{F(A)}(a_1, \ldots, a_m)\big) = f(a_2) \neq f(a_1) = i_{F(B)}\big(f(a_1), \ldots, f(a_m)\big),$$

a contradiction again.

In the proof of Theorem 1, to prove that f is either injective or constant we made use of a ternary operation. Hence, if $n \geqslant 3$, F is a realization in H_Δ.

COROLLARY 3. *If $n \geqslant 2$ and a type $\Delta = (n_i)_{i \in I}$ is such that $n_i \leqslant n$ for all $i \in I$, then Theorem 3 holds if we replace* (α) *by* (α_n).

Remark 1. In the case $n = 2$, Theorem 3 does not hold if we replace L_Δ by H_Δ. For example, let K have a single object A; let $\operatorname{Card} \dot{A} \geqslant 4$ and let $\langle f, A, A \rangle$ be a K-morphism if and only if f is either an injective or a constant mapping of $\dot{A}$ into $\dot{A}$. K satisfies (U), (α_2), (β), (γ) and (δ). However, if B is an algebra with at most binary operations, $\dot{B} = \dot{A}$ and every constant and every injective mapping of $\dot{A}$ into $\dot{A}$ is an endomorphism of B, then every mapping of $\dot{A}$ into $\dot{A}$ is an endomorphism of B.

THEOREM 4. *Let K be a small concrete category such that if $\langle f, A, B \rangle$ is a K-morphism, then f is either injective or constant. K is realizable in some L_Δ with $\Delta = (n_i)_{i \in I}$ satisfying $n_i \leqslant 1$ for all $i \in I$ if and only if conditions* (U), (β), (γ) *and* (δ) *are satisfied and*

(α_1') *if A, B are K-objects, f is an injective mapping of $\dot{A}$ into $\dot{B}$, and $\langle f, A, B \rangle$ is not a K-morphism, then there exists an $a \in \dot{A}$ such that either $\langle O_a^{\dot{A}}, A, A \rangle$ or $\langle O_{f(a)}^{\dot{B}}, B, B \rangle$ is not a K-morphism and $\hat{a}|_{C_A^K(\{a\})} = f|_{C_A^K(\{a\})}$ for no path a from A to B.*

Proof. Define Δ as follows: I is the set of all $\langle e, u, E \rangle$ such that E is a K-object, $e \in \dot{E}$, $u \in C_E^K(\{e\}) - \{e\}$ and $\langle O_e^{\dot{E}}, E, E \rangle$ is not a K-morphism; $n_i = 1$ for all i.

Define a realization F as follows: if A is a K-object, $i = \langle e, u, E \rangle \in I$ and $a \in \dot{A}$, then $i_{F(A)}(a) = \hat{a}(u)$ if there exists a path a from E to A such that $\hat{a}(e) = a$, while $i_{F(A)}(a) = a$ if a does not exist.

Again, we prove only that if f is an injective homomorphism of $F(A)$ into $F(B)$, then $\langle f, A, B \rangle$ is a K-morphism. Suppose that this is not true. There exists an $a \in \dot{A}$ as in (α_1'). If $\langle O_a^{\dot{A}}, A, A \rangle$ is not a K-morphism and $\hat{a}(a) = f(a)$ for no path a, we choose a $b \in C_A^K(\{a\}) - \{a\}$ and put $i = \langle a, b, A \rangle$; we get

$$f\big(i_{F(A)}(a)\big) = f(b) \neq f(a) = i_{F(B)}\big(f(a)\big),$$

a contradiction. If $\langle O_a^{\dot{A}}, A, A \rangle$ is not a K-morphism and $\hat{a}(a) = f(a)$ for some path a, there exists a $b \in C_A^K(\{a\}) - \{a\}$ such that $\hat{a}(b) \neq f(b)$; we put $i = \langle a, b, A \rangle$ and get

$$f\big(i_{F(A)}(a)\big) = f(b) \neq \hat{a}(b) = i_{F(B)}\big(f(a)\big),$$

a contradiction again. Finally, if $\langle O_a^{\dot{A}}, A, A \rangle$ is a K-morphism and $\langle O_{f(a)}^{\dot{B}}, B, B \rangle$ is not, then there does not exist any path a from A to B

satisfying $\hat{a}(a) = f(a)$; choosing $b \in C_B^K(\{f(a)\}) - \{f(a)\}$ and putting $i = \langle f(a), b, B\rangle$, we get

$$f\big(i_{F(A)}(a)\big) = f(a) \neq b = i_{F(B)}\big(f(a)\big),$$

a contradiction.

Remark 2. It can be analogously proved that if type $\Delta = (n_i)_{i \in I}$ is such that $n_i \leqslant 1$ for all i, then Theorem 2 holds if we replace (α) by the following condition:

(α_1'') *if* A, B *are* K*-objects and* f *is an injective mapping of* $\dot{A}$ *into* $\dot{B}$ *such that, for every* $a \in \dot{A}$ *satisfying either* $C_A^K(\{a\}) \neq \{a\}$ *or* $C_B^K(\{f(a)\}) \neq \{f(a)\}$, *there exists a path* a *from* A *to* B *satisfying* $\hat{a}|_{C_A^K(\{a\})} = f|_{C_A^K(\{a\})}$, *then* $\langle f, A, B\rangle$ *is a* K*-morphism.*

REFERENCES

[1] J. Ježek, *On categories of structures and classes of algebras*, Dissertationes Mathematicae 75, Warszawa 1970.

[2] B. Jónsson, *Algebraic structures with prescribed automorphism groups*, Colloquium Mathematicum 19 (1968), p. 1-4.

[3] — *Topics in universal algebra*, Lecture Notes in Mathematics 250 (1970).

[4] E. Płonka, *A problem of Bjarni Jónsson concerning automorphisms of a general algebra*, Colloquium Mathematicum 19 (1968), p. 5-8.

[5] A. Pultr, *On full embeddings of concrete categories with respect to forgetful functors*, Commentationes Mathematicae Universitatis Carolinae 9 (1968), p. 281-305.

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