REALIZATION OF SMALL CONCRETE CATEGORIES
BY ALGEBRAS AND INJECTIVE HOMOMORPHISMS

BY

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0. Jónsson and Płonka (see [2]-[4]) characterized permutation groups which are automorphism groups of a universal algebra. The purpose of this paper is to extend their results to the case where — roughly speaking — not one, but a family of permutation groups and bijections between their underlying sets is given; more precisely, a small concrete category $K$ with only injective morphisms is given and we find (in Theorem 2) some necessary and sufficient conditions for $K$ to be realizable by algebras and injective homomorphisms. Theorem 2 is an easy application of Theorem 1 in which necessary and sufficient conditions are found for $K$ to be realizable by algebras and arbitrary homomorphisms. We suppose that $K$ has only injective and constant morphisms. At present, it seems hopeless to prove a similar result for $K$ arbitrary, since no characterization is known as yet even if $K$ has only one object. On the other hand, a characterization of small concrete categories which are realizable by partial algebras and homomorphisms is given in [5] and [1]. Theorems 3 and 4 extend Płonka’s results on automorphism groups of algebras with bounded arities of operations.

1. If $A$ is an ordered pair, then its first member is denoted by $\hat{A}$.

By a concrete category we mean a category $K$ satisfying the following three conditions:

(1) every $K$-object is an ordered pair, its first member being a non-empty set;

(2) if $a$ is a $K$-morphism, $a: A \to B$, then $a = \langle f, A, B \rangle$ for some mapping $f$ of $\hat{A}$ into $\hat{B}$;

(3) the mapping assigning to every $K$-object $A$ the set $\hat{A}$ and to every $K$-morphism $\langle f, A, B \rangle$ the triple $\langle f, \hat{A}, \hat{B} \rangle$ is a covariant functor of $K$ into the category of non-empty sets.

The functor defined in (3) is called the forgetful functor of $K$. 
Evidently, a category $K$ is concrete if and only if the following four conditions are satisfied:

(1') as (1);
(2') every $K$-morphism is an ordered triple $\langle f, A, B \rangle$, where $A$ and $B$ are $K$-objects and $f$ is a mapping of $A$ into $B$;
(3') for every $K$-object $A$, the triple $\langle 1_A, A, A \rangle$ is a $K$-morphism;
(4') if $\langle f, A, B \rangle$ and $\langle g, B, C \rangle$ are $K$-morphisms, then $\langle g \circ f, A, C \rangle$ is a $K$-morphism.

Let a type $\Delta$ be given, i.e., $\Delta = (n_i)_{i \in I}$, where $I$ is a set and every $n_i$ is a non-negative integer. An algebra of type $\Delta$ is an ordered pair $A = \langle X, (f_i)_{i \in I} \rangle$, where $X$ is a non-empty set and $f_i$ is an $n_i$-ary operation in $X$ for every $i \in I$. We denote $f_i$ by $i_A$.

Let $A$ and $B$ be two algebras of type $\Delta$. By a homomorphism of $A$ into $B$ we mean a mapping $f$ of $A$ into $B$ such that

$$f(i_A(a_1, \ldots, a_{n_i})) = i_B(f(a_1), \ldots, f(a_{n_i}))$$

for all $i \in I$ and $a_1, \ldots, a_{n_i} \in A$.

With every type $\Delta$ we associate two concrete categories $H_\Delta$ and $M_\Delta$: objects of both categories are algebras of type $\Delta$; $H_\Delta$-morphisms are triples $\langle f, A, B \rangle$ such that $A$ and $B$ are algebras of type $\Delta$ and $f$ is a homomorphism of $A$ into $B$; $M_\Delta$-morphisms are $H_\Delta$-morphisms $\langle f, A, B \rangle$ such that $f$ is injective.

Let two concrete categories $K_1$ and $K_2$ be given. A one-to-one functor $F$ of $K_1$ onto a full subcategory of $K_2$ is said to be a realization of $K_1$ in $K_2$ if the following two conditions are satisfied:

(I) if $F(A) = B$ for some $K_1$-object $A$, then $A = B$;
(II) if $F(\langle f, A, B \rangle) = \langle g, C, D \rangle$, then $f = g$.

Evidently, a realization $F$ is defined if and only if $F(A)$ is defined for all $K_1$-objects $A$ and the following there conditions are satisfied:

(\text{T}' as (I));
(\text{II}') $F(A) = F(B)$ implies $A = B$ for all $K_1$-objects $A$ and $B$;
(\text{III}') for every two $K_1$-objects $A$ and $B$, $\langle f, A, B \rangle$ is a $K_1$-morphism if and only if $\langle f, F(A), F(B) \rangle$ is a $K_2$-morphism.

We call $K_1$ realizable in $K_2$ if there exists a realization of $K_1$ in $K_2$.

Let $K$ be a concrete category. By a path in $K$ we mean a finite sequence

$$a = \langle A_0, f_1, A_1, f_2, A_2, \ldots, f_n, A_n \rangle$$

such that $n$ is an even positive integer, $A_0, A_1, A_2, \ldots, A_n$ are $K$-objects, $f_1, f_2, \ldots, f_n$ are injective mappings, $\langle f_k, A_{k-1}, A_k \rangle$ is a $K$-morphism for
all odd numbers \( k \in \{1, \ldots, n\} \), and \( \langle f_1, A_k, A_{k-1} \rangle \) is a \( K \)-morphism for all even \( k \in \{1, \ldots, n\} \). We call \( a \) a path from \( A_0 \) to \( A_n \). Write

\[
\hat{a} = f_{n}^{-1} \circ f_{n-1} \circ \ldots \circ f_{2}^{-1} \circ f_{1},
\]

so that \( \hat{a} \) is an injective mapping of a subset \( D(\hat{a}) \) of \( \hat{A}_0 \) into \( \hat{A}_n \).

Let \( A \) be a \( K \)-object and \( X \subseteq \hat{A} \). We denote by \( C^X_A(X) \) the set of all \( x \in \hat{A} \) such that

(i) if \( a \) is a path in \( K \) from \( A \) to some \( B \) and \( X \subseteq D(\hat{a}) \), then \( x \in D(\hat{a}) \);
(ii) if \( a \) and \( \beta \) are two paths in \( K \) from \( A \) to \( B \) and \( \hat{a} |_X = \hat{\beta} |_X \) (so that \( X \subseteq D(\hat{a}) \cap D(\hat{\beta}) \)), then \( \hat{a} (x) = \hat{\beta} (x) \).

Evidently, \( X \subseteq C^X_A(X) \).

If there are given a set \( \mathcal{A} \) and an element \( a \), then the (unique) mapping of \( \mathcal{A} \) into \( \{a\} \) is denoted by \( O^a_{\mathcal{A}} \).

An \( n \)-ary operation \( g \) is called quasi-trivial if

\[
g(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\} \quad \text{for all } a_1, \ldots, a_n.
\]

2. Theorem 1. Let \( K \) be a small concrete category such that if \( \langle f, A, B \rangle \) is a \( K \)-morphism, then the mapping \( f \) is either injective or constant. \( K \) is realizable in some \( H_\mathcal{A} \) if and only if the following conditions are satisfied:

- \((U)\) if \( A \) and \( B \) are \( K \)-objects such that \( \hat{A} = \hat{B} \) and \( \langle 1_{\hat{A}}, A, B \rangle \) and \( \langle 1_{\hat{A}}, B, A \rangle \) are \( K \)-morphisms, then \( A = B \);
- \((\alpha)\) if \( A \) and \( B \) are \( K \)-objects and \( f \) is an injective mapping of \( \hat{A} \) into \( \hat{B} \) such that, for every finite \( X \subseteq \hat{A} \), there exists a path \( a \) in \( K \) from \( A \) to \( B \) satisfying \( \hat{a} |_X = f |_X \), then \( \langle f, A, B \rangle \) is a \( K \)-morphism;
- \((\beta)\) if \( A, B \) are \( K \)-objects and \( a \in \hat{A} \), then \( \langle O^a_{A}, A, A \rangle \) is a \( K \)-morphism if and only if \( \langle O^B_{a}, B, A \rangle \) is a \( K \)-morphism;
- \((\gamma)\) if \( A, B \) are \( K \)-objects, \( a \in \hat{A} \), \( f \) is an injective mapping of \( \hat{A} \) into \( \hat{B} \), and \( \langle f, A, B \rangle \) and \( \langle O^B_{f(a)}, B, B \rangle \) are \( K \)-morphisms, then \( \langle O^A_{a}, A, A \rangle \) is a \( K \)-morphism;
- \((\delta)\) if \( A \) is a \( K \)-object, \( a \in \hat{A} \), and \( \langle O^a_{A}, A, A \rangle \) is not a \( K \)-morphism, then \( C^X_A(\{a\}) \) has at least two elements.

If these conditions are satisfied, then there exists a realization \( F \) such that every \( F(\mathcal{A}) \) is an algebra with one unary and some quasi-trivial operations.

Proof. Let \( F \) be a realization of \( K \) in \( H_\mathcal{A} \). \((U)\) is evident and \((\alpha)\) is easy. To prove \((\beta)\) and \((\gamma)\) it is sufficient to notice that \( O^B_{a} \) is a homomorphism of \( F(B) \) into \( F(A) \) if and only if \( a \) is an idempotent of \( F(\mathcal{A}) \), i.e., \( i_{F}(a, \ldots, a) = a \) for all \( i \in I \). The set \( C^X_A(\{a\}) \) contains \( a \) and all \( i_{F}(a, \ldots, a) \); if \( a \) is not an idempotent, some \( i_{F}(a, \ldots, a) \) is different from \( a \) and we get \((\delta)\).
Let \((U), (\alpha), (\beta), (\gamma)\) and \((\delta)\) be satisfied. Denote by \(V\) the set of all \(\langle A, a \rangle\) such that \(A\) is a \(K\)-object, \(a \in \hat{A}\), and \(\langle O^A_\alpha, A, A \rangle\) is not a \(K\)-morphism.

Define a binary relation \(\sim\) on \(V\) as follows: \(\langle A_1, a_1 \rangle \sim \langle A_2, a_2 \rangle\) if and only if there exists a path \(a\) in \(K\) from \(A_1\) to \(A_2\) such that \(a_1 \in D(\hat{a})\) and \(\hat{a}(a_1) = a_2\). Evidently, \(\sim\) is an equivalence relation. There exists a subset \(Z\) of \(V\) such that every element of \(V\) is equivalent to exactly one element of \(Z\).

Define a mapping \(h\) with domain \(Z\) as follows: if \(\langle A, a \rangle \in Z\), then \(h(\langle A, a \rangle)\) is an element of \(C^K_{\chi}(\langle a \rangle) - \{a\}\) (this set is non-empty by \((\delta)\)).

Denote by \(I_0\) the set of all \(\langle e_1, \ldots, e_m, E \rangle\) such that \(E\) is a \(K\)-object, \(m \geq 2\) and \(e_1, \ldots, e_m\) are pairwise different elements of \(\hat{E}\). Put \(I = \{\lambda\} \cup I_0\), where \(\lambda\) is an element not belonging to \(I_0\). Define the type \(\Delta = (n_i)_{i \in I}\) by

\[
n_{\lambda} = 1, \quad n_{\langle e_1, \ldots, e_m, E \rangle} = m.
\]

For every \(K\)-object \(A\), define an algebra \(F(A)\) of type \(\Delta\): the first member of \(F(A)\) is \(\hat{A}\); if \(a \in \hat{A}\) and \(\langle A, a \rangle \notin V\), then \(\lambda_{F(A)}(a) = a\); if \(\langle A, a \rangle \in V\), we denote by \(\langle D, d \rangle\) the element of \(Z\) equivalent to \(\langle A, a \rangle\), choose a path \(d\) from \(A\) to \(A\) satisfying \(\hat{a}(d) = a\) and put \(\lambda_{F(A)}(a) = \hat{a}(h(\langle D, d \rangle))\); if

\[
i = \langle e_1, \ldots, e_m, E \rangle \in I_0 \quad \text{and} \quad a_1, \ldots, a_m \in \hat{A},
\]

then

\[
i_{F(A)}(a_1, \ldots, a_m) = a_2
\]

if there exists a path \(a\) from \(E\) to \(A\) such that \(\hat{a}(e_1) = a_1, \ldots, \hat{a}(e_m) = a_m\), while

\[
i_{F(A)}(a_1, \ldots, a_m) = a_1
\]

if such a path does not exist. (It is easy to see that the definition of \(\lambda_{F(A)}\) is correct.)

We shall prove that \(F\) is a realization of \(K\) in \(H_\Delta\). (I) is evident. By \((U), (I')\) will be proved if we prove (II').

Let \(\langle f, \hat{A}, \hat{B} \rangle\) be a \(K\)-morphism. We shall prove that \(f\) is a homomorphism of \(F(A)\) into \(F(B)\). If \(f\) is constant, \(f = O^A_\beta\) for some \(b \in \hat{B}\); then \(\langle B, b \rangle \notin V\) by \((\beta)\), so that \(\lambda_{F(B)}(b) = b\); of course, we have

\[
i_{F(B)}(b, \ldots, b) = b \quad \text{for all} \quad i \in I_0
\]

and, consequently, \(f\) is a homomorphism of \(F(A)\) into \(F(B)\). The case of \(f\) injective remains. If \(\langle A, a \rangle \notin V\), then \(\langle B, f(a) \rangle \notin V\), so that

\[
f(\lambda_{F(A)}(a)) = f(a) = \lambda_{F(B)}(f(a))
\]
If \( \langle A, a \rangle \in V \), then \( \langle B, f(a) \rangle \in V \) by \((\gamma)\). Denote by \( \langle D, d \rangle \) the element of \( Z \) equivalent to \( \langle A, a \rangle \) and by

\[
a = \langle D, f_1, A_1, \ldots, f_n, A \rangle
\]
a path from \( D \) to \( A \) such that \( \hat{a}(d) = a \). Evidently,

\[
\beta = \langle D, f_1, A_1, \ldots, f_n, A, f, B, 1_{\hat{B}}, B \rangle
\]
is a path from \( D \) to \( B \) such that \( \hat{\beta}(d) = f(a) \). We get

\[
f(\lambda_{F(d)}(a)) = f(\hat{a}(h(\langle D, d \rangle))) = \hat{\beta}(h(\langle D, d \rangle)) = \lambda_{F(B)}(f(a)).
\]

Let \( i = \langle e_1, \ldots, e_m, E \rangle \in I_0 \) and \( a_1, \ldots, a_m \in \hat{A} \). If there exists a path

\[
a = \langle E, f_1, A_1, \ldots, f_n, A \rangle
\]
from \( E \) to \( A \) such that \( \hat{a}(e_1) = a_1, \ldots, \hat{a}(e_m) = a_m \), then

\[
\beta = \langle E, f_1, A_1, \ldots, f_n, A, f, B, 1_{\hat{B}}, B \rangle
\]
is a path from \( E \) to \( B \) such that \( \hat{\beta}(e_1) = f(a_1), \ldots, \hat{\beta}(e_m) = f(a_m) \), so that

\[
f(i_{F(A)}(a_1, \ldots, a_m)) = f(a_2) = i_{F(B)}(f(a_1), \ldots, f(a_m)).
\]

Assume that \( a \) does not exist. If

\[
\beta = \langle E, g_1, B_1, \ldots, g_n, B \rangle
\]
were a path such that

\[
\hat{\beta}(e_1) = f(a_1), \ldots, \hat{\beta}(e_m) = f(a_m),
\]
then \( \langle E, g_1, B_1, \ldots, g_n, B, 1_{\hat{B}}, B, f, A \rangle \) would have the properties of \( a \). Hence, \( \beta \) does not exist, and we have

\[
f(i_{F(A)}(a_1, \ldots, a_m)) = f(a_1) = i_{F(B)}(f(a_1), \ldots, f(a_m)).
\]

We have proved that \( f \) is a homomorphism of \( F(A) \) into \( F(B) \).

Let \( A, B \) be two \( K \)-objects and \( f \) a homomorphism of \( F(A) \) into \( F(B) \). We shall prove that \( \langle f, A, B \rangle \) is a \( K \)-morphism. Suppose that there exist three different elements \( a, b, c \) of \( \hat{A} \) such that \( f(a) \neq f(b) = f(c) \). Put \( i = \langle a, b, c, A \rangle \), so that \( i \in I_0 \). There does not exist any path \( a \) from \( A \) to \( B \) such that \( \hat{a}(a) = f(a) \), \( \hat{a}(b) = f(b) \) and \( \hat{a}(c) = f(c) \), as \( \hat{a} \) is injective. On the other hand, \( \langle A, 1_{\hat{A}}, A, 1_{\hat{A}}, A \rangle \) is a path and we get

\[
f(i_{F(A)}(a, b, c)) = f(b) \neq f(a) = i_{F(B)}(f(a), f(b), f(c)),
\]
a contradiction. This shows that \( f \) is either constant or injective.

Let \( f \) be constant, \( f = O_{b}^{d} \) for some \( b \in \hat{B} \). Evidently, \( \lambda_{F(B)}(b) = b \), so that \( \langle B, b \rangle \notin V \). Hence, \( \langle f, A, B \rangle \) is a \( K \)-morphism.
Let $f$ be not constant, and thus injective. Suppose that $\langle f, A, B \rangle$ is not a $K$-morphism. By (a), there exists a finite $X \subseteq \hat{A}$ such that $\hat{a}|_X = f|_X$ for no path $a$ from $A$ to $B$. Put $m = \text{Card}X$. Since $f$ is not constant, we have $\text{Card} \hat{A} \geq 2$ and we may suppose $m \geq 2$. Denote by $a_1, \ldots, a_m$ the elements of $X$ and put $i = \langle a_1, \ldots, a_m, A \rangle$, so that $i \in I_0$. We have

\[
f(i_{F(A)}(a_1, \ldots, a_m)) = f(a_2) \neq f(a_1) = i_{F(B)}(f(a_1), \ldots, f(a_m)),
\]

a contradiction.

**Theorem 2.** Let $K$ be a small concrete category such that if $\langle f, A, B \rangle$ is a $K$-morphism, then the mapping $f$ is injective. $K$ is realizable in some $M \hat{A}$ if and only if conditions (U) and (a) are satisfied and

(8') if $A, B$ are $K$-objects, $a \in \hat{A}$, $\text{Card} \hat{B} = 1$, and $\langle O_a^\hat{B}, B, A \rangle$ is not a $K$-morphism, then $O_a^K(\langle a \rangle)$ has at least two elements.

**Proof.** The direct implication is easy (as in Theorem 1). Let (U), (a) and (8') be fulfilled. Construct a new concrete category $L$ as follows: $L$ has the same class of objects as $K$; if $\text{Card} \hat{D} \geq 2$ for all $K$-objects $D$, then $\langle f, A, B \rangle$ is an $L$-morphism if and only if it is a $K$-morphism or else $f$ is constant; if $\text{Card} \hat{D} = 1$ for some $D$, then $\langle f, A, B \rangle$ is an $L$-morphism if and only if it is a $K$-morphism or else $f = O_b^\hat{D}$ for some $b \in \hat{B}$ and $\langle O_b^D, D, B \rangle$ is a $K$-morphism. It is easy to see that $L$ satisfies conditions (U), (a), (8'), (e) and (8), so that, by Theorem 1, there exists a type $\hat{A}$ and a realization $F$ of $L$ in $H_\hat{A}$. Evidently, $F$ is a realization of $K$ in $M \hat{A}$.

**3. Corollary 1.** Let $K$ be a small concrete category such that $\text{Card} \hat{A} \geq 2$ for all $K$-objects $A$. If $\langle f, A, B \rangle$ is a $K$-morphism, then $f$ is injective. $K$ is realizable in some $M \hat{A}$ if and only if it satisfies (U) and (a). The constructed realization $F$ is such that, for every $K$-object $A$, all subsets of $\hat{A}$ are subuniverses of $F(\hat{A})$.

**Corollary 2.** Let $H$ be a set of injective mappings of a set $A$ into $A$, closed under composition and containing $1_A$. $H$ is the set of all injective endomorphisms of some algebra if and only if $H$ contains any $f$ such that, for every finite $X \subseteq A$, there exists a finite sequence $f_1, \ldots, f_{2n}$ of elements of $H$ such that

\[
f_{2n}^{-1} \circ f_{2n-1} \circ \cdots \circ f_2^{-1} \circ f_1
\]

is defined and coincides with $f$ on $X$.

Denote by $L_\hat{A}$ the concrete category whose objects are algebras of type $\hat{A}$ and morphisms are triples $\langle f, A, B \rangle$ such that $f$ is either an injective or constant homomorphism of $A$ into $B$.

**Theorem 3.** Let a natural number $n \geq 2$ be given and let $K$ be a small concrete category such that if $\langle f, A, B \rangle$ is a $K$-morphism, then $f$ is either
injective or constant. Then $K$ is realizable in some $L_\Delta$ with $\Delta = (n_i)_{i \in I}$ satisfying $n_i \leq n$ for all $i \in I$ if and only if conditions (U), (B), (\gamma) and (\delta) are satisfied and

$(x_n)$ if $A, B$ are $K$-objects and $f$ is an injective mapping of $A$ into $B$ such that, for every $X \subseteq \hat{A}$ of cardinality not greater than $n$, there exists a path $a$ in $K$ from $A$ to $B$ satisfying $\hat{a}|_{C_{A}^K(X)} = f|_{C_{A}^K(X)}$, then $\langle f, A, B \rangle$ is a $K$-morphism.

If $n \geq 3$, we may replace $L_\Delta$ by $H_\Delta$.

Proof. The direct implication is easy. For the converse, define $\lambda_{F(\Delta)}$ as in the proof of Theorem 1; define $I_0$ as the set of all $\langle e_1, \ldots, e_m, e, E \rangle$ such that $E$ is a $K$-object, $2 \leq m \leq n$, $e_1, \ldots, e_m$ are pairwise different elements of $E$, $e \in C_{E}^K(\{e_1, \ldots, e_m\})$; put $I = \{\lambda\} \cup I_0$; if

$$i = \langle e_1, \ldots, e_m, e, E \rangle \in I_0,$$

put $n_i = m$ and define $i_{F(\Delta)}$ as follows: if there exists a path $a$ from $E$ to $A$ such that

$$\hat{a}(e_1) = a_1, \ldots, \hat{a}(e_m) = a_m,$$

then

$$i_{F(\Delta)}(a_1, \ldots, a_m) = \hat{a}(e);$$

if $a$ does not exist, then

$$i_{F(\Delta)}(a_1, \ldots, a_m) = a_1.$$

Let us prove only that if $f$ is an injective but not constant homomorphism of $F(A)$ into $F(B)$, then $\langle f, A, B \rangle$ is a $K$-morphism. Suppose that $\langle f, A, B \rangle$ is not a $K$-morphism. By $(x_n)$, there exists a set $X \subseteq \hat{A}$ of cardinality $m \leq n$ with the prescribed properties. We may suppose $m \geq 2$. Denote by $a_1, \ldots, a_m$ the elements of $X$. If there exists a path $a$ from $A$ to $B$ such that $\hat{a}|_{X} = f|_{X}$, choose an $a \in C_{A}^K(X) - X$ such that $\hat{a}(a) \neq f(a)$, and put

$$i = \langle a_1, \ldots, a_m, a, A \rangle;$$

we have

$$f(i_{F(\Delta)}(a_1, \ldots, a_m)) = f(a) \neq \hat{a}(a) = i_{F(B)}(f(a_1), \ldots, f(a_m)),$$

a contradiction. If $a$ does not exist, put

$$i = \langle a_1, \ldots, a_m, a_2, A \rangle;$$

we have

$$f(i_{F(\Delta)}(a_1, \ldots, a_m)) = f(a_2) \neq f(a_1) = i_{F(B)}(f(a_1), \ldots, f(a_m)),$$

a contradiction again.
In the proof of Theorem 1, to prove that \( f \) is either injective or constant we made use of a ternary operation. Hence, if \( n \geq 3 \), \( F \) is a realization in \( H_\Delta \).

**Corollary 3.** If \( n \geq 2 \) and a type \( \Delta = (n_i)_{i \in I} \) is such that \( n_i \leq n \) for all \( i \in I \), then Theorem 3 holds if we replace \( (x) \) by \( (x_n) \).

**Remark 1.** In the case \( n = 2 \), Theorem 3 does not hold if we replace \( L_\Delta \) by \( H_\Delta \). For example, let \( K \) have a single object \( A \); let \( \text{Card}A \geq 4 \) and let \( \langle f, A, A \rangle \) be a \( K \)-morphism if and only if \( f \) is either an injective or a constant mapping of \( A \) into \( A \). \( K \) satisfies (U), (\( \alpha_2 \)), (\( \beta \)), (\( \gamma \)) and (\( \delta \)). However, if \( B \) is an algebra with at most binary operations, \( B = A \) and every constant and every injective mapping of \( \hat{A} \) into \( \hat{A} \) is an endomorphism of \( B \), then every mapping of \( \hat{A} \) into \( \hat{A} \) is an endomorphism of \( B \).

**Theorem 4.** Let \( K \) be a small concrete category such that if \( \langle f, A, B \rangle \) is a \( K \)-morphism, then \( f \) is either injective or constant. \( K \) is realizable in some \( L_\Delta \) with \( \Delta = (n_i)_{i \in I} \) satisfying \( n_i \leq 1 \) for all \( i \in I \) if and only if conditions (U), (\( \beta \)), (\( \gamma \)) and (\( \delta \)) are satisfied and

\[
\alpha \quad \text{if} \ A, B \text{ are } K\text{-objects, } f \text{ is an injective mapping of } \hat{A} \text{ into } \hat{B}, \text{ and } \\
\beta \quad \text{if } A, B \text{ are } K\text{-morphisms, then there exists an } a \in A \text{ such that either } \\
\gamma \quad \text{if } O^A_e, A, A \rangle \text{ is not a } K\text{-morphism and } \\
\delta \quad \text{if } O^B_e, B, B \rangle \text{ is not a } K\text{-morphism and } \hat{a} |_{C^K_A(a)} = f |_{C^K_A(a)}
\]

**Proof.** Define \( A \) as follows: \( I \) is the set of all \( (e, u, E) \) such that \( E \) is a \( K \)-object, \( e \in \hat{E}, u \in C^K_E(\{e\}) - \{e\} \) and \( O^E_e, E, E \rangle \) is not a \( K \)-morphism; \( n_i = 1 \) for all \( i \).

Define a realization \( F \) as follows: if \( A \) is a \( K \)-object, \( i = (e, u, E) \in I \) and \( a \in \hat{A} \), then \( i_{F(A)}(a) = \hat{a}(u) \) if there exists a path \( a \) from \( E \) to \( A \) such that \( \hat{a}(e) = a \), while \( i_{F(A)}(a) = a \) if \( a \) does not exist.

Again, we prove only that if \( f \) is an injective homomorphism of \( F(A) \) into \( F(B) \), then \( \langle f, A, B \rangle \) is a \( K \)-morphism. Suppose that this is not true. There exists an \( a \in \hat{A} \) as in \( (\alpha) \). If \( O^A_a, A, A \rangle \) is not a \( K \)-morphism and \( \hat{a}(a) = f(a) \) for no path \( a \), we choose a \( b \in C^K_A(\{a\}) - \{a\} \) and put \( i = \langle a, b, A \rangle \); we get

\[
f(i_{F(A)}(a)) = f(b) \neq f(a) = i_{F(B)}(f(a)),
\]

a contradiction. If \( O^A_a, A, A \rangle \) is not a \( K \)-morphism and \( \hat{a}(a) = f(a) \) for some path \( a \), there exists a \( b \in C^K_A(\{a\}) - \{a\} \) such that \( \hat{a}(b) \neq f(b) \); we put \( i = \langle a, b, A \rangle \) and get

\[
f(i_{F(A)}(a)) = f(b) \neq \hat{a}(b) = i_{F(B)}(f(a)),
\]

a contradiction again. Finally, if \( O^A_a, A, A \rangle \) is a \( K \)-morphism and \( O^B_{f(a)}, B, B \rangle \) is not, then there does not exist any path \( a \) from \( A \) to \( B \).
satisfying \( \hat{a}(a) = f(a) \); choosing \( b \in C^K_B(\{f(a)\}) \setminus \{f(a)\} \) and putting \( i = \langle f(a), b, B \rangle \), we get
\[
f(i_{F(A)}(a)) = f(a) \neq b = i_{F(B)}(f(a)),
\]
a contradiction.

Remark 2. It can be analogously proved that if type \( A = (n_i)_{i \in I} \) is such that \( n_i \leq 1 \) for all \( i \), then Theorem 2 holds if we replace \( (x) \) by the following condition:
\( (x''_i) \) if \( A, B \) are \( K \)-objects and \( f \) is an injective mapping of \( \hat{A} \) into \( \hat{B} \) such that, for every \( a \in \hat{A} \) satisfying either \( C^K_A(\{a\}) \neq \{a\} \) or \( C^K_B(\{f(a)\}) \neq \{f(a)\} \), there exists a path \( a \) from \( A \) to \( B \) satisfying \( \hat{a}|_{C^K_A(\{a\})} = f|_{C^K_B(\{a\})} \), then \( \langle f, A, B \rangle \) is a \( K \)-morphism.

REFERENCES


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