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THE TOPOLOGICAL STRUCTURE OF THE SET OF SUBSUMS OF AN INFINITE SERIES

BY

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1. Introduction. Let $\sum a_n$ be a convergent series with $0 < a_{n+1} \leqslant a_n$ for all n and let

$$E = \{\sum \varepsilon_n a_n : \varepsilon_n = 0 \text{ or } 1 \ (n = 1, 2, 3, \ldots)\}$$

denote its set of subsums. Also, let

$$r_n = \sum_{k=n+1}^{\infty} a_k$$

denote the n-th "tail" of the series.

The following three facts about the set E were discovered in 1914 by Kakeya [4], and rediscovered by Hornich [3] in 1941 (see also the 1948 paper of Menon [7]):

A. E is a perfect set.

B. E is the finite union of closed intervals if and only if $a_n \leqslant r_n$ for n sufficiently large. (Also, E is an interval if and only if $a_n \leqslant r_n$ for all n.)

C. If $a_n > r_n$ for n sufficiently large, then E is homeomorphic to the Cantor set.

In the same paper [4], Kakeya conjectured that if $a_n > r_n$ for infinitely many n, then E is nowhere dense (and hence homeomorphic to the Cantor set). It appears that the first counterexample to this conjecture was the one of Weinstein and Shapiro [9] given, without proof, in 1980. Independently, Ferens [2] gave another example, including proof, in 1984. Both of these examples have $a_n > r_n$ if $n \equiv 0 \pmod 5$, while $a_n < r_n$ otherwise, and yet the set E contains a closed interval, and therefore is not nowhere dense. In Section 2 we give another example which is much simpler than the two mentioned above.

In at least two cases, authors [5], [8] have claimed that the set E of subsums must always be either a finite union of closed intervals or homeomorphic to the Cantor set. The examples cited show that this claim is false. Another paper ([1]) has a result on the subject which is neither complete nor precise. We show, however, in Section 3 that the set E is always a finite union of closed intervals, homeomorphic to the Cantor set or homeomorphic to the set of subsums of the examples above.

In Section 4 we generalize that result to show that the range of any finite measure is either a finite set or a set of one of the three types just mentioned.

2. An example. The example given in this section shows that $a_n > r_n$ for infinitely many n is not a sufficient condition for the set of subsums to be homeomorphic to the Cantor set.

Let $a_{2n-1} = 3/4^n$ and $a_{2n} = 2/4^n$ $(n = 1, 2, \ldots)$. Then

$$r_{2n} = (5/3)(1/4^n) < 2/4^n = a_{2n}.$$

We will show that $[3/4, 1] \subseteq E$. Let

$$\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k} \rangle = \sum_{i=1}^{k} \left(\varepsilon_{2i-1}(3/4^i) + \varepsilon_{2i}(2/4^i)\right), \qquad \varepsilon_i = 0 \text{ or } 1,$$

and

$$(\delta_1, \delta_2, \ldots, \delta_k) = \sum_{i=1}^{k} \delta_i/4^i, \qquad \delta_i = 0, 1, 2, 3.$$

To prove $[3/4, 1] \subseteq E$ it is sufficient to show that every $x = (\delta_1, \delta_2, \ldots, \delta_k)$ with $\delta_1 = 3$ is of the form $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k} \rangle$. This will be done by induction on k. The conclusion is clearly true for $k = 1$. Now suppose it is true for all integers from 1 to k. Let $x = (\delta_1, \delta_2, \ldots, \delta_{k+1})$ with $\delta_1 = 3$. The induction step is clear if $\delta_{k+1} = 0$, 2 or 3, so now assume $\delta_{k+1} = 1$. Choose n to be the largest integer for which $1 \leqslant n \leqslant k$ and $\delta_n > 0$. Then, by the induction hypothesis, we can write

$$(\delta_1, \delta_2, \ldots, \delta_{n-1}) = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n} \rangle.$$

Hence

$$(\delta_1, \delta_2, \ldots, \delta_{k+1}) = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}, 1, 0, 1, 0, \ldots, 1, 0, 1, 1 \rangle.$$

3. Topological structure of E. Before stating the theorem which characterizes the topological structure of the set of subsums of a positive term series, we make the following simple observation which will be used in the proof of the theorem. If E is the set of subsums of $\sum a_n$, and E_1 is the set of subsums of some tail of $\sum a_n$, then E is a finite union of translates of E_1.

THEOREM 1. *If E is the set of subsums of a positive term convergent series $\sum a_n$, then E is one of the following*:

(i) *a finite union of closed intervals*;

(ii) *homeomorphic to the Cantor set*;

(iii) *homeomorphic to the set T of subsums of the example in Section* 2.

Proof. Suppose that E is neither a finite union of intervals nor homeomorphic to the Cantor set. Then it is clear that the complement of E must contain infinitely many intervals. E must contain infinitely many intervals as well, for if there were only finitely many, then either $E \cap [0, \varepsilon)$ is an interval for some $\varepsilon > 0$ or $E \cap [0, \varepsilon)$ contains no interval for some $\varepsilon > 0$. If the former is true, then there is some tail of $\sum a_n$ which has an interval as its set of subsums, and therefore E would be a finite union of intervals. If the latter holds, then $E \cap [0, \varepsilon]$ is homeomorphic to the Cantor set, and there is a tail of $\sum a_n$ which has the Cantor set as its set of subsums. Thus E would be homeomorphic to the Cantor set. This is again a contradiction to our initial supposition. Thus E contains infinitely many intervals.

In fact, $E \cap [a, b]$ cannot be homeomorphic to the Cantor set for any $a, b \in E$, since every tail of $\sum a_n$ must have intervals in its set of subsums. Suppose then that, for some $x \in E$,

$$E \cap (x, x+\varepsilon) = \emptyset \quad \text{for some } \varepsilon > 0.$$

Then, since E is perfect, $E \cap (x-\varepsilon, x) \neq \emptyset$ for every $\varepsilon > 0$, and therefore there are intervals in E arbitrarily close to x.

We now define a strictly increasing mapping f from the union of all intervals of T onto the union of all intervals in E. We can define the mapping inductively. Begin by mapping the longest interval in T in a strictly increasing way onto the longest interval in E. There can be at most finitely many intervals of the same length in either set, so we may choose the left-most interval in case no one interval is longest.

After the n-th step, $2^n - 1$ intervals $[\alpha_j, \beta_j] \subset T$ $(1 \leqslant j \leqslant 2^n - 1,\ \beta_j < \alpha_{j+1})$ will have been identified, in a strictly increasing way, with intervals $[\alpha'_j, \beta'_j] \subset E$. Now repeat the above process on each subset of T lying in $[\beta_j, \alpha_{j+1}]$ $(j < 2^n - 1)$ or in $[0, \alpha_1]$ or in $[\beta_{2^n-1}, 5/3]$. That is, map the longest interval in every such portion of T to the longest interval in E lying in $[\beta'_j, \alpha'_{j+1}]$ $(j < 2^n - 1)$ or in $[0, \alpha'_1]$ or in $[\beta'_{2^n-1}, \sum a_k]$, respectively.

When f is defined in this way, it is a strictly increasing mapping of the union of all intervals in T onto the union of all intervals in E. The property verified above that each point of E (and of T) is the limit of a sequence chosen from the intervals of the set allows us to extend f continuously to all of T, and guarantees that the extension will be onto E. The extension will be strictly increasing and, therefore, one-to-one. Since T is compact, f is the desired homeomorphism.

We now give an example of a set which is homeomorphic to the set T of Theorem 1. The set will help us to visualize the set T.

Let C denote the Cantor ternary set and let S_n denote the union of the 2^{n-1} open middle thirds which are removed from $[0, 1]$ at the n-th step in the construction of C. Then

$$C = [0, 1] \sim \bigcup_{n=1}^{\infty} S_n.$$

From the proof of Theorem 1 it is not difficult to see that T is homeomorphic to

$$C \cup \bigcup_{n=1}^{\infty} S_{2n-1}.$$

In Section 1 we noted that $a_n \leqslant r_n$ for n sufficiently large is a necessary and sufficient condition for the set of subsums to be a finite union of closed intervals and that $a_n > r_n$ for n sufficiently large is a sufficient condition for the set of subsums to be homeomorphic to the Cantor set. These facts together with Theorem 1 show that for the set of subsums to be homeomorphic to the set T of Theorem 1 it is necessary that $a_n \leqslant r_n$ for infinitely many n and $a_n > r_n$ for infinitely many n. That this is not a sufficient condition is seen in the following example.

Let $a_{2n-1} = a_{2n} = 2/5^n$ for $n = 1, 2, \ldots$ Then $r_{2n-1} = 3/5^n > a_{2n-1}$ while $r_{2n} = 1/5^n < a_{2n}$. The set of subsums of $\sum a_n$ is clearly the set of those numbers in [0, 1] which have a base-5 expansion using only 0, 2 and 4. This set is clearly homeomorphic to the Cantor set.

It would be of interest to obtain useful necessary and sufficient conditions for the set of subsums to be homeomorphic to the Cantor set.

4. The range of a finite measure. Let $\sum a_n$ be a positive term convergent series and let N denote the set of positive integers. If $A \subseteq N$, define μ by

$$\mu(A) = \sum_{n \in A} a_n.$$

Then μ is a finite measure, defined on the power set of N, whose range is the set of subsums of $\sum a_n$. In this section we prove the following generalization of Theorem 1 to ranges of arbitrary finite measures.

THEOREM 2. *The range of any finite measure is one of the following*:

(i) *a finite set*;

(ii) *a finite union of intervals*;

(iii) *homeomorphic to the Cantor set*;

(iv) *homeomorphic to the set of subsums of the example in Section* 2.

Proof. Let $(\Omega, \mathscr{F}, \mu)$ be an arbitrary finite measure space. A result from measure theory (see, e.g., [6], p. 100) assures us that we can find disjoint sets C and D for which $\Omega = C \cup D$ and, if we let

$$\mu_C(A) = \mu(A \cap C) \quad \text{and} \quad \mu_D(A) = \mu(A \cap D)$$

for all $A \in \mathscr{F}$, then the range of μ_D is the set of all subsums of a finite (possibly empty) sum or an infinite series, and the range of μ_C is the interval $[0, \mu_C(\Omega)]$ (possibly $\{0\}$ if $\mu(C) = 0$). We note then that the range of μ is the algebraic sum of the range of μ_C and the range of μ_D. The proof is completed by considering cases, keeping in mind that the range of μ_D is a finite set or one of the three types of sets of Theorem 1, and the range of μ_C is an interval or $\{0\}$.

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