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An optimal solution of Nicoletti's boundary value problem

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Introduction. In the present note we are concerned with a system of ordinary differential equations

$$x_i' = f_i(t, x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{0.1}$$

and the boundary value problem

$$x_i(t_i) = r_i, \quad 0 \leqslant t_i \leqslant h, \quad i = 1, \ldots, n. \tag{0.2}$$

Without any loss of generality we assume in the sequel that

$$0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant h. \tag{0.3}$$

Problem (0.1)-(0.2) can be written shortly in vector notation as

$$x' = f(t, x), \quad Nx = r, \tag{0.4}$$

where $x = (x_1, \ldots, x_n)$, $f = (f_1, \ldots, f_n)$, $r = (r_1, \ldots, r_n)$ and N is the linear operator of $C^n_{\langle 0,h\rangle}$ into R^n defined by $Nx = \big(x_1(t_1), \ldots, x_n(t_n)\big)$, where $C^n_{\langle 0,h\rangle}$ is the space of n-vector valued continuous functions defined on $\langle 0, h\rangle$ and R^n is the n-dimentional Euclidean space.

Problem (0.4) was posed by O. Nicoletti ([3]) in 1897 and he gave a solution to the problem using the method of successive approximations. Since that time a considerable attention has been paid in the literature to problem (0.4). Let us mention only three more recent papers by R. Conti [1]. M. Švec [6] and by V. P. Skripnik [5]. In Conti's paper [1] the reader can find an extensive list of references concerning this and related boundary value problems for system (0.1).

Our aim is to present two results concerning the uniqueness and the existence of solution of problem (0.4). The sufficient conditions we are giving are the best possible in a certain class of the right-hand sides of the system.

Throughout the paper we assume for $f(t, x)$, in (0.4), the Carathéodory conditions; that is we assume $f(t, x)$ to be defined for $t \in \langle 0, h\rangle$ and $x \in R^n$, continuous in x for each $t \in \langle 0, h\rangle$ and Lebesgue measurable in t for each

$x \in R^n$. By a solution of (0.1) we mean any absolutely continuous function $x(t)$ defined on the interval $\langle 0, h\rangle$ and satisfying (0.1) almost everywhere on $\langle 0, h\rangle$.

THEOREM 1. *Suppose function $f(t, x)$ of $\langle 0, h\rangle \times R^n$ into R^n satisfies the inequality*

$$|f(t, u)-f(t, v)| \leqslant p(t)|u-v| \quad \text{for each } u, v \in R^n , \tag{0.5}$$

where $|\ |$ stands for the Euclidean norm in R^n and $p(t)$ is Lebesgue integrable on $\langle 0, h\rangle$ and such that

$$\int_0^h p(t)\,dt < \pi/2 . \tag{0.6}$$

Then a solution of (0.4), *if exists, is unique for any $0 \leqslant t_i \leqslant h$ and any r.*

THEOREM 2. *Suppose $f(t, x)$, in* (0.4), *satisfies the Carathéodory conditions and the inequality*

$$|f(t, x)| \leqslant p(t)|x|+g(t, x) \quad \text{for} \quad t \in \langle 0, h\rangle ,\ x \in R^n , \tag{0.7}$$

where $p(t)$ is Lebesgue integrable function satisfying condition (0.6) *and $g(t, x)$ is a function of $\langle 0, h\rangle \times R^n$ satisfying Carathéodory conditions and the following assumption*

$$\lim_{k\to\infty} \frac{1}{k} \int_0^h \sup_{|x|\leqslant k} g(t, x)\,dt = 0 . \tag{0.8}$$

Under these assumptions problem (0.4) *has at least one solution for each $t_i \in \langle 0, h\rangle$, $i = 1, \dots, n$, and each $r \in R^n$.*

From Theorems 1 and 2 follows the following

COROLLARY 1. *If function $f(t, x)$ satisfies Carathéodory conditions, inequalities* (0.5) *and* (0.6) *hold and $f(t, 0)$ is Lebesgue integrable over $\langle 0, h\rangle$ then there exists a solution of problem* (0.4) *and is unique.*

Proofs of Theorem 1 and Theorem 2 are given in Section 3 and 4, respectively. Theorem 1 is a consequence of Theorem 3, given in Section 2, concerning a differential inequality. Section 1 contains a lemma which we need to prove Theorem 3. This lemma admits as a special case the following simple but worthwhile noticing purely geometrical statement.

Let $p_i = (p_{i1}, \dots, p_{in})$, $i = 1, \dots, n$, be n given points on the unit sphere S of the Euclidean n-space R^n. If $p_{ii} = 0$ for $i = 1, \dots, n$, then $\sum_{i=1}^{n-1} \varrho(p_i, p_{i+1}) \geqslant \pi/2$, where $\varrho(x, y)$ if $x, y \in S$ stands for the length of the (shorter) great circle's arc joining x and y.

As the reader will notice, the proof of the lemma is reduced to a proof of the above statement.

Theorem 2 is a consequence of Theorem 3 and a general result of [2] due to the first of the authors. For convenience of the reader this result is stated in Section 4. Finally Section 5 contains some concluding remarks. There we give an example showing that if "$<$" in (0.6) is replaced by "$\leqslant$" then both Theorems 1 and 2 fail to hold true. This example proves that for the class of right-hand members $f(t,x)$ of (0.1) satisfying (0.5) (or (0.7) and (0.8)) the condition (0.6) is the best possible for the uniqueness (or the existence) of solution of (0.4).

1. A lemma. Let $x, y \in R^n$; by (x, y) we denote the scalar product of x and y. Then $|x| = (x, x)^{1/2}$ is the Euclidean norm of x. Denote $S = \{x \in R^n\colon\ |x| = 1\}$.

LEMMA. *Suppose an absolutely continuous function φ of $\langle 0, h\rangle$ into S is given. Assume there is a sequence $0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant h$ such that*

$$\varphi_i(t_i) = 0\,, \quad i = 1, \ldots, n\,, \tag{1.1}$$

where φ_i, $i = 1, \ldots, n$, are coordinates of φ. Then

$$\int_{t_1}^{t_n} |\varphi'(t)|\,dt \geqslant \pi/2\,. \tag{1.2}$$

Proof. Set $p_i = (p_{i1}, \ldots, p_{in}) = \varphi(t_i)$, $i = 1, \ldots, n$. We have by (1.1)

$$p_{ii} = 0 \quad \text{for} \quad i = 1, \ldots, n\,. \tag{1.3}$$

Since $|p_i| = 1$ for $i = 1, \ldots, n$, therefore it follows from (1.3) that $p_i \neq p_j$ at least for one pair i, j, $i \neq j$, and in consequence $t_1 < t_n$. Let $x, y \in S$. We set

$$\varrho(x, y) = \inf \int_a^b |\psi'(t)|\,dt\,, \tag{1.4}$$

where the infimum is taken over all absolutely continuous ψ of $\langle a, b\rangle$ into S such that $\psi(a) = x$ and $\psi(b) = y$. Since the integral in (1.4) is the length of an arc on S connecting x and y, it is easy to see that

$$\varrho(x, y) = \arccos\langle(x, y)\rangle \tag{1.5}$$

and that

$$\varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y) \text{ for any } x, y, z \in S\,. \tag{1.6}$$

By (1.4) and (1.5) we have

$$\int_{t_1}^{t_n} |\varphi'(t)|\,dt \geqslant \sum_{i=1}^{n-1} \varrho(p_i, p_{i+1}) = \sum_{i=1}^{n-1} \arccos\langle(p_i, p_{i+1})\rangle\,. \tag{1.7}$$

Hence to show (1.2) it is enough to prove that

$$\sum_{i=1}^{n-1} \varrho(p_i, p_{i+1}) \geqslant \pi/2 . \tag{1.8}$$

For the proof of inequality (1.8) we will show the existence of an integer $k < n$ and a sequence $q_i = (q_{i1}, \ldots, q_{in})$, $i = 1, \ldots, k$, such that

(i) $\quad |q_i| = 1 \quad \text{and} \quad q_{ij} = 0 \quad \text{for} \quad j \leqslant i\,,\ i = 1, \ldots, k\,,$

(ii) $\quad \varrho(q_i, p_{i+1}) \leqslant \sum_{j=1}^{i} \varrho(p_j, p_{j+1}) \quad \text{for} \quad i = 1, \ldots, k\,,$

(iii) $\quad \varrho(q_k, p_{k+1}) \geqslant \pi/2\,.$

If such integer k and sequence q_i exist then (1.8) is an obvious consequence of (iii) and (ii) for $i = k$. Therefore to complete the proof of the Lemma we need to prove the existence of k and q_i satisfying (i)-(iii). For this purpose we shall use the induction argument. We take $q_1 = p_1$. Then (i) and (ii) are satisfied for $i = 1$. If $(p_1, p_2) = (q_1, p_2) \leqslant 0$ then by (1.5) also (iii) holds for $k = 1$ and we are done. Suppose then that $(p_1, p_2) > 0$ and suppose we have defined q_i for $i = 1, \ldots, m$ such that (i) and (ii) hold for $i = 1, \ldots, m$ and

$$(q_i, p_{i+1}) > 0 \quad \text{for} \quad i = 1, \ldots, m\,, \tag{1.9}$$

where $m \geqslant 1$. Note that $m < n-1$. Indeed if $m = n-1$ then by (i) for $i = n-1$ and (1.3) $(q_{n-1}, p_n) = 0$ which contradics (1.9). Let $s = (s_1, \ldots, s_n)$ and put $s_i = 0$ for $i = 1, \ldots, m+1$ and $s_i = q_{mi}$ for $i = m+2, \ldots, n$. We have by (i) for $i = m$ that $|s| \leqslant |q_m| = 1$ and by (1.3) and (1.9) that

$$(s, p_{m+1}) = (q_m, p_{m+1}) > 0\,, \tag{1.10}$$

which implies that $|s| > 0$, hence $0 < |s| \leqslant 1$. We put $q_{m+1} = s/|s|$. Clearly (i) is satisfied for $i = m+1$. By (1.10) we get the inequality $(q_{m+1}, p_{m+1}) \geqslant (q_m, p_{m+1})$, therefore by (1.5)

$$\varrho(q_{m+1}, p_{m+1}) \leqslant \varrho(q_m, p_{m+1})\,. \tag{1.11}$$

By (1.11), (1.6) and (ii) for $i = m$ we have

$$\varrho(q_{m+1}, p_{m+2}) \leqslant \varrho(q_m, p_{m+1}) + \varrho(p_{m+1}, p_{m+2}) \leqslant \sum_{j=1}^{m+1} \varrho(p_j, p_{j+1})\,.$$

Hence (ii) also holds for $i = m+1$. Now either $(q_{m+1}, p_{m+2}) \leqslant 0$ then $k = m+1$ and we are done or (1.9) holds for $i = m+1$ and we can define q_{m+2}. It is clear that either there is $k < n-1$ such that (iii) holds or q_i satisfying (i) and (ii) can be defined for $i = 1, \ldots, n-1$. In the last case, as we already noticed, (q_{n-1}, p_n) must be equal zero. Thus $\varrho(q_{n-1}, p_n) = \pi/2$ hence (iii) holds for $k = n-1$. Therefore we proved the existence of k and q_i, $i = 1, \ldots, k$, such that (i)-(iii) hold, which completes the proof of the Lemma.

Remark. Let us observe that the equality in (1.2) can hold only if

$$\varrho(p_1, p_n) = \sum_{i=1}^{n-1} \varrho(p_i, p_{i+1}) = \pi/2 \quad \text{and} \quad \int_{t_i}^{t_{i+1}} |\varphi'(t)|\,dt = \varrho(p_i, p_{i+1})$$

for $i = 1, \ldots, n-1$. This is the case if

$$\varphi(t) = p_1 \cos P(t) + p_n \sin P(t),$$

where $P(t) = \int_0^t |\varphi'(t)|\,dt$, $|p_1| = |p_n| = 1$ and $(p_1, p_n) = 0$.

2. A differential inequality. Theorem 1 stated in the Introduction is a consequence of the following result.

THEOREM 3. *Suppose the function* p *of* $\langle 0, h\rangle$ *into* R^1 *is Lebesgue integrable and nonnegative. Consider the differantial inequality for an* n*-vector valued function*

$$|x'(t)| \leqslant p(t)\,|x(t)|, \quad 0 \leqslant t \leqslant h, \tag{2.1}$$

and the homogeneous condition corresponding to (0.2); *that is*

$$x_i(t_i) = 0, \quad 0 \leqslant t_i \leqslant h, \quad i = 1, \ldots, n. \tag{2.2}$$

If $p(t)$, *in* (2.1), *satisfies the inequality*

$$\int_0^h p(t)\,dt < \pi/2 \tag{2.3}$$

then $x(t) \equiv 0$ *is the only one absolutely continuous function of* $\langle 0, h\rangle$ *into* R^n *which satisfies condition* (2.2) *and inequality* (2.1) *almost everywhere in* $\langle 0, h\rangle$.

Proof. Let us note first that if an absolutely continuous function $x(t)$ satisfies (2.1) and $|x(0)| \neq 0$ then $|x(t)| \neq 0$ for each $t \in \langle 0, h\rangle$. Therefore any absolutely continuous solution of (2.1) and (2.2) is either trivial ($x(t) \equiv 0$) or $|x(t)| \neq 0$ for each $t \in \langle 0, h\rangle$. Suppose there exists a nontrivial solution $x(t)$ of (2.1) satisfying (2.2). Put $\varphi(t) = x(t)/|x(t)|$, then φ is a map of $\langle 0, h\rangle$ into S and it satisfies assumption (1.1) of the Lemma since $x(t)$ satisfies (2.2). Hence owing to the Lemma

$$\int_0^h |\varphi'(t)|\,dt \geqslant \int_{t_1}^{t_n} |\varphi'(t)|\,dt \geqslant \pi/2. \tag{2.4}$$

On the other hand we have

$$|\varphi'(t)| = \left|\frac{x'(t)}{|x(t)|} - \frac{x(t)\big(x(t), x'(t)\big)}{|x(t)|^3}\right| = \left(\frac{|x'(t)|^2}{|x(t)|^2} - \frac{\big(x(t), x'(t)\big)^2}{|x(t)|^4}\right)^{1/2}.$$

But

$$0 \leqslant |x'(t)|^2|x(t)|^2 - \big(x'(t), x(t)\big)^2 \leqslant |x'(t)|^2|x(t)|^2.$$

Therefore by (2.1)

$$(2.5) \qquad |\varphi'(t)| \leqslant \frac{|x'(t)|}{|x(t)|} \leqslant p(t).$$

Integrating (2.5) over the interval $\langle 0, h\rangle$ and making use of (2.3), we obtain a contradiction with (2.4). Hence there cannot exist a non-trivial solution of (2.1) and (2.2), which finishes the proof of Theorem 3.

3. Proof of Theorem 1. Suppose $\bar{x}(t)$ and $\bar{\bar{x}}(t)$ are two absolutely continuous solutions of problem (0.4). Then $x(t) = \bar{x}(t) - \bar{\bar{x}}(t)$ satisfies the homogeneous condition (2.2) and by (0.5) inequality (2.1) holds, too. Since assumption (0.6) of Theorem 1 is the same as (2.3) in Theorem 3, therefore by Theorem 3 $x(t) \equiv 0$, which proves Theorem 1.

4. Proof of Theorem 2. The proof of Theorem 2 is based on a result concerning homogeneous contingent equations. This result will be now stated.

Denote by $\mathrm{cf}(R^n)$ the metric space of all closed and convex subsets of R^n, where the metric function is given by the Hausdorff distance of two sets; that is by the function

$$d(A, B) = \max\big(\sup_{x\in B}\delta(x, A), \sup_{x\in A}\delta(x, B)\big), \qquad A, B \in \mathrm{cf}(R^n),$$

where $\delta(x, A)$ denotes the Euclidean distance of point x from set A.

The metric d and Lebesgue measure on $\langle 0, h\rangle$ allow to introduce the class of Lebesgue measurable maps of $\langle 0, h\rangle$ into $\mathrm{cf}(R^n)$. Here we adopt the following convienient definition due to A. Pliś ([4]). We say that a map $F(t)$ of $\langle 0, h\rangle$ into $\mathrm{cf}(R^n)$ is *Lebesgue measurable* if for each closed $A \subset R^n$ the set $\{t\colon A \cap F(t) \neq \emptyset\}$ is Lebesgue measurable.

Let $F(t, x)$ be a map of $\langle 0, h\rangle \times R^n$ into $\mathrm{cf}(R^n)$, $f(t, x)$ a map of $\langle 0, h\rangle \times R^n$ into R^n and $L(x)$ a map of $C^n_{\langle 0,h\rangle}$ into R^n. Concerning F, f and L let us suppose the following assumptions.

(i) Assume $F(t, x)$ satisfies the Carathéodory conditions; that is F is measurable in t for each $x \in R^n$ and continuous in x for each $t \in \langle 0, h\rangle$. Assume $F(t, x)$ is homogeneous in x; that is $F(t, \lambda x) = \lambda F(t, x)$ for each real λ, and finally suppose there is Lebesgue integrable real function $\varphi(t)$, $t \in \langle 0, h\rangle$ such that for each fixed t

$$|y| \leqslant \varphi(t) \quad \text{for each} \quad y \in \bigcup_{|x|=1} F(t, x).$$

(ii) The map $f(t, x)$ satisfies the Carathéodory conditions and is such that

$$\lim_{k\to\infty} \frac{1}{k} \int_0^h \sup_{|x|\leqslant k} \delta\big(f(t, x), F(t, x)\big)\, dt = 0 . \tag{4.1}$$

(iii) The map $L(x)$ is continuous and homogeneous; that is $L(\lambda x) = \lambda L(x)$ for each real λ, where the topology in $C^n_{\langle 0,h\rangle}$ is given by the uniform convergence.

Consider a contingent equation; that is the condition

$$x' \in F(t, x) , \tag{4.2}$$

where $F(t, x)$ is a map of $\langle 0, h\rangle \times R^n$ into $\mathrm{cf}(R^n)$. By a solution of (4.2) we mean any absolutely continuous function of an interval, say $\langle 0, h\rangle$, into R^n which satisfies (4.2) almost everywhere.

The following theorem is due to A. Lasota ([2]).

THEOREM 4. *Assume F, f and L satisfy assumptions* (i), (ii) *and* (iii), *respectively.*

If $x(t) \equiv 0$ is the only one solution of (4.2) *defined on $\langle 0, h\rangle$, which satisfies the homogeneous boundary value condition*

$$L(x) = 0 ,$$

then the boundary value problem

$$x' = f(t, x) , \quad L(x) = r$$

has at least one solution for each $r \in R^n$.

Let us go back now to the proof of Theorem 2. Let us set

$$F(t, x) = \{y\colon\ |y| \leqslant p(t)\,|x|\} . \tag{4.3}$$

For each t, x, $F(t, x)$ is a closed n-ball of center 0 and diameter $p(t)\,|x|$. Therefore it is clear that $F(t, x) \in \mathrm{cf}(R^n)$, is homogeneous and continuous in x for each t. If $A \subset R^n$ is closed then for each fixed x we have

$$\{t\colon\ F(t, x) \cap A \neq \emptyset\} = \{t\colon\ p(t)\,|x| \geqslant \delta(0, A)\} .$$

Hence $F(t, x)$ is Lebesgue measurable in t. Now the last part of (i) is satisfied by F given by (4.3) with $\varphi(t) = p(t)$, since $p(t)$ is assumed to be Lebesgue measurable. Hence condition (i) holds.

Condition (ii) for $f(t, x)$, in (0.4), is assumed in Theorem 2. Note that, if F is given by (4.3), by (0.7) we have $\delta\big(f(t, x), F(t, x)\big) \leqslant g(t, x)$. Therefore (4.1) holds because of (0.8) assumed in Theorem 2.

Finally, condition (iii) is satisfied for $L(x) = Nx$ (cf. (0.4)) since N is linear. Since assumption (2.3) is satisfied by $p(t)$ in Theorem 2, therefore by Theorem 3 $x(t) \equiv 0$ is unique solution of (4.2), with F given by (4.3), which satisfies (2.2); that is the homogeneous condition $Nx = 0$. Hence by Theorem 4 we conclude that problem (0.4) has at least one solution for each $r \in R^n$, which was to be proved.

5. Concluding remarks. As we have mentioned in Introduction our results (Theorems 1 and 2) are best possible for the class of right-hand sides of (0.1) satisfying inequality (0.5) (or (0.7) with condition (0.8)) with $p(t)$ and h being fixed. More precisely, if we fix h and $p(t) \geqslant 0$ such that

$$\int_0^h p(t)\,dt = \pi/2 \tag{5.1}$$

then there are $t_i \in \langle 0, h\rangle$, $i = 1, \ldots, n$, and $f(t, x)$ satisfying (0.5) (or (0.7) and (0.8)) such that there are at least two different solutions of problem (0.4) (or there does not exist any solution of (0.4)) for some $r \in R^n$. This is shown by the following example.

Consider system (0.1) with $f_i(t, x)$ given by

$$f_1(t, x) = -p(t)x_2\,, \quad f_2(t, x) = p(t)x_1\,, \quad f_i(t, x) = 0 \quad \text{for} \quad i > 2\,. \tag{5.2}$$

It is obvious that $f(t, x)$ defined by (5.2) satisfies (0.5) (or (0.7) with $g(t, x) = 0$). It is easy to check that

$$\begin{aligned} x_1(t) &= c_1\cos P(t) - c_2\sin P(t)\,, \\ x_2(t) &= c_1\sin P(t) + c_2\cos P(t)\,, \\ x_i(t) &= c_i\,, \quad i = 3, \ldots, n\,, \end{aligned} \tag{5.3}$$

where $P(t) = \int_0^t p(\tau)\,d\tau$ and c_i are constants, is the general solution of (0.1) when $f(t, x)$ is given by (5.2).

Consider now the boundary value condition

$$x_1(0) = a\,, \quad x_2(h) = x_i(t_i) = 0\,, \quad 0 \leqslant t_i \leqslant h\,,\ i = 3, \ldots, n\,, \tag{5.4}$$

If $a = 0$ then because of (5.1) solution (5.3) satisfies (5.4) if $c_1 = c_3 = \ldots = c_n = 0$ and c_2 arbitrary. Hence a solution of the system satisfying (5.4) is not unique. If $a \neq 0$ then there is no solution of the system satisfying (5.4). Indeed, since $x_1(0) \neq 0$ thus by (5.3) c_1 must be different from zero. Therefore by (5.1) and (5.3) we get that $x_2(h) = c_1 \neq 0$ and (5.4) cannot be satisfied.

Note that the example we gave is essentially two-dimensional. The first two equations of the system make the example working and the remaining are only for a decoration. It is also clear that as $f_3, \ldots, f_n$ we could take as well arbitrary functions but homogeneous and such that (0.5) or (0.7) would be satisfied. On the other hand, as follows from the Remark of Section 1, if we replace "$<$" in (2.3) by "$\leqslant$" then a non-trivial solution

$x(t)$ of (2.1) satisfying the homogeneous condition (2.2) must be two-dimensional in the sense that there is a two-dimensional subspace of R^n containing $x(t)$ for $t \in \langle 0, h \rangle$.

In the formulation of Theorem 1 and 2 we gave the condition (0.6) concerns the integral of $p(t)$ over the interval $\langle 0, h \rangle$ and because of that our results are valid for any $t_i \in \langle 0, h \rangle$, $i = 1, \ldots, n$. It is clear that if we fix t_i in (0.2) or (0.4) then what we really need for the validity of Theorem 1 and 2 is the inequality

$$\int_{\alpha}^{\beta} p(t)\,dt < \pi/2 ,$$

where $\alpha = \min_i t_i$ and $\beta = \max_i t_i$.

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