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METRIZABILITY AND WEIGHT OF INVERSES UNDER CONFLUENT MAPPINGS

BY

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We say that a mapping $f: X \to Y$ of a topological space X onto a topological space Y is *confluent* provided for every connected closed subset C of Y, and points $x \in f^{-1}(C)$ and $y \in C$, the set $f^{-1}(C)$ is connected between $\{x\}$ and $f^{-1}(y)$, i.e. every open-and-closed neighbourhood of x in $f^{-1}(C)$ meets $f^{-1}(y)$. This notion was introduced in [4]. We say that a mapping $f: X \to Y$ is *locally confluent* provided every point $y \in Y$ has a neighbourhood V_y in Y such that $f|f^{-1}(V_y)$ is confluent. A routine argument shows that the class of confluent mappings contains all mappings which are either open-and-closed, or monotone and closed, or monotone and open. Thus within the class of continuous mappings of compact Hausdorff spaces, confluent mappings constitute a common generalization for monotone mappings and open mappings. In the theory of compact Hausdorff spaces an important role is played also by 0-dimensional mappings, i.e. those having 0-dimensional inverses of points. A counterpart to 0-dimensional mappings, suitable for the non-compact case, can be defined as follows. We say that a mapping $f: X \to Y$ is *separative* provided for every point $x \in X$ and its neighbourhood U in X there exists a neighbourhood V of $f(x)$ in Y such that the set $f^{-1}(V)$ is not connected between $\{x\}$ and $f^{-1}(V) \setminus U$, i.e there is an open-and-closed neighbourhood of x in $f^{-1}(V)$ which is contained in U. The concept of separative mappings is due to Zarelua [10]. A continuous mapping of a locally compact Hausdorff space is separative if and only if it is 0-dimensional.

The problem of estimating the weight of inverses under 0-dimensional mappings has been investigated by Mardešić [5]. More precisely, his theorem says that if $f: X \to Y$ is a 0-dimensional continuous mapping of a locally connected compact Hausdorff space X onto a Hausdorff (infinite) space Y, then $w(X) = w(Y)$. An analogue of this theorem for non-compact spaces, involving separative mappings, has been given by Proizvolov [6]. In the present paper we prove that a similar estimation of weights is possible under the assumption that the space Y is locally connected in lieu of the much stronger assumption that its inverse X is locally connected. Then, however, it is necessary to assume some additional conditions concerning the function f, and we have done this by assuming, among other things, that f is locally confluent (see Theorem 1).

The problem of metrizing inverses of metrizable spaces has been raised by Proizvolov [7] who asked whether or not a compact Hausdorff space must be metrizable if it admits an open continuous mapping onto a metrizable space such that all inverses of points are metrizable (see also [1], p. 170, Problem 5.5). A negative solution of this problem has been given by Veličko [8] who constructed a non-metrizable compact Hausdorff space X and an open continuous mapping of X onto the unit segment I such that all inverses of points are homeomorphic to I. What we propose is a positive solution of the Proizvolov problem in a special case: the mapping is assumed to be separative and the image is assumed to possess a σ-locally finite base consisting of connected sets (see Theorem 2).

We start with an example of a countable-to-one open continuous mapping of a non-metrizable compact Hausdorff space onto a metrizable space. In this way we replace the monotoneity of the function achieved in [8] by the requirement that all inverses of points are countable. The notation and the terminology come from [2]. Let Q denote the Cantor quinary set, i.e. the set of all real numbers t such that

$$t = \sum_{i=1}^{\infty} \frac{t_i}{5^i},$$

where $t_i = 0$ or 4 for $i = 1, 2, \ldots$ We provide Q with the natural topology inherited from the real line. Actually, in our example just the space Q will be an open image of a non-metrizable compact Hausdorff space, and a special feature of the Cantor quinary set will be exploited: we have $t \neq u \pm 2/5^i$ for $t, u \in Q$ and $i = 1, 2, \ldots$ It follows that no point $(t, 2/5^i)$ lies on the line $x = u \pm y$ for any pair of points $t, u \in Q$.

Example. Let D be the Cantor set Q equipped with the discrete topology and let $i: D \to Q$ be the identity mapping. *There exist a compact Hausdorff space D^* containing D as a subspace and an open continuous extension i^*: $D^* \to Q$ of i over D^* such that D^* satisfies the first axiom of countability and $i^{*-1}(q) \setminus D$ is a countable discrete subspace for $q \in Q$.* Given real numbers x_0 and y_0, we denote by $A(x_0)$, $B_i(x_0, y_0)$, $C(y_0)$ the plane sets

$$A(x_0) = \{(x, y) \colon x_0 - y \leqslant x \leqslant x_0 + y\},$$

$$B_i(x_0, y_0) = \{(x, y) \colon (x - x_0)^2 + (y - y_0)^2 < 1/4^i\},$$

$$C(y_0) = \{(x, y_0) \colon x \in Q\},$$

for $i = 1, 2, \ldots$ We put

$$D^* = C(0) \cup C(1) \cup \bigcup_{i=1}^{\infty} C\left(\frac{2}{5^i}\right)$$

and determine the topology in D^* be generated by a neighbourhood system $\{\mathfrak{B}(p)\}$ which is defined as follows. Let $p = (x_0, y_0) \in D^*$; we distinguish three cases.

Case 1: $p \in C(0)$. Then we define

$$\mathfrak{B}(p) = \{A(x_0) \cap B_i(x_0, 0) \cap D^*\}_{i=1}^{\infty}.$$

Case 2: $p \in C(1)$. Then we define

$$\mathfrak{B}(p) = \{(B_i(x_0, 1) \cup [B_i(x_0, 0) \setminus A(x_0)]) \cap D^*\}_{i=1}^{\infty}.$$

Case 3: $p \in D^* \setminus C(0) \setminus C(1)$. Then

$$\mathfrak{B}(p) = \{B_i(x_0, y_0) \cap D^*\}_{i=1}^{\infty}.$$

It is easily seen that D^* with this topology is a Hausdorff space. Moreover, the subspace $C(0)$ is Q with the discrete topology, i.e. it coincides with D. To see that D^* is compact, let us observe that the subspace $C(1)$ is Q with the natural topology, thus it is compact. Since elements of $\mathfrak{B}(p)$ do not meet $C(1)$ for $p \notin C(1)$, every collection covering the space D^* and consisting of sets from our neighbourhood system contains a finite subcollection covering the whole set $C(1)$ and also the set $C(0)$ except for a finite number of points. We only have to adjoin some neighbourhoods of these points in order to obtain a finite subcollection of sets whose union U contains $C(0) \cup C(1)$. But then the subspace $D^* \setminus U$ is compact which yields the compactness of D^*.

The mapping $i^*\colon D^* \to Q$ is now defined by the formula

$$i^*((x, y)) = x$$

for $(x, y) \in D^*$. Clearly, i^* is continuous and open. If $q \in Q$, the set $i^{*-1}(q)$ consists of a countable number of points which converge to $(q, 0) \in D$.

Remarks. It seems to be worth noticing that our space D^* contains a countable dense subset, and that the subspace $D^* \setminus C(1)$ is very much like the Niemytzki plane (see [2], p. 34, Example 2). On the other hand, the subspace $C(0) \cup C(1)$ is the "double Cantor set" (see [3], p. 629; see also [2], p. 109, Exercise E). The subspace D being uncountable and discrete, the space D^* is non-metrizable. Nevertheless D^* is transformed onto the Cantor set Q via the open continuous mapping i^* under which the inverses of points are countable, and thus they all are metrizable. By a result of Proizvolov [7], the inverses of points under such a mapping cannot be finite. Taking a standard extension of i^* over the cones built up over D^* and Q, we obtain a countable-to-one open continuous mapping of a non-metrizable continuum onto a metrizable continuum. The latter continuum is not locally connected, and this is essential here (see Corollary 3.3).

LEMMA. *Let $f: X \to Y$ be a separative continuous mapping of a topological space X onto a regular space Y and let $\{G_s\}_{s \in S}$ be a base in Y. If $F_s \subset Y$ is a set such that $G_s \subset F_s \subset \bar{G}_s$ and $\mathfrak{F}_s$ denotes the collection of all open-and-closed non-void subsets of the subspace $f^{-1}(F_s)$ for $s \in S$, then the collection $\mathfrak{B}$ defined by the formula*

$$\mathfrak{B} = \bigcup_{s \in S} \{\mathrm{Int} Q\colon Q \in \mathfrak{F}_s\}$$

is a base in X.

Proof. Suppose $U \subset X$ is an open set and $x \in U$. Since f is separative, there exist a neighbourhood V of $f(x)$ in Y and an open-and-closed neighbourhood P of x in $f^{-1}(V)$ such that $P \subset U$. Since Y is regular, there exists a neighbourhood V_0 of $f(x)$ in Y such that $\bar{V}_0 \subset V$. Then there exists an index $s_0 \in S$ satisfying $f(x) \in G_{s_0} \subset V_0$. Thus $F_{s_0} \subset V$ whence $f^{-1}(F_{s_0}) \subset f^{-1}(V)$ and the set $Q = P \cap f^{-1}(F_{s_0})$ is open-and-closed in $f^{-1}(F_{s_0})$. But the sets $f^{-1}(F_{s_0})$ and $f^{-1}(V)$ are neighbourhoods of x in X, and so are the sets P and Q. Consequently, we have $Q \in \mathfrak{F}_{s_0}$ and $Q \subset P \subset U$, whence $x \in \mathrm{Int} Q \in \mathfrak{B}$ and $\mathrm{Int} Q \subset U$.

THEOREM 1. *Let $f\colon X \to Y$ be a separative locally confluent continuous mapping of a topological space X onto a locally connected regular space Y. If $w(Y) \leqslant \mathfrak{m}$ (where $\aleph_0 \leqslant \mathfrak{m}$) and there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a compact Hausdorff subspace with the weight $w[f^{-1}(y)] \leqslant \mathfrak{m}$ for $y \in A$, then $w(X) \leqslant \mathfrak{m}$.*

Proof. For any point $y \in Y$, let us denote by V_y a neighbourhood of y in Y such that $f|f^{-1}(V_y)$ is confluent. Since Y is locally connected and the weight of Y does not exceed $\mathfrak{m}$, there exists a base $\{G_s\}_{s \in S}$ in Y such that $\overline{\overline{S}} \leqslant \mathfrak{m}$ and the set G_s is connected and contained in a set V_{y_s} for $s \in S$. Let F_s be the closure of G_s in V_{y_s}. Thus $f|f^{-1}(F_s)$ is confluent ($s \in S$) and it suffices to show that the base $\mathfrak{B}$ described in the lemma above has the cardinality $\overline{\overline{\mathfrak{B}}} \leqslant \mathfrak{m}$. Since $\mathfrak{m}$ is infinite, it is enough to verify the inequality $\overline{\overline{\mathfrak{F}_s}} \leqslant \mathfrak{m}$ for $s \in S$.

In fact, let us choose a point $a_s \in A \cap G_s$. Thus $a_s \in F_s$ and since $f|f^{-1}(F_s)$ is confluent, every set belonging to $\mathfrak{F}_s$ meets $f^{-1}(a_s)$. If $Q, Q' \in \mathfrak{F}_s$ and $Q \setminus Q' \neq \emptyset$, then $Q \setminus Q' \in \mathfrak{F}_s$, whence $Q \setminus Q'$ meets $f^{-1}(a_s)$. It follows that if we assign to each set $Q \in \mathfrak{F}_s$ the intersection $Q \cap f^{-1}(a_s)$, we get a one-to-one correspondence between the sets from $\mathfrak{F}_s$ and some open-and-closed subsets of the subspace $f^{-1}(a_s)$. But the space $f^{-1}(a_s)$ being compact Hausdorff, the collection of all open-and-closed subsets of $f^{-1}(a_s)$ has the cardinality not greater than $w[f^{-1}(a_s)]$. Since $a_s \in A$, we conclude that $\overline{\overline{\mathfrak{F}}}_s \leqslant \mathfrak{m}$, and the proof of Theorem 1 is complete.

Remarks. In this proof, we have utilized the compactness of $f^{-1}(y)$ for $y \in A$ only when estimating how many open-and-closed subsets are in $f^{-1}(y)$. Instead of compactness of $f^{-1}(y)$ we could as well assume a weaker condition: namely that $f^{-1}(y)$ is completely regular and the space of quasi-components of $f^{-1}(y)$ is compact for $y \in A$ (see [9], p. 49). We check briefly the necessity of some other conditions in Theorem 1: the example of the mapping constructed in [8] shows the condition saying that f is separative cannot be omitted. The projection of the "double segment" (see [2], p. 107, Example 2) onto I shows the necessity of the condition saying that f is locally confluent. The mapping i^* in our example above applies to argue that Y must be assumed to be locally connected. Also, the inequality for the weight of $f^{-1}(y)$ cannot be omitted in Theorem 1 as it is shown by projection of $\omega D \times I$ onto I, where ωD stands for the Alexandroff compactification of a discrete uncountable space D.

COROLLARY 1.1. *Let $f: X \to Y$ be a separative open-and-closed continuous mapping of a topological space X onto a locally connected regular space Y. If $w(Y) \leqslant \mathfrak{m}$ (where $\aleph_0 \leqslant \mathfrak{m}$) and there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a compact Haüsdorff subspace with $w[f^{-1}(y)] \leqslant \mathfrak{m}$ for $y \in A$, then $w(X) \leqslant \mathfrak{m}$.*

COROLLARY 1.2. *Let $f: X \to Y$ be a 0-dimensional open perfect mapping of a Hausdorff space X onto a locally connected space Y. If $w(Y) \leqslant \mathfrak{m}$ (where $\aleph_0 \leqslant \mathfrak{m}$) and there exists a dense subset $A \subset Y$ such that $w[f^{-1}(y)] \leqslant \mathfrak{m}$ for $y \in A$, then $w(X) \leqslant \mathfrak{m}$.*

COROLLARY 1.3. *Let $f: X \to Y$ be a 0-dimensional open continuous mapping of a compact Hausdorff space X onto a locally connected Hausdorff (infinite) space Y. If there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is metrizable for $y \in A$, then $w(X) = w(Y)$.*

THEOREM 2. *Let $f: X \to Y$ be a separative locally confluent continuous mapping of a regular space X onto a regular space Y. If Y admits a σ-locally finite base consisting of connected sets and there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a metrizable compact subspace for $y \in A$, then X is metrizable.*

Proof. For $y \in Y$, let V_y be a neighbourhood of y in Y such that $f|f^{-1}(V_y)$ is confluent. Let $\{G_{si}\}_{s \in S_i}$ be a locally finite family of connected sets $G_{si} \subset Y$ $(i = 1, 2, \ldots)$ such that the collection $\{G_{si}\colon s \in S_i,\ i = 1, 2, \ldots\}$ is a base in Y. We can assume that each set G_{si} is contained in a set $V_{y_{si}}$ for $s \in S_i$. Let F_{si} be the closure of G_{si} in $V_{y_{si}}$, and let $\mathfrak{F}_{si}$ be the collection of all open-and-closed non-void subsets of the subspace $f^{-1}(F_{si})$, for $s \in S_i$ and $i = 1, 2, \ldots$ Since $f^{-1}(y)$ is metrizable compact for $y \in A$, we have

$w[f^{-1}(y)] \leqslant \aleph_0$ for $y \in A$. It now follows exactly in the same way as in the proof of Theorem 1 that $\overline{\overline{\mathfrak{F}_{si}}} \leqslant \aleph_0$ for $s \in S_i$ and $i = 1, 2, \ldots$ Thus we can write

$$\mathfrak{F}_{si} = \{Q_{si1}, Q_{si2}, \ldots\},$$

and, according to the Lemma, the collection $\mathfrak{B}$ defined by the formula

$$\mathfrak{B} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{\mathrm{Int} Q_{sij}: s \in S_i\}$$

is a base in X. Since each family $\{G_{si}\}_{s \in S_i}$ is locally finite in Y, so is $\{F_{si}\}_{s \in S_i}$. Hence each family $\{f^{-1}(F_{si})\}_{s \in S_i}$ is locally finite in X. But we have

$$\mathrm{Int} Q_{sij} \subset Q_{sij} \subset f^{-1}(F_{si})$$

for $s \in S_i$ and $i, j = 1, 2, \ldots$ Consequently, the base $\mathfrak{B}$ is σ-locally finite. This yields the metrizability of X (see [2], p. 196), and the proof of Theorem 2 is complete.

Remarks. Like in our remarks to Theorem 1, let us observe that instead of metrizability and compactness of $f^{-1}(y)$ in Theorem 2 we could assume only that $f^{-1}(y)$ is completely regular with $w[f^{-1}(y)] \leqslant \aleph_0$ and that the space of quasi-components of $f^{-1}(y)$ is compact for $y \in A$. Also, the same four examples mentioned in the remarks to Theorem 1 apply here to show that each of the corresponding four conditions is essential in Theorem 2. Finally, let us point out that analogues of Corollaries 1.1-1.3 can easily be formulated as consequences of Theorem 2.

THEOREM 3. *Let X be a metrizable space. If X is locally connected and locally separable, then X admits a σ-locally finite base consisting of connected sets.*

Proof. Since X is metrizable, X admits a σ-locally finite base. Let $\{G_{si}\}_{s \in S_i}$ be a locally finite family of sets $G_{si} \subset X$ $(i = 1, 2, \ldots)$ such that the collection $\{G_{si}: s \in S_i, i = 1, 2, \ldots\}$ is a base in X. We can assume that each set G_{si} is contained in a separable subspace of X. Then the collection of components of G_{si} is countable; and let $C_{si1}, C_{si2}, \ldots$ be all components of G_{si}. Hence

$$\mathfrak{B} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{C_{sij}: s \in S_i\}$$

is also a base in X, and it follows from the inclusion $C_{sij} \subset G_{si}$ that the family $\{C_{sij}\}_{s \in S_i}$ is locally finite in X. Thus $\mathfrak{B}$ is σ-locally finite and the elements of $\mathfrak{B}$ are connected.

COROLLARY 3.1. *Let $f: X \to Y$ be a separative locally confluent continuous mapping of a regular space X onto a metrizable locally connected locally separable space Y. If there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a metrizable compact subspace for $y \in A$, then X is metrizable.*

COROLLARY 3.2. *Let $f\colon X \to Y$ be a separative open-and-closed continuous mapping of a regular space X onto a metrizable locally connected locally separable space Y. If there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a metrizable compact subspace for $y \in A$, then X is metrizable.*

COROLLARY 3.3. *Let $f\colon X \to Y$ be a 0-dimensional open perfect mapping of a Hausdorff space X onto a metrizable locally connected space Y. If there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is metrizable for $y \in A$, then X is metrizable.*

THEOREM 4. *Let X be a metrizable space. If X is locally connected and X has a countable closed covering whose elements admit σ-locally finite bases consisting of connected sets, then X admits a σ-locally finite base consisting of connected sets.*

Proof. Let $\{C_i\}_{i=1}^{\infty}$ denote this countable closed covering of X, and let $\{G_{sij}\}_{s \in S_{ij}}$ be a locally finite family of connected sets $G_{sij} \subset C_i$ $(j = 1, 2, \ldots)$ such that the collection $\{G_{sij}\colon s \in S_{ij},\ j = 1, 2, \ldots\}$ is a base in C_i $(i = 1, 2, \ldots)$. Then the family $\{\overline{G}_{sij}\}_{s \in S_{ij}}$ is locally finite in X, and there exist open subsets $V_{sij} \subset X$ such that $\overline{G}_{sij} \subset V_{sij}$ for $s \in S_{ij}$ and the family $\{V_{sij}\}_{s \in S_{ij}}$ is locally finite in X (see [2], p. 214). Since X is locally connected, every point $x \in \overline{G}_{sij}$ has a connected open neighbourhood $W_k(x)$ such that $W_k(x) \subset B(x, k^{-1}) \cap V_{sij}$. Since $\overline{G}_{sij}$ is connected, so is the union

$$U_{sijk} = \bigcup_{x \in \overline{G}_{sij}} W_k(x)$$

for $k = 1, 2, \ldots$ It readily follows that

$$\mathfrak{B} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \{U_{sijk}\colon s \in S_{ij}\}$$

is a base in X. But we have $U_{sijk} \subset V_{sij}$ whence the family $\{U_{sijk}\}_{s \in S_{ij}}$ is locally finite in X. Thus $\mathfrak{B}$ is σ-locally finite which completes the proof of Theorem 4.

P 688. *Does each metrizable locally connected space admit a σ-locally finite base consisting of connected sets?*

P 689. *Suppose $f\colon X \to Y$ is a separative locally confluent continuous mapping of a regular space X onto a metrizable locally connected space Y such that $f^{-1}(y)$ is metrizable compact for every point $y \in Y$. Is X metrizable?*

According to Theorem 2, a positive solution of P 688 would imply a positive solution of P 689.

Added in proof. Problem 689 was solved affirmatively by T. Przymusiński, the solution will appear in one of the next issues of this journal.

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