METRIZABILITY AND WEIGHT OF INVERSES UNDER CONFLUENT MAPPINGS

BY

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We say that a mapping \( f : X \to Y \) of a topological space \( X \) onto a topological space \( Y \) is confluent provided for every connected closed subset \( C \) of \( Y \), and points \( x \in f^{-1}(C) \) and \( y \in C \), the set \( f^{-1}(C) \) is connected between \( \{x\} \) and \( f^{-1}(y) \), i.e. every open-and-closed neighbourhood of \( x \) in \( f^{-1}(C) \) meets \( f^{-1}(y) \). This notion was introduced in [4]. We say that a mapping \( f : X \to Y \) is locally confluent provided every point \( y \in Y \) has a neighbourhood \( V_y \) in \( Y \) such that \( f|f^{-1}(V_y) \) is confluent. A routine argument shows that the class of confluent mappings contains all mappings which are either open-and-closed, or monotone and closed, or monotone and open. Thus within the class of continuous mappings of compact Hausdorff spaces, confluent mappings constitute a common generalization for monotone mappings and open mappings. In the theory of compact Hausdorff spaces an important role is played also by 0-dimensional mappings, i.e. those having 0-dimensional inverses of points. A counterpart to 0-dimensional mappings, suitable for the non-compact case, can be defined as follows. We say that a mapping \( f : X \to Y \) is separative provided for every point \( x \in X \) and its neighbourhood \( U \) in \( X \) there exists a neighbourhood \( V \) of \( f(x) \) in \( Y \) such that the set \( f^{-1}(V) \) is not connected between \( \{x\} \) and \( f^{-1}(V) \setminus U \), i.e. there is an open-and-closed neighbourhood of \( x \) in \( f^{-1}(V) \) which is contained in \( U \). The concept of separative mappings is due to Zarelua [10]. A continuous mapping of a locally compact Hausdorff space is separative if and only if it is 0-dimensional.

The problem of estimating the weight of inverses under 0-dimensional mappings has been investigated by Mardesić [5]. More precisely, his theorem says that if \( f : X \to Y \) is a 0-dimensional continuous mapping of a locally connected compact Hausdorff space \( X \) onto a Hausdorff (infinite) space \( Y \), then \( w(X) = w(Y) \). An analogue of this theorem for non-compact spaces, involving separative mappings, has been given by Proizvolov [6]. In the present paper we prove that a similar estimation of weights is possible under the assumption that the space \( Y \) is locally
connected in lieu of the much stronger assumption that its inverse $X$
is locally connected. Then, however, it is necessary to assume some additional
conditions concerning the function $f$, and we have done this by assuming,
among other things, that $f$ is locally confluent (see Theorem 1).

The problem of metrizing inverses of metrizable spaces has been raised
by Proizvolov [7] who asked whether or not a compact Hausdorff space
must be metrizable if it admits an open continuous mapping onto a metrizable
space such that all inverses of points are metrizable (see also [1],
p. 170, Problem 5.5). A negative solution of this problem has been given
by Veličko [8] who constructed a non-metrizable compact Hausdorff
space $X$ and an open continuous mapping of $X$ onto the unit segment $I$
such that all inverses of points are homeomorphic to $I$. What we propose
is a positive solution of the Proizvolov problem in a special case: the
mapping is assumed to be separative and the image is assumed to possess
a $\sigma$-locally finite base consisting of connected sets (see Theorem 2).

We start with an example of a countable-to-one open continuous
mapping of a non-metrizable compact Hausdorff space onto a metrizable
space. In this way we replace the monotoneity of the function achieved
in [8] by the requirement that all inverses of points are countable. The
notation and the terminology come from [2]. Let $Q$ denote the Cantor
quinary set, i.e. the set of all real numbers $t$ such that

$$
t = \sum_{i=1}^{\infty} \frac{t_i}{5^i},
$$

where $t_i = 0$ or $4$ for $i = 1, 2, \ldots$ We provide $Q$ with the natural topology
inherited from the real line. Actually, in our example just the space $Q$
will be an open image of a non-metrizable compact Hausdorff space,
and a special feature of the Cantor quinary set will be exploited: we have
$t \neq u \pm 2/5^i$ for $t, u \in Q$ and $i = 1, 2, \ldots$ It follows that no point $(t, 2/5^i)$
lies on the line $x = u \pm y$ for any pair of points $t, u \in Q$.

Example. Let $D$ be the Cantor set $Q$ equipped with the discrete
topology and let $i : D \rightarrow Q$ be the identity mapping. There exist a compact
Hausdorff space $D^*$ containing $D$ as a subspace and an open continuous
extension $i^* : D^* \rightarrow Q$ of $i$ over $D^*$ such that $D^*$ satisfies the first axiom
of countability and $i^*^{-1}(q) \setminus D$ is a countable discrete subspace for $q \in Q$.
Given real numbers $x_0$ and $y_0$, we denote by $A(x_0), B_i(x_0, y_0), C(y_0)$
the plane sets

$$
A(x_0) = \{(x, y) : x_0 - y \leq x \leq x_0 + y\},
$$

$$
B_i(x_0, y_0) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < 1/4^i\},
$$

$$
C(y_0) = \{(x, y_0) : x \in Q\},
$$
for $i = 1, 2, \ldots$ We put

$$D^* = C(0) \cup C(1) \cup \bigcup_{i=1}^{\infty} C \left( \frac{2}{5^i} \right)$$

and determine the topology in $D^*$ be generated by a neighbourhood system $\{B(p)\}$ which is defined as follows. Let $p = (x_0, y_0) \in D^*$; we distinguish three cases.

Case 1: $p \in C(0)$. Then we define

$$B(p) = \{ A(x_0) \cap B_i(x_0, 0) \cap D^* \}_{i=1}^{\infty}.$$ 

Case 2: $p \in C(1)$. Then we define

$$B(p) = \{ [B_i(x_0, 1) \cup [B_i(x_0, 0) \setminus A(x_0)] \cap D^* \}_{i=1}^{\infty}.$$ 

Case 3: $p \in D^* \setminus C(0) \setminus C(1)$. Then

$$B(p) = \{ B_i(x_0, y_0) \cap D^* \}_{i=1}^{\infty}.$$ 

It is easily seen that $D^*$ with this topology is a Hausdorff space. Moreover, the subspace $C(0)$ is $\mathbb{Q}$ with the discrete topology, i.e. it coincides with $D$. To see that $D^*$ is compact, let us observe that the subspace $C(1)$ is $\mathbb{Q}$ with the natural topology, thus it is compact. Since elements of $B(p)$ do not meet $C(1)$ for $p \notin C(1)$, every collection covering the space $D^*$ and consisting of sets from our neighbourhood system contains a finite subcollection covering the whole set $C(1)$ and also the set $C(0)$ except for a finite number of points. We only have to adjoin some neighbourhoods of these points in order to obtain a finite subcollection of sets whose union $U$ contains $C(0) \cup C(1)$. But then the subspace $D^* \setminus U$ is compact which yields the compactness of $D^*$.

The mapping $i^*: D^* \to \mathbb{Q}$ is now defined by the formula

$$i^*(x, y) = x$$

for $(x, y) \in D^*$. Clearly, $i^*$ is continuous and open. If $q \in Q$, the set $i^{-1}(q)$ consists of a countable number of points which converge to $(q, 0) \in D$.

Remarks. It seems to be worth noticing that our space $D^*$ contains a countable dense subset, and that the subspace $D^* \setminus C(1)$ is very much like the Niemytzezki plane (see [2], p. 34, Example 2). On the other hand, the subspace $C(0) \cup C(1)$ is the "double Cantor set" (see [3], p. 629; see also [2], p. 109, Exercise E). The subspace $D$ being uncountable and discrete, the space $D^*$ is non-metrizable. Nevertheless $D^*$ is transformed onto the Cantor set $Q$ via the open continuous mapping $i^*$ under which the inverses of points are countable, and thus they all are metrizable. By a result of Proizvolov [7], the inverses of points under such a mapping cannot be finite. Taking a standard extension of $i^*$ over the cones built
up over $D^*$ and $Q$, we obtain a countable-to-one open continuous mapping of a non-metrizable continuum onto a metrizable continuum. The latter continuum is not locally connected, and this is essential here (see Corollary 3.3).

**Lemma.** Let $f: X \to Y$ be a separative continuous mapping of a topological space $X$ onto a regular space $Y$ and let $\{G_s\}_{s \in S}$ be a base in $Y$. If $F_s \subseteq Y$ is a set such that $G_s \subseteq F_s \subseteq \tilde{G}_s$ and $\tilde{G}_s$ denotes the collection of all open-and-closed non-void subsets of the subspace $f^{-1}(F_s)$ for $s \in S$, then the collection $\mathcal{B}$ defined by the formula

$$\mathcal{B} = \bigcup_{s \in S} \{\text{Int} Q : Q \in \tilde{G}_s\}$$

is a base in $X$.

**Proof.** Suppose $U \subseteq X$ is an open set and $x \in U$. Since $f$ is separative, there exist a neighbourhood $V$ of $f(x)$ in $Y$ and an open-and-closed neighbourhood $P$ of $x$ in $f^{-1}(V)$ such that $P \subseteq U$. Since $Y$ is regular, there exists a neighbourhood $V_0$ of $f(x)$ in $Y$ such that $V_0 \subseteq V$. Then there exists an index $s_0 \in S$ satisfying $f(x) \in G_{s_0} \subseteq V_0$. Thus $F_{s_0} \subseteq V$ whence $f^{-1}(F_{s_0}) \subseteq f^{-1}(V)$ and the set $Q = P \cap f^{-1}(F_{s_0})$ is open-and-closed in $f^{-1}(F_{s_0})$. But the sets $f^{-1}(F_{s_0})$ and $f^{-1}(V)$ are neighbourhoods of $x$ in $X$, and so are the sets $P$ and $Q$. Consequently, we have $Q \in \tilde{G}_{s_0}$ and $Q \subseteq P \subseteq U$, whence $x \in \text{Int} Q \in \mathcal{B}$ and $\text{Int} Q \subseteq U$.

**Theorem 1.** Let $f: X \to Y$ be a separative locally confluent continuous mapping of a topological space $X$ onto a locally connected regular space $Y$. If $w(Y) \leq m$ (where $\aleph_0 \leq m$) and there exists a dense subset $A \subseteq Y$ such that $f^{-1}(y)$ is a compact Hausdorff subspace with the weight $w[f^{-1}(y)] \leq m$ for $y \in A$, then $w(X) \leq m$.

**Proof.** For any point $y \in Y$, let us denote by $V_y$ a neighbourhood of $y$ in $Y$ such that $f|f^{-1}(V_y)$ is confluent. Since $Y$ is locally connected and the weight of $Y$ does not exceed $m$, there exists a base $\{G_s\}_{s \in S}$ in $Y$ such that $S \subseteq m$ and the set $G_s$ is connected and contained in a set $V_{y_s}$ for $s \in S$. Let $F_s$ be the closure of $G_s$ in $V_{y_s}$. Thus $f|f^{-1}(F_s)$ is confluent $(s \in S)$ and it suffices to show that the base $\mathcal{B}$ described in the lemma above has the cardinality $\mathfrak{b} \leq m$. Since $m$ is infinite, it is enough to verify the inequality $\mathfrak{b} \leq m$ for $s \in S$.

In fact, let us choose a point $x_s \in A \cap G_s$. Thus $x_s \in F_s$ and since $f|f^{-1}(F_s)$ is confluent, every set belonging to $\tilde{G}_s$ meets $f^{-1}(x_s)$. If $Q, Q' \in \tilde{G}_s$ and $Q \cap Q' \neq \emptyset$, then $Q \cap Q' \in \tilde{G}_s$, whence $Q \cap Q'$ meets $f^{-1}(x_s)$. It follows that if we assign to each set $Q \in \tilde{G}_s$ the intersection $Q \cap f^{-1}(x_s)$, we get a one-to-one correspondence between the sets from $\tilde{G}_s$ and some open-and-closed subsets of the subspace $f^{-1}(x_s)$. But the space $f^{-1}(x_s)$ being compact Hausdorff,
the collection of all open-and-closed subsets of $f^{-1}(a_i)$ has the cardinality not greater than $w[f^{-1}(a_i)]$. Since $a_i \in A$, we conclude that $\overline{\omega}_s \leq m$, and the proof of Theorem 1 is complete.

Remarks. In this proof, we have utilized the compactness of $f^{-1}(Y)$ for $y \in A$ only when estimating how many open-and-closed subsets are in $f^{-1}(Y)$. Instead of compactness of $f^{-1}(Y)$ we could as well assume a weaker condition: namely that $f^{-1}(Y)$ is completely regular and the space of quasi-components of $f^{-1}(Y)$ is compact for $y \in A$ (see [9], p. 49). We check briefly the necessity of some other conditions in Theorem 1: the example of the mapping constructed in [8] shows the condition saying that $f$ is separative cannot be omitted. The projection of the “double segment” (see [2], p. 107, Example 2) onto $I$ shows the necessity of the condition saying that $f$ is locally confluent. The mapping $i*$ in our example above applies to argue that $Y$ must be assumed to be locally connected. Also, the inequality for the weight of $f^{-1}(Y)$ cannot be omitted in Theorem 1 as it is shown by projection of $\omega D \times I$ onto $I$, where $\omega D$ stands for the Alexandroff compactification of a discrete uncountable space $D$.

**Corollary 1.1.** Let $f: X \to Y$ be a separative open-and-closed continuous mapping of a topological space $X$ onto a locally connected regular space $Y$. If $w(Y) \leq m$ (where $\omega_0 \leq m$) and there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a compact Hausdorff subspace with $w[f^{-1}(y)] \leq m$ for $y \in A$, then $w(X) \leq m$.

**Corollary 1.2.** Let $f: X \to Y$ be a 0-dimensional open perfect mapping of a Hausdorff space $X$ onto a locally connected space $Y$. If $w(Y) \leq m$ (where $\omega_0 \leq m$) and there exists a dense subset $A \subset Y$ such that $w[f^{-1}(y)] \leq m$ for $y \in A$, then $w(X) \leq m$.

**Corollary 1.3.** Let $f: X \to Y$ be a 0-dimensional open continuous mapping of a compact Hausdorff space $X$ onto a locally connected Hausdorff (infinite) space $Y$. If there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is metrizable for $y \in A$, then $w(X) = w(Y)$.

**Theorem 2.** Let $f: X \to Y$ be a separative locally confluent continuous mapping of a regular space $X$ onto a regular space $Y$. If $Y$ admits a $\sigma$-locally finite base consisting of connected sets and there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a metrizable compact subspace for $y \in A$, then $X$ is metrizable.

**Proof.** For $y \in Y$, let $V_y$ be a neighbourhood of $y$ in $Y$ such that $f|f^{-1}(V_y)$ is confluent. Let $\{G_{si}\}_{s \in S_i}$ be a locally finite family of connected sets $G_{si} \subset Y$ ($i = 1, 2, \ldots$) such that the collection $\{G_{si}: s \in S_i, i = 1, 2, \ldots\}$ is a base in $Y$. We can assume that each set $G_{si}$ is contained in a set $V_{v_{si}}$ for $s \in S_i$. Let $F_{si}$ be the closure of $G_{si}$ in $V_{v_{si}}$, and let $\bar{G}_{si}$ be the collection of all open-and-closed non-void subsets of the subspace $f^{-1}(F_{si})$, for $s \in S_i$ and $i = 1, 2, \ldots$ Since $f^{-1}(y)$ is metrizable compact for $y \in A$, we have
$w[f^{-1}(y)] \leq \aleph_0$ for $y \in A$. It now follows exactly in the same way as in the proof of Theorem 1 that $\overline{\mathcal{B}_{S_i}} \leq \aleph_0$ for $s \in S_i$ and $i = 1, 2, \ldots$ Thus we can write

$$\mathcal{B}_{S_i} = \{Q_{s_{i1}}, Q_{s_{i2}}, \ldots\},$$

and, according to the Lemma, the collection $\mathcal{B}$ defined by the formula

$$\mathcal{B} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{\text{Int}Q_{s_{ij}}: s \in S_i\}$$

is a base in $X$. Since each family $\{G_{s_i}\}_{s \in S_i}$ is locally finite in $X$, so is $\{F_{s_i}\}_{s \in S_i}$. Hence each family $\{f^{-1}(F_{s_i})\}_{s \in S_i}$ is locally finite in $X$. But we have

$$\text{Int}Q_{s_{ij}} \subset Q_{s_{ij}} \subset f^{-1}(F_{s_i})$$

for $s \in S_i$ and $i, j = 1, 2, \ldots$ Consequently, the base $\mathcal{B}$ is $\sigma$-locally finite. This yields the metrizability of $X$ (see [2], p. 196), and the proof of Theorem 2 is complete.

Remarks. Like in our remarks to Theorem 1, let us observe that instead of metrizability and compactness of $f^{-1}(y)$ in Theorem 2 we could assume only that $f^{-1}(y)$ is completely regular with $w[f^{-1}(y)] \leq \aleph_0$ and that the space of quasi-components of $f^{-1}(y)$ is compact for $y \in A$. Also, the same four examples mentioned in the remarks to Theorem 1 apply here to show that each of the corresponding four conditions is essential in Theorem 2. Finally, let us point out that analogues of Corollaries 1.1-1.3 can easily be formulated as consequences of Theorem 2.

**Theorem 3.** Let $X$ be a metrizable space. If $X$ is locally connected and locally separable, then $X$ admits a $\sigma$-locally finite base consisting of connected sets.

**Proof.** Since $X$ is metrizable, $X$ admits a $\sigma$-locally finite base. Let $\{G_{s_i}\}_{s \in S_i}$ be a locally finite family of sets $G_{s_i} \subset X$ ($i = 1, 2, \ldots$) such that the collection $\{G_{s_i}: s \in S_i, i = 1, 2, \ldots\}$ is a base in $X$. We can assume that each set $G_{s_i}$ is contained in a separable subspace of $X$. Then the collection of components of $G_{s_i}$ is countable; and let $C_{s_{i1}}, C_{s_{i2}}, \ldots$ be all components of $G_{s_i}$. Hence

$$\mathcal{B} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{C_{s_{ij}}: s \in S_i\}$$

is also a base in $X$, and it follows from the inclusion $C_{s_{ij}} \subset G_{s_i}$ that the family $\{C_{s_{ij}}\}_{s \in S_i}$ is locally finite in $X$. Thus $\mathcal{B}$ is $\sigma$-locally finite and the elements of $\mathcal{B}$ are connected.

**Corollary 3.1.** Let $f: X \to Y$ be a separative locally confluent continuous mapping of a regular space $X$ onto a metrizable locally connected locally separable space $Y$. If there exists a dense subset $A \subset Y$ such that $f^{-1}(y)$ is a metrizable compact subspace for $y \in A$, then $X$ is metrizable.
COROLLARY 3.2. Let \( f: X \to Y \) be a separative open-and-closed continuous mapping of a regular space \( X \) onto a metrizable locally connected locally separable space \( Y \). If there exists a dense subset \( A \subseteq Y \) such that \( f^{-1}(y) \) is a metrizable compact subspace for \( y \in A \), then \( X \) is metrizable.

COROLLARY 3.3. Let \( f: X \to Y \) be a 0-dimensional open perfect mapping of a Hausdorff space \( X \) onto a metrizable locally connected space \( Y \). If there exists a dense subset \( A \subseteq Y \) such that \( f^{-1}(y) \) is metrizable for \( y \in A \), then \( X \) is metrizable.

THEOREM 4. Let \( X \) be a metrizable space. If \( X \) is locally connected and \( X \) has a countable closed covering whose elements admit \( \sigma \)-locally finite bases consisting of connected sets, then \( X \) admits a \( \sigma \)-locally finite base consisting of connected sets.

Proof. Let \( \{C_i\}_{i=1}^\infty \) denote this countable closed covering of \( X \), and let \( \{G_{sij}\}_{s \in S_{ij}} \) be a locally finite family of connected sets \( G_{sij} \subseteq C_i \) \((j = 1, 2, \ldots)\) such that the collection \( \{G_{sij}: s \in S_{ij}, j = 1, 2, \ldots\} \) is a base in \( C_i \) \((i = 1, 2, \ldots)\). Then the family \( \{G_{sij}\}_{s \in S_{ij}} \) is locally finite in \( X \), and there exist open subsets \( V_{sij} \subseteq X \) such that \( G_{sij} \subseteq V_{sij} \) for \( s \in S_{ij} \) and the family \( \{V_{sij}\}_{s \in S_{ij}} \) is locally finite in \( X \) (see [2], p. 214). Since \( X \) is locally connected, every point \( x \in G_{sij} \) has a connected open neighbourhood \( W_k(x) \) such that \( W_k(x) \subseteq B(x, k^{-1}) \cap V_{sij} \). Since \( G_{sij} \) is connected, so is the union

\[ U_{sijk} = \bigcup_{x \in G_{sij}} W_k(x) \]

for \( k = 1, 2, \ldots \) It readily follows that

\[ B = \bigcup_{i=1}^\infty \bigcup_{j=1}^\infty \bigcup_{k=1}^\infty \{U_{sijk}: s \in S_{ij}\} \]

is a base in \( X \). But we have \( U_{sijk} \subseteq V_{sij} \) whence the family \( \{U_{sijk}\}_{s \in S_{ij}} \) is locally finite in \( X \). Thus \( B \) is \( \sigma \)-locally finite which completes the proof of Theorem 4.

P 688. Does each metrizable locally connected space admit a \( \sigma \)-locally finite base consisting of connected sets?

P 689. Suppose \( f: X \to Y \) is a separative locally confluent continuous mapping of a regular space \( X \) onto a metrizable locally connected space \( Y \) such that \( f^{-1}(y) \) is metrizable compact for every point \( y \in Y \). Is \( X \) metrizable?

According to Theorem 2, a positive solution of P 688 would imply a positive solution of P 689.

Added in proof. Problem 689 was solved affirmatively by T. Przymusiński, the solution will appear in one of the next issues of this journal.
REFERENCES


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