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A boundary value problem for non-linear differential equations with a retarded argument

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The present paper concerns the existence and uniqueness of solutions of boundary value problems for non-linear differential equations with a retarded argument. The proofs are based on the idea, due to A. Lasota and Z. Opial, of an application of the contingent equations technique to the theory of boundary value problems (see [6], [7], [8]).

Boundary value problems for differential equations with lag have been studied in [1]-[5], [9], [11]. The usual assumptions are continuity, boundedness or the Lipschitz condition of the right-hand sides of the equations. Our hypotheses are weaker (of the Carathéodory type), but the results are obtained under the condition that the initial functions are constant. Similar restrictions are in [4], [5], [9].

In Section 1 we give the notation and introduce the notion. Sections 2 and 3 contain the main result of the paper: Theorem 2.1 on the existence and Theorem 3.1 concerning the uniqueness of the boundary value problem for a differential equation with delay. In Section 4 we consider the approximation of the boundary value problem for ordinary differential equations by the boundary value problems for functional differential equations. In Section 5 we give some applications of differential inequalities to the boundary value problem for differential equations with a retarded argument. Next, in Section 6, as applications of there theorems, we give the existence theorems for the so-called aperiodic problem. In the last Section we give a generalization of the problem.

1. Let R^m denote the m-dimensional Euclidean space with norm $|\cdot|$ and let $\mathrm{cf}(R^m)$ be the family of all non-empty, closed and convex subsets of R^m. For $A \in \mathrm{cf}(R^m)$ we put $|A| = \sup\{|p|: p \in A\}$.

A mapping F: $R^m \to \mathrm{cf}(R^m)$ is called *continuous*, if it is continuous in the Hausdorff metric

$$d(A, D) = \max\big(\sup_{p\in D} \delta(p, A), \sup_{p\in A} \delta(p, D)\big), \qquad A, D \subset R^m,$$

where $\delta(p, A)$ denotes the Euclidean distance of the point x from the set A.

We say that a map $F(t)$ of the interval $[a, b]$ into $\mathrm{cf}(R^m)$ is *Lebesgue measurable* if for each closed $A \subset R^m$ the set $\{t: A \cap F(t) = 0\}$ is Lebesgue measurable [10].

We say that a map $F(t, p)$ (resp. $f(t, p)$) of $[a, b] \times R^m$ into $\mathrm{cf}(R^m)$ (resp. R^m) *satisfies the Carathéodory conditions* if it is measurable in t for each $p \in R^m$ and continuous in p for each $t \in [a, b]$.

Let $[a, b]$ be a compact interval of R^1 and let $C^m_{[a,b]}$ denote the space of all continuous functions $x: [a, b] \to R^m$ with the norm of the uniform convergence, $\|x\| = \max\{|x(t)|: t \in [a, b]\}$.

Consider the differential equations with a retarded argument

$$x'(t) = f\big(t, x(t-\tau)\big) \quad (a \leqslant t \leqslant b, \tau > 0) \tag{1.1}$$

and a boundary condition

$$Nx = r \quad (r \in R^m), \tag{1.2}$$

where $f: [a, b] \times R^m \to R^m$ and $N: C^m_{[a,b]} \to R^m$.

Side by side with problem (1.1), (1.2), consider the differential equations with a retarded argument and with multi-valued right-hand sides

$$x'(t) \in F\big(t, x(t-\tau)\big) \quad (a \leqslant t \leqslant b, \tau > 0), \tag{1.3}$$

where $F: [a, b] \times R^m \to \mathrm{cf}(R^m)$, and the homogeneous condition

$$Nx = 0. \tag{1.4}$$

By a solution of equation (1.1) (resp. equation (1.3)) we mean any absolutely continuous function $x: [a-\tau, b] \to R^m$ which satisfies (1.1) (resp. (1.3)) a.e. (almost everywhere) on the interval $[a, b]$ and is constant on the interval $[a-\tau, a]$.

2. THEOREM 2.1. *Suppose that F, f, N satisfy the following assumptions:*

(i) *$F(t, p)$ satisfies the Carathéodory conditions, is homogeneous in p and, moreover,*

$$\sup |F(t, p)| \leqslant \varphi(t),$$

where $\varphi(t)$ is an integrable function on $[a, b]$;

(ii) *$f(t, p)$ satisfies the Carathéodory conditions and, furthermore,*

$$\lim_{n\to\infty} \frac{1}{n} \int_a^b \sup_{|p|\leqslant n} \delta\big(f(t, p), F(t, p)\big)\, dt = 0;$$

(iii) *the mapping N is continuous and homogeneous;*

If $x(t) \equiv 0$ is the only solution of problem (1.3), (1.4), *then there exists at least one solution of problem* (1.1), (1.2).

Proof. Consider the mapping h of $B = C^m_{[a,b]} \times R^m$ into itself such that for every point (x, p) its image $h(x, p)$ is a pair $(\bar{y}, \bar{q})$ given by the formulae

$$\bar{y}(t) = \int_a^t f\big(s, x(s-\tau)\big)ds + p, \qquad \bar{q} = p - Nx + r$$

and the set-valued mapping H of B into B such that, for every point (x, p), its image $H(x, p)$ is a set of all pairs $(\bar{\bar{y}}, \bar{\bar{q}})$ given by the formulae

$$\bar{\bar{y}}(t) = \int_a^t u(s)ds + p, \qquad \bar{\bar{q}} = p - Nx,$$

where u is any measurable function satisfying $u(s) \in F\big(s, x(s-\tau)\big)$.

It is easy to see that if $(x, p) = h(x, p)$, then x is a solution of problem (1.1), (1.2). Similarly, if $(x, p) \in H(x, p)$, then x is a solution of problem (1.3), (1.4) and, by our assumptions, we immediately deduce that $(x, p) = (0, 0)$.

Put

$$\|(x, p)\|_0 = \|x\| + |p| \qquad (x \in C^m_{[a,b]}, p \in R^m).$$

Since the sets $F(t, p)$ are non-empty and convex, $H(x, p)$ is also non-empty and convex.

Thus, to apply Theorem 1.1 from [6] it remains to verify that

1° $\lim\limits_{\|(x,p)\|_0 \to \infty} \dfrac{\varrho\big(h(x, p), H(x, p)\big)}{\|(x, p)\|_0} = 0;$

2° H is homogeneous;

3° H and h are completely continuous.

1° follows from (ii). 2° is evident. It is obvious that h is completely continuous. In order to prove that H is completely continuous, it suffices to show that H is compact and upper semi-continuous.

It is easy to see that $\bigcup\limits_{\|(x,p)\|_0 = 1} H(x, p)$ is contained in a set Z_φ of all pairs $(\tilde{x}, \tilde{p})$ given by the formulae

$$\tilde{x} = \int_a^t u(s)ds + p, \qquad \text{where } |u(s)| \leqslant \varphi(s),\ |p| < 1,$$

$$|\tilde{p}| \leqslant \sup_{\|x\|=1} |Nx| + 1.$$

Since the function φ is integrable on $[a, b]$, the functions $\tilde{x}$ are bounded and equicontinuous on $[a, b]$. Since N is continuous and homogeneous, $\sup\limits_{\|x\|=1} |Nx|$ is bounded and, consequently, the closure of Z_φ is compact. Therefore, the mapping H is compact.

In order to show that H is upper semi-continuous, we should show that for all sequences $\{(x^k, p^k)\}$, $\{(y^k, q^k)\} \subset C^m_{[a,b]} \times R^m$ the conditions $(x^k, p^k) \to (x, p)$, $(y^k, q^k) \to (y, q)$ and $(y^k, q^k) \in H(x^k, p^k)$ $(k = 1, 2, \ldots)$ imply that $(y, q) \in H(x, p)$.

Obviously,

$$y^k(t) = \int_a^t u^k(s)\,ds + p^k, \quad \text{where } u^k(s) \in F(s, x^k(s-\tau)) \tag{2.1}$$

and

$$q^k = p^k + Nx^k. \tag{2.2}$$

Hence

$$(y^k(t))' \in F(t, x^k(t-\tau)).$$

By the semi-continuouity of F and from the last condition we have

$$\lim_{k\to\infty} \varrho\big((y^k(t))',\ F(t, x(t-\tau))\big) = 0.$$

In virtue of the Pliś lemma [10]

$$y'(t) \in F(t, x(t-\tau)). \tag{2.3}$$

In addition, upon passing to the limit in (2.1) (for $t = a$) and (2.2), we have

$$y(a) = p, \quad q = p + Nx, \tag{2.4}$$

and consequently

$$y(t) = \int_a^t y'(t)\,dt + p. \tag{2.5}$$

From (2.3), (2.4) and (2.5) it follows that $(y, q) \in H(x, p)$. This completes the proof of Theorem 2.1.

3. In this section we shall formulate conditions for the existence and uniqueness of solutions of a boundary value problem (1.1), (1.2).

Namely, we make the following assumptions:

(iv) *$f(t, p)$ is measurable in t for each $p \in R^m$ and satisfies the conditions*

$$f(t, p) - f(t, q) \in F(t, p-q), \qquad \int_a^b |f(t, 0)|\,dt < +\infty;$$

(v) *N is linear and continuous.*

THEOREM 3.1. *Suppose that F, f, N, satisfy* (i), (iv), (v) *and that $x(t) \equiv 0$ is the unique solution of problem* (1.3), (1.4). *Then there exists exactly one solution of problem* (1.1), (1.2).

Proof. Since conditions (iv), (v) imply (ii), (iii), the existence of solutions of problem (1.1), (1.2) follows from Theorem 2.1.

Now, suppose that $\bar{x}(t)$ and $\bar{\bar{x}}(t)$ are two absolutely continuous solutions of problem (1.1), (1.2). By (iv) and (v), $\bar{x}(t)-\bar{\bar{x}}(t)$ is a solution of (1.3), (1.4). From the uniqueness of this problem it follows that $\bar{x}(t) = \bar{\bar{x}}(t)$ on $[a, b]$.

4. Consider the ordinary differential equation

$$x'(t) = f\big(t, x(t)\big) \quad (a \leqslant t \leqslant b) \tag{4.1}$$

with boundary value condition (1.2) and the contingent equations

$$x'(t) \in F\big(t, x(t)\big) \quad (a \leqslant t \leqslant b) \tag{4.2}$$

with boundary value condition (1.4).

THEOREM 4.1. *Assume that function F satisfies* (i). *If $x(t) \equiv 0$ is the only solution of problem* (4.1), (1.4), *then there is a $\tau_0 > 0$ such that for $0 < \tau \leqslant \tau_0$, $x(t) \equiv 0$ is the unique solution of problem* (1.3), (1.4).

Proof. On the contrary, suppose that for each k there exist a $\tau_k < 1/k$ and a corresponding non-trivial solution x^k of problem (1.3), (1.4) such that

$$\big(x^k(t)\big)' \in F\big(t, x^k(t-\tau_k)\big), \qquad Nx^k = 0 \ (k = 1, 2, \ldots). \tag{4.3}$$

Since N and F are homogeneous, we may assume that

$$\|x^k(t)\| = 1.$$

Hence, from (4.3) and (i) it follows that the functions $x^k(t)$ are uniformly bounded and equicontinuous on $[a, b]$. By the Arzelá theorem, passing if necessary to suitable subsequences, we may assume that

$$x^k(t) \to x(t), \qquad \text{as } k \to \infty.$$

Since the functions $x^k(t)$ are absolutely continuous and $\tau_k < 1/k$, we have

$$x^k(t-\tau_k) \to x(t), \qquad \text{as } k \to \infty. \tag{4.4}$$

Furthermore,

$$\big(x^k(t)\big)' \in F\big(t, x^k(t-\tau_k)\big). \tag{4.5}$$

By (4.4), (4.5) and (i) we have

$$\lim_{k\to\infty} \varrho\big((x^k(t))', F\big(t, x(t)\big)\big) = 0 \qquad \text{a.e.}$$

The Pliś lemma [10] yields

$$x'(t) \in F\big(t, x(t)\big).$$

Moreover, it is easy to see that $Nx = 0$ and $\|x(t)\| = 1$, which is impossible. This proves the theorem.

The next corollary is an immediate consequence of the preceding theorem.

COROLLARY. *Suppose that* $x(t) \equiv 0$ *is the unique solution of problem* (4.2), (1.4). *Then there exists a* $\tau_0 > 0$ (which depends upon F, f, N) *such that for* $\tau \leqslant \tau_0$ *problem* (1.1), (1.2) *has*

1° *at least one solution if the functions* F, f, N *satisfy conditon* (i), (ii), (iii);

2° *exactly one solution if the functions* F, f, N *satisfy conditions* (i), (iv), (v).

THEOREM 4.2. *Assume that* F, f, N *satisfy conditions* (i), (iv), (v). *If problem* (4.2), (1.4) *has only the trivial solution* $x(t) \equiv 0$, *then the solutions of the boundary value problem* (1.1), (1.2) *converge to the solutions of the boundary value problem* (4.1), (1.2) *as* $\tau \to 0$.

Proof. The proof consists in showing that

$$\lim_{k\to\infty} |\tilde{x}^k(t) - \mathring{x}(t)| = 0,$$

where $\mathring{x}(t)$, $\tilde{x}^k(t)$ denote the solutions of problems (4.1), (1.2) and (1.1), (1.2) respectively; that is,

$$\big(\mathring{x}(t)\big)' = f\big(t, \mathring{x}(t)\big) \qquad \text{a.e. on } [a, b],\ N\mathring{x} = r \ (r \in R^m)$$

and

$$\big(\tilde{x}^k(t)\big)' = f\big(t, x^k(t-\tau_k)\big) \qquad \text{a.e. on } [a, b],\ Nx^k = r \ (\tau_k < 1/k,\ r \in R^m). \tag{4.6}$$

Put

$$u^k(t) = \tilde{x}^k(t) - \mathring{x}(t);$$

we have

$$\begin{aligned}\big(u^k(t)\big)' &= f\big(t, \tilde{x}^k(t-\tau_k)\big) - f\big(t, \mathring{x}(t)\big)\\ &= f\big(t, \tilde{x}^k(t-\tau_k)\big) - f\big(t, \mathring{x}(t-\tau_k)\big) + f\big(t, \mathring{x}(t-\tau_k)\big) - f\big(t, \mathring{x}(t)\big).\end{aligned}$$

Hence by (iv)

$$\big(u^k(t)\big)' \in F\big(t, u^k(t-\tau_k)\big) + \varepsilon(t, \tau_k) \qquad \text{a.e. on } [a, b],$$

where

$$\varepsilon(t, \tau_k) = f\big(t, \mathring{x}(t-\tau_k)\big) - f\big(t, \mathring{x}(t)\big).$$

Since $\mathring{x}$ absolutely continuous, for t fixed f is continuous in x, inequality $\tau_k < 1/k$ implies that for each fixed t

$$\lim_{k\to\infty} \varepsilon(t, \tau_k) = 0. \tag{4.7}$$

Now, suppose that there exists a sequence of functions $\{u^k(t)\}$ satisfying (4.6) and such that the sequence $\{\|u^k\|\}$ is not convergent to zero. Replacing $\{\|u^k\|\}$, if necessary, by a subsequence, we may assume that $\|u^k\| \to c$, $c \in (0, +\infty)$. Setting

$$v^k(t) = \frac{u^k(t)}{\|u^k\|},$$

we obtain

$$(4.8) \qquad \big(v^k(t)\big)' \in F\big(t, v^k(t-\tau_k)\big) + \frac{\varepsilon^k}{\|u^k\|}, \qquad Nv^k(t) = 0, \ \|v^k\| = 1.$$

The functions $v^k(t)$ are uniformly bounded and equicontinuous, hence, by the Arzelá Theorem, passing if necessary to a suitable subsequences, we may assume that $v^k(t) \to v(t)$. Since $\tau_k < 1/k$ and the functions $v^k(t)$ are absolutely continuous,

$$(4.9) \qquad v^k(t-\tau_k) \to v(t), \qquad \text{as } k \to \infty.$$

By (4.7), (4.8), (4.9) and (i)

$$\lim_{k\to\infty} \varrho\Big(\big(v^k(t-\tau_k)\big)', \ F\big(t, v(t)\big)\Big) = 0 \qquad \text{a.e.}$$

Thus, from the lemma of Pliś it follows immediately that

$$v'(t) \in F\big(t, v(t)\big),$$

and obviously $Nv = 0$, $\|v\| = 1$, which gives a contradiction. Thus $\|u^k\| \to 0$. This completes the proof of Theorem 4.2.

5. Theorems 2.1 and 3.1 are particularly simple if equation (1.3) reduces to a differential inequality

$$(5.1) \qquad |x'(t)| \leqslant \omega\big(t, |x(t-\tau)|\big),$$

where the function $\omega\colon [a, b] \times C^m_{[a-h,a]} \to [0, \infty)$ satisfies the condition:

(vi) *for each* $t \in [a, b]$, $\omega(t, u)$ *is continuous and homogeneous with respect to* u; *for each* $u \in [0, \infty)$, $\omega(t, u)$ *is measurable with respect to* t; *the functions* $\sup\{\omega(t, u)\colon u \leqslant k\}$ *are integrable for* $k = 1, 2, \ldots$

Setting

$$F(t, p) = \{q \in R^m\colon |q| \leqslant \omega(t, |p|)\},$$

we immediately obtain from Theorem 2.1

THEOREM 5.1. *Let the functions N and ω satisfy conditions* (iii) *and* (vi), *let f satisfy the Carathéodory conditions and, in addition, the inequality*

$$f(t, p) \leqslant \omega(t, |p|) + \varphi(t),$$

where the function $\varphi(t)$ is integrable on $[a, b]$. If problem (5.1), (1.4) *has only the trivial solution $x(t) \equiv 0$, then for each $r \in R^m$ there exists at least one solution of problem* (1.1), (1.2).

As a consequence of Theorem 3.1 we obtain

THEOREM 5.2. *Suppose that N satisfies* (v) *and*

$$|f(t, p) - f(t, q)| \leqslant \omega(t, |p - q|),$$

where ω is as in Theorem 5.1. *If problem* (5.1), (1.4) *has only the trivial solution $x(t) \equiv 0$, then for each $r \in R^m$ problem* (1.1), (1.2) *has exactly one solution.*

6. As an application of Theorems 5.1 and 5.2 consider for (1.1) the following aperiodic boundary value condition:

$$x(a) + \lambda x(b) = r \quad (\lambda > 0, r \in R^m). \tag{6.1}$$

Before stating the Theorem of this section, we will prove the following

LEMMA 6.1. *If*

$$1 - \lambda e^{K(b-a)} > 0, \tag{6.2}$$

then $x(t) \equiv 0$ is the unique absolutely continuous function satisfying the condition

$$|x'(t)| \leqslant K|x(t-\tau)|, \quad x(a) + \lambda x(b) = 0 \ (\lambda > 0) \tag{6.3}$$

almost everywhere on $[a, b]$.

Proof. Integrating inequality (6.3) over $[a, t]$, we obtain

$$|x(t) - x(a)| \leqslant K \int_a^t |x(s-\tau)|\, ds.$$

Hence

$$|x(t)| \leqslant |x(a)| + K \int_a^t |x(s-\tau)|\, ds. \tag{6.4}$$

Let

$$u(s) = \sup_{\theta \in [s-h, s]} |x(\theta)|.$$

From (6.4) we obtain

$$u(t) \leqslant |x(a)| + K \int_a^s u(s)\, ds.$$

By Gronwall's inequality,

$$u(t) \leqslant |x(a)| e^{K(t-a)}, \tag{6.5}$$

which implies that

(6.6) $$|x(b)| \leqslant |x(a)|e^{K(b-a)}.$$

From (6.3) we have

$$|x(a)| = \lambda |x(b)|.$$

From this and (6.6) it follows that

(6.7) $$|x(a)| \leqslant \lambda |x(a)| e^{K(b-a)}.$$

By (6.2) and (6.7) it follows that $|x(a)| = 0$, and by (6.5) $u(t) \equiv 0$. This completes the proof of the lemma.

Using Lemma 6.1 and Theorems 5.1 and 5.2, we obtain

THEOREM 6.1. *Let the function f satisfy the Carathéodory conditions and let the following assumptions hold:*

1° *f satisfies the inequality*

$$|f(t, x(t))| \leqslant K|x(t-\tau)| + \varphi(t),$$

where $\varphi(t)$ is integrable and (6.2) *holds true; then problem* (1.1), (6.1) *has at most one solution.*

2° *f satisfies the condition*

$$|f(t, p) - f(t, q)| \leqslant K|p-q|.$$

If λ satisfies (6.2), then problem (1.1), (6.1) has exactly one solution.

7. It is easy to see that the results of the present paper hold true also in the case of differential equations with a retarded argument

$$x'(t) = f\big(t, x(t-\tau(t))\big),$$

and the contingent equations

$$x'(t) \in F\big(t, x(t-\tau(t))\big),$$

provided that $\tau(t)$ is non-negative, continuous and $\tau(t) \to 0$ as $t \to 0$.

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