Approximate fixed points for nonexpansive mappings in uniformly convex spaces

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Abstract. Let $X$ be a uniformly convex Banach space and $K$ a nonempty bounded closed and convex subset of $X$. For $\varepsilon > 0$, let $\mathfrak{I}_\varepsilon$ denote the family of all nonexpansive mappings defined on $K$ and taking values in an $\varepsilon$-neighborhood of $K$, and let $\mathfrak{I}$ denote the family of all nonexpansive self-mappings of $K$. It is shown that if $T \in \mathfrak{I}_\varepsilon$ and if $d = \text{diam} (K)$, then

$$\inf \{ \| x - T(x) \| : x \in K \} \leq (d + 2\varepsilon)\delta^{-1}(2\varepsilon/(d + 2\varepsilon)) + \varepsilon,$$

where $\delta$ denotes the modulus of convexity of $X$. It is also shown that if $n \in \mathbb{N}$ is chosen so that $(1 - \delta(2\varepsilon/d))^n \leq \varepsilon/d$, then for each $T \in \mathfrak{I}$ and $x \in K$, $\| S^n(x) - S^{n+1}(x) \| \leq \varepsilon$, where $S = (1/2)(I + T)$.

1. Notation and definitions. Throughout the paper, $X$ will denote a Banach space. For $x \in X$ and $r > 0$, let $B(x; r)$ denote the closed ball centered at $x$ with radius $r$. The modulus of convexity of $X$ is the function $\delta: [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf \{ 1 - (1/2)\| x + y \| : x, y \in B(0; 1), \| x - y \| \geq \varepsilon \}.$$

The space $X$ is said to be uniformly convex ([1]) if $\delta(\varepsilon) > 0$ for each $\varepsilon > 0$. For such spaces it is known that the function $\delta$ is continuous and strictly increasing on $[0,2)$ (see [4], [9]). Also:

(1.1) $t\delta(\varepsilon/t)$ increases as $t > 0$ decreases (with $\varepsilon$ fixed; see [5]).

We shall use the following routine consequence of the definition of $\delta$ (cf. [10]).

(1.2) If $u, v \in X$ satisfy $\| u \| < q$, $\| v \| \leq q$ and $\| u - v \| \geq \varepsilon$, then

$$(1/2)\| u + v \| \leq q(1 - \delta(\varepsilon/q)).$$

For $K \subseteq X$ and $x \in X$, let

$$\text{dist}(x, K) = \inf \{ \| x - u \| : u \in K \}; \quad \text{diam}(K) = \sup \{ \| u - v \| : u, v \in K \};$$

and for $\varepsilon > 0$, let $N_\varepsilon(K)$ denote the closed $\varepsilon$-neighborhood of $K$:

$$N_\varepsilon(K) = \{ x \in X : \text{dist}(x, K) \leq \varepsilon \}.$$
Recall that a mapping $T: K \to X$ is said to be nonexpansive if $\|T(u) - T(v)\| \leq \|u - v\|$ for each $u, v \in K$. (Much of the extensive fixed point theory for nonexpansive mappings is summarized in [6], [7].)

2. Approximate fixed points for approximate self-maps. Let $K$ be a fixed bounded closed and convex subset of a Banach space $(K \neq \emptyset)$ and for $\varepsilon > 0$ let $I_{\varepsilon}$ denote the family of all nonexpansive mappings $T$ for which $T: K \to N_{\varepsilon}(K)$. Let

$$q(\varepsilon) = \sup_{T \in I_{\varepsilon}} \left\{ \inf_{x \in K} \|x - T(x)\| : x \in K \right\}.$$  

In [8] it is shown that if $K$ has a nonempty interior and if $\bar{r} = \sup\{r > 0: B(x; r) \subset K \text{ for some } x \in K\}$, then

$$q(\varepsilon) \leq \left[ \frac{\text{diam}(K) - \bar{r} + \varepsilon}{\bar{r} + \varepsilon} \right] \varepsilon.$$

(Note that this implies $q(\varepsilon) = \varepsilon$ if $K$ itself is a ball.) For arbitrary $K$, the following qualitative result is also proved in [8].

$$q(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

In this section we show that the above results may be further refined if $X$ is assumed to be uniformly convex. Specifically, we have:

**Theorem 1.** Let $X$ be a uniformly convex Banach space with modulus of convexity $\delta$, let $K$ a nonempty bounded closed and convex subset of $X$ with $\text{diam}(K) = d$, and let $I_{\varepsilon}$ denote the family of all nonexpansive mappings $T$ for which $T: K \to N_{\varepsilon}(K)$. Then for $T \in I_{\varepsilon}$,

$$\inf_{x \in K} \|x - T(x)\| \leq (d + 2\varepsilon) \delta^{-1} \left[ \frac{2\varepsilon}{d + 2\varepsilon} \right] + \varepsilon.$$

**Proof.** Let $\mathcal{R}$ denote the family of all nonempty closed and convex subsets of $K$ with the property $C \in \mathcal{R}$ if and only if $T(C) \subset N_{\varepsilon}(C)$. Partially order $\mathcal{R}$ by set inclusion and let $\mathcal{C} = \{C_\alpha : \alpha \in A\}$ be any descending chain in $\mathcal{R}$. The set $C_0 = \bigcap \limits_{\alpha} C_\alpha$ is closed convex and nonempty (by weak compactness).

Also, if $x \in C_0$, then for each $\alpha$ there exists $x_\alpha \in C_\alpha$ such that $\|x_\alpha - T(x)\| \leq \varepsilon$. Since $K$ is weakly compact, the net $\{x_\alpha : \alpha \in A\}$ has a subnet which converges weakly, say to $w$, with $w \in C_0$. Since $\|w - T(x)\| \leq \sup \|x_\alpha - T(x)\| \leq \varepsilon$, it follows that $T(C_0) \subset N_{\varepsilon}(C_0)$, i.e., $C_0 \in \mathcal{R}$. This proves that every chain in $\mathcal{R}$ has a lower bound, so by Zorn's Lemma $\mathcal{R}$ has a minimal element $K_0$. Also, since

$$\inf_{x \in K} \|x - T(x)\| \leq \inf_{x \in K_0} \|x - T(x)\| : x \in K_0,$$

we may assume at the outset that $K$ itself is minimal (with $\text{diam}(K) = d' \leq d$).
Now let \( z \) and \( r \) denote, respectively, the Chebyshev center and radius of \( K \). (Thus \( z \in K \) and \( B(z; \ r) \) is the smallest ball centered at any point of \( K \) which contains \( K \).) Since \( X \) is uniformly convex, \( 1/2 \leq r < d \). Set

\[
H = B(T(z); \ r + \varepsilon) \cap K.
\]

We claim \( H = K \). To see this it suffices, by minimality of \( K \), to prove \( T(H) \subset N_\varepsilon(H) \). Let \( x \in H \). By assumption there exists \( y \in K \) such that \( \| T(x) - y \| \leq \varepsilon \). Thus

\[
\| T(z) - y \| \leq \| T(z) - T(x) \| + \| T(x) - y \| \leq \| z - x \| + \| T(x) - y \| \leq r + \varepsilon,
\]

proving \( y \in H \). Hence \( T(x) \in N_\varepsilon(H) \), establishing the claim.

Since \( K = H = B(T(z); r + \varepsilon) \cap K \), we conclude \( K \subset B(T(z); r + \varepsilon) \). Also, by assumption, there exists \( p \in K \) such that \( \| p - T(z) \| \leq \varepsilon \), so it follows that \( K \subset B(p; r + 2\varepsilon) \). Also, since \( r \) is the Chebyshev radius of \( K \) there exists \( x \in K \) such that

\[
r \leq \left\| \frac{p + z}{2} - x \right\|.
\]

We now have \( \| x - p \| \leq r + 2\varepsilon \) and \( \| x - z \| \leq r + 2\varepsilon \). By (1.2),

\[
\left\| \frac{p + z}{2} - x \right\| \leq 1 - \delta \left[ \frac{\| p - z \|}{r + 2\varepsilon} \right] (r + 2\varepsilon).
\]

Therefore (assuming \( \| z - T(z) \| \geq \varepsilon \)),

\[
\delta \left[ \frac{\| z - T(z) \| - \varepsilon}{r + 2\varepsilon} \right] \leq \delta \left[ \frac{\| p - z \|}{r + 2\varepsilon} \right] \leq 1 - \frac{r}{r + 2\varepsilon} = \frac{2\varepsilon}{r + 2\varepsilon}.
\]

Consequently,

\[
(d + 2\varepsilon) \delta \left[ \frac{\| z - T(z) \| - \varepsilon}{d + 2\varepsilon} \right] \leq (r + 2\varepsilon) \delta \left[ \frac{\| z - T(z) \| - \varepsilon}{r + 2\varepsilon} \right] \leq 2\varepsilon.
\]

It follows that

\[
\| z - T(z) \| \leq (d + 2\varepsilon) \delta^{-1} \left[ \frac{2\varepsilon}{d + 2\varepsilon} \right] + \varepsilon.
\]

3. Uniform iteration. It is shown in [5] (cf. also [2]) that if \( K \) is any nonempty bounded closed and convex subset of a Banach space and if \( \varepsilon > 0 \), then there exists an integer \( N \) such that if \( T: K \to K \) is nonexpansive, if \( x_0 \in K \), and if \( n \geq N \), then \( \| S^n(x_0) - S^{n+1}(x_0) \| \leq \varepsilon \), where \( S \) denotes the mapping \( \frac{1}{2}(I + T) \). The proof given in [5] is purely an existence proof offering no estimate on the magnitude of \( N \). Indeed, it seems unlikely that such estimates would be easy to obtain in general settings. Although the problem appears to have received little attention, it would appear to be a tractable one in special settings.
(In a conversation with the first author, J. Alexander observed that it suffices to take \( N \geq \varepsilon^{-1} - 2 \) if \( K \) is the interval \([0, 1]\) in \( R^1 \).)

Here we give an estimate for \( N \) in terms of the modulus of convexity of a uniformly convex space.

**Theorem 2.** Let \( X \) be a uniformly convex Banach space with modulus of convexity \( \delta \), let \( K \) be a nonempty bounded close and convex subset of \( X \) with \( \text{diam}(K) = d \), and let \( \varepsilon > 0 \) (\( \varepsilon \leq d/2 \)). If \( T: K \to K \) is nonexpansive and if \( S = (1/2)(I + T) \), then for any \( x \in K \), \( \| S^n(x) - S^{n+1}(x) \| \leq \varepsilon \) for all \( n \in \mathbb{N} \) satisfying
\[
(1 - \delta(2\varepsilon/d))^n \leq \varepsilon/d.
\]

**Proof.** Under the assumptions of the theorem it is well known that \( T \) (hence \( S \)) has at least one fixed point \( p \in K \). Suppose \( n \) satisfies (4). Since \( \{ \| S^k(x) - S^{k+1}(x) \| \} \) is monotone decreasing, if \( \| S^k(x) - S^{k+1}(x) \| \leq \varepsilon \) holds for some \( k < n \) there is nothing to prove. So we may assume
\[
\varepsilon < \| S^{n-1}(x) - S^n(x) \| \leq \| S^{n-2}(x) - S^{n-1}(x) \| \leq \ldots \leq \| x - S(x) \| \leq d.
\]
Now, \( \| S^{n-1}(x) - S^n(x) \| > \varepsilon \) implies \( \| S^{n-1}(x) - T(S^{n-1}(x)) \| > 2\varepsilon \). Also, we have
\[
\| T(S^{n-1}(x)) - p \| \leq \| S^{n-1}(x) - p \|.
\]
Thus, since \( S^n(x) = (1/2)(S^{n-1}(x) + T(S^{n-1}(x))) \),
\[
\| S^n(x) - p \| \leq 1 - \delta \left[ \frac{2\varepsilon}{\| S^{n-1}(x) - p \|} \right] \| S^{n-1}(x) - p \|.
\]
In view of (6),
\[
\| S^n(x) - p \| \leq \prod_{j=1}^{n} \left[ 1 - \delta \left[ \frac{2\varepsilon}{\| S^{n-j}(x) - p \|} \right] \right] \| x - p \|,
\]
and by monotonicity of \( \delta \),
\[
\| S^n(x) - p \| \leq \left[ 1 - \delta \left[ 2\varepsilon/d \right] \right]^n d \leq (\varepsilon/d)d = \varepsilon.
\]
Since \( p \) is a fixed point of \( T \) with \( T \) nonexpansive, \( \| T(S^n(x)) - p \| \leq \varepsilon \); hence \( \| S^n(x) - T(S^n(x)) \| \leq 2\varepsilon \) yielding
\[
\| S^n(x) - S^{n+1}(x) \| = (1/2)\| S^n(x) - T(S^n(x)) \| \leq \varepsilon.
\]

4. Remarks.

(4.1) Since \( r \leq d(1 - \delta(1)) \), implicit in (3) is a sharper estimate for Theorem 1, namely:
\[
\inf\{ \| x - T(x) \| : x \in K \} \leq \left[ d(1 - \delta(1)) + 2\varepsilon \right] \left[ \delta^{-1} \frac{2\varepsilon}{d(1 - \delta(1)) + 2\varepsilon} \right] + \varepsilon.
\]
However, even this estimate is not precise since it does not yield the known (see [8]) fact that if \( X \) is a Hilbert space,
\[
\inf\{ \| x - T(x) \| : x \in K \} \leq \varepsilon.
\]

(4.2) In cases where the modulus of convexity is explicitly known, the estimate of Theorem 2 can be improved. For example, if \( X = l^p \), \( 2 \leq p < \infty \),
then $\delta(\varepsilon) = 1 - (1 - (\varepsilon/2)p)^{1/p}$. Computing directly from (6):

$\|S(x) - p\| \leq (1 - \delta(2\varepsilon/d))d = (1 - (\varepsilon/d)p)^{1/p}d = (d^p - e^p)^{1/p}$;

$\|S^2(x) - p\| \leq \left[ 1 - \delta \left( \frac{2\varepsilon}{(d^p - e^p)^{1/p}} \right) \right] (d^p - e^p)^{1/p} = (d^p - 2e^p)^{1/p}$;

and continuing:

$\|S^n(x) - p\| \leq (d^p - ne^p)^{1/p}, \quad n = 1, 2, \ldots$

Also, $(d^p - ne^p)^{1/p} \leq \varepsilon$ if and only if $n \geq (d/e)^p - 1$. We note that this estimate for $n$ is better than that given by (4).

Also, we note that for $X = R^1$ and $K = [0, 1]$, $\delta(\varepsilon) = \varepsilon/2$, and a repetition of the above argument yields $|S^n(x) - p| \leq 1 - ne$; hence $|S^n(x) - p| \leq \varepsilon$ provided $n \geq \varepsilon^{-1} - 1$.

References


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