Lagrangian formalism in the classical field theory

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Abstract. A variational problem with a fixed boundary in the geometric form is considered. Equations of extremals are formulated in the language of differential forms on the Grassmannian manifold. A discussion on the transformation properties of the classical form of the Euler–Lagrange equations is presented and a new interpretation of these equations is proposed. The Legendre transformation in the field theory is defined and several physical examples of it are given. Invariance problems are considered and the Noether theorem is proved. Problems with constraints are formulated geometrically.

It is well known that a variational principle is an appropriate tool for the formulation of physical laws. Equations of mechanics, scalar field theory, electrodynamics can be derived from “the least action principle”. In mechanics this approach allows us to obtain two formulations of the theory: the Lagrangian and the canonical one. A mapping which gives an isomorphism between these formulations is called the Legendre transformation [1]. In the present paper we consider the Lagrangian approach to the classical field theory based on the geometrical theory of the calculus of variations. This theory was given by P. Dedecker [2] in the fifties. Recently H. Goldschmidt and S. Sternberg have given a modern exposition of the geometrical theory of the calculus of variations [5]. Their results are equivalent to ours, but are given in another formulation. The starting point of our considerations is a bundle $W$ over an $n$-dimensional manifold $B$. In relativistic field theories $B$ is the space-time. We construct, for a given Lagrangian function $\mathcal{L}$, an $n$-form $\mathfrak{v}$ on the Grassmannian bundle $G^n(W)$. Equations of motions take the form $(X \cdot d\mathfrak{v})|C = 0$, where $C$ is an $n$-dimensional submanifold of $G^n(W)$ and $X$ is any vector field defined on $C$, tangent to $G^n(W)$. This approach is in a 1–1 correspondence (when some conditions of regularity of $\mathcal{L}$ are fulfilled) with the canonical formulation. In the canonical formulation (cf. [4], [6]) a subspace $\mathcal{P} \subset \wedge^n T^*(W)$ and the canonical $n$-form $\omega$ on $\wedge^n T^*(W)$ are given. Equations of motion are $(Y \cdot d\omega)|\mathcal{S} = 0$, where $\mathcal{S}$ is an $n$-dimensional submanifold of $\mathcal{P}$ and $Y$ is any vector field defined on $\mathcal{S}$, tangent to $\mathcal{P}$. A diffeomorphism
\( L : G^n(\mathcal{C}) \rightarrow \mathcal{P} \) such that \( L^* \omega = \psi \) is called the Legendre transformation. We present several examples of Lagrangian functions and Legendre transformations.

In section 6 we study a new approach to the Euler–Lagrange equations. The analysis of geometrical properties of the classical form of the Euler–Lagrange equations in the language of differential forms is given. Especially simple are in our approach problems connected with the invariance of the Lagrangian function with respect to an \( m \)-parameter family of transformations of \( \mathcal{C} \). In section 9 we present an elegant proof of the Noether theorem. The results of that section are equivalent to the results of A. Trautman [10], who has given an exposition in the language of jet-bundles.

In section 10 we consider a variational principle with constraints in the “velocity space”. Such problems appear for instance in the hydrodynamics of an incompressible fluid, cf. [9].

We do not investigate the canonical structure of field theories. Results concerning that problem were published in papers of J. Kijowski and K. Gawędzki [6], [4]. Completely new results on the canonical structure of the classical field theory were recently obtained by J. Kijowski and the author [7]. Paper [7] gives the natural symplectic structure in an infinite dimensional manifold of solutions of the given field equations. This paper is an essential generalization of [4], [6] and provides a simple and elegant definition of physical quantities and Poisson brackets.

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1. NOTATION

In this paper we shall use notions of modern differential geometry. All these notions can be found in [8]. Let us recall some definitions.

If \( V_1, V_2 \) are smooth manifolds and \( f \) is a smooth mapping from \( V_1 \) into \( V_2 \), then \( f_* \) denotes the tangent map to \( f \). For any \( x \in V_1 \), \( f_* \) is a linear map from \( T_x(V_1) \) into \( T_{f(x)}(V_2) \), where \( T_x(V_1) \) denotes the space tangent to \( V_1 \) at the point \( x \). \( f_* \) is a linear map from \( T_{f(x)}(V_2) \) into \( T^*_x(V_1) \). The maps \( f_* \) can be extended onto exterior products:

\[
\begin{align*}
  (1.1) & \quad f_* : \Lambda^n T_x(V_1) \rightarrow \Lambda^n T_{f(x)}(V_2), \\
  (1.2) & \quad f_* : \Lambda^n T_{f(x)}(V_2) \rightarrow \Lambda^n T_x(V_1).
\end{align*}
\]

Let \( K^n(V_i) \) denote the bundle of all simple, non-vanishing \( n \)-vectors tangent to \( V_i \) (\( i = 1, 2 \)). There exist natural projections

\[
(1.3) \quad K^n(V_i) \rightarrow G^n(V_i), \quad i = 1, 2,
\]
where $G^n(V_i)$ denotes the Grassmannian bundle of all oriented $n$-planes tangent to $V_i$. There exists a map generated by (1.1)

\[(1.4) \quad f_\ast : G^n_x(V_1) \rightarrow G^n_{f(x)}(V_2).\]

For every $f : V_1 \rightarrow V_2$ the map (1.2) induces a linear map $f^*$ from $C^\infty(\wedge^n T^*(V_2))$ into $C^\infty(\wedge^n T^*(V_1))$, where $C^\infty(\wedge^n T^*(V_i))$ are the vector spaces of sections of the bundle $\wedge^n T^*(V_i) \rightarrow V_i$, $i = 1, 2$ (the "pull-back" of $n$-forms).

If $f$ is an injection, it generates a linear map $f^*$ from $C^\infty(\wedge^n T(V_1))$ into $C^\infty(\wedge^n T(V_2) | f(V_1))$. For

\[
v = v_1 \wedge \ldots \wedge v_n, \quad v_i \in T_x(V), \quad i = 1, \ldots, n,
\]

\[u^* = u_1^* \wedge \ldots \wedge u_n^*, \quad u_j^* \in T_x^*(V), \quad j = 1, \ldots, n,
\]

we define a bilinear form:

\[(1.5) \quad \langle v | u^* \rangle = \langle v_1 \wedge \ldots \wedge v_n | u_1^* \wedge \ldots \wedge u_n^* \rangle = \det \langle v_i | u_j^* \rangle = n! \cdot u_1^* \wedge \ldots \wedge u_n^* (v_1 \wedge \ldots \wedge v_n).
\]

We shall use the following definition of the interior product:

\[(1.6) \quad \langle v_1 \wedge \ldots \wedge v_n | v_1 \_ u^* \rangle = \langle v_1 \wedge \ldots \wedge v_n | u^* \rangle,
\]

\[v_1 \_ u^* \in \wedge^n T^*(V).
\]

Let $C \subset V$ be an embedded submanifold of $V$ and let $i : C \rightarrow V$ be the natural injection. For every $n$-form $\omega$ on $V$ we shall write

\[(1.7) \quad \omega | C := i^*(\omega).
\]

Let $\pi : W \rightarrow B$ be a bundle and $v \in T_{w}(W)$, $w \in W$. We call $v$ a $\pi$-vertical vector if $\pi^* v = 0$. The subspace of all $\pi$-vertical vectors tangent to $W$ at $w$ we denote $\pi\text{-}ver T_w(W)$.

By $\pi\text{-}hor T_w^*(W)$ we denote a subspace of $T_w^*(W)$ which annihilates $\pi\text{-}ver T_w(W)$. Elements of $\pi\text{-}hor T_w^*(W)$ are called $\pi\text{-}horizontal covectors.$

In this paper we shall use the summation convention in formulae which contain sums with respect to upper and lower indexes.

2. THE ACTION INTEGRAL

Let $W$ be an $r$-dimensional, smooth manifold (with boundary) and let $B$ be an $n$-dimensional, smooth, compact, orientable and connected manifold with boundary ($n \leq r$). Let $K^n(W)$ be a bundle of all simple, non-zero $n$-vectors tangent to $W$. Let $\omega_B$ be an $n$-form giving an orientation of $B$ (volume $n$-form) and let $v_B$ be the dual field of $n$-vectors on $B$, i.e., $\langle v_B | \omega_B \rangle = 1.$
DEFINITION. A Lagrangian function $\mathcal{L}$ is a positive homogeneous function on $K^n(W)$, i.e., $\mathcal{L} : K^n(W) \rightarrow \mathbb{R}$, for $\lambda \in \mathbb{R}_+$, $\nu \in K^n(W)$, $\lambda \cdot \mathcal{L}(\nu) = \mathcal{L}(\lambda \nu)$.

We shall assume that $\mathcal{L}$ is at least of class $C^2$.

For every smooth embedding $f : B \rightarrow W$ ($f$ is a diffeomorphism $B$ onto $f(B)$) we can define the action integral

$$I_f = \int_B \mathcal{L}(f \nu_B) \omega_B,$$

where $f_* : \wedge^n T(B) \rightarrow K^n(W)$ is induced by the following diagram:

$$\begin{array}{ccc}
\wedge^n T(B) & \xrightarrow{f_*} & K^n(W) \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & W
\end{array}$$

(2.1)

LEMMA 1. If $\omega_B$ belongs to a given orientation class of $B$, then $I_f$ defined by (2.1) does not depend on the particular choice of $\omega_B$.

Proof. If $\omega_B = \nu \cdot \omega_B$, $\nu \in C^\infty(B)$, $\nu > 0$, then from the positive homogeneity of $\mathcal{L}$ we have

$$\int_B \mathcal{L}(f \nu_B) \omega_B = \int_B \mathcal{L}(f \nu_B) \omega_B.$$

LEMMA 2. Let $B_1$ be a manifold diffeomorphic to $B$ and let $\lambda : B_1 \rightarrow B$ be an orientation-preserving diffeomorphism. Let $f_1 : B_1 \rightarrow W$ be an embedding such that $f_1 = f \circ \lambda$. Then for any $\omega_{B_1}, \omega_B$ which belong to the orientation classes corresponding to one another by $\lambda$ we have

$$\int_{B_1} \mathcal{L}(f_1 \nu_{B_1}) \omega_{B_1} = \int_{B_1} \mathcal{L}(f_1 \nu_{B_1}) \nu_{B_1}.$$

Proof. If $\omega_{B_1} = \lambda^* \omega_B$, $\nu_{B_1} = (\lambda^*)^{-1} \nu_B$ we have

$$\int_{B_1} \mathcal{L}(f_1 \nu_{B_1}) \omega_{B_1} = \int_{B_1} \mathcal{L}((f_1 \circ \lambda^{-1}) \nu_B) \lambda^* \omega_B = \int_B \mathcal{L}(f \nu_B) \omega_B.$$

For arbitrary $\omega_{B_1}, \omega_B$ the result follows from lemma 1.

LEMMA 3. If $\varphi$ is an orientation-preserving diffeomorphism $f(B)$ onto $f(B)$, then

$$\int_B \mathcal{L}((\varphi \circ f) \nu_B) \omega_B = \int_B \mathcal{L}(f \nu_B) \omega_B.$$
Proof. For simplicity we assume that $W = B$, $f = \text{id}_B$, $\varphi \in \text{Dif}(B)$. Then

$$\varphi^*_B(v_B(\varphi)) = \psi(\varphi(\varphi)) \cdot v_B(\varphi(\varphi)), \quad \varphi \in B, \quad 0 < \psi \in C^\infty(B),$$

$$\varphi(\varphi(\varphi)) \cdot (\varphi^{-1})^*_B(\omega_B(\varphi)) = \omega_B(\varphi(\varphi)),$$

$$\int_B \mathcal{L}(\varphi^*_B) \omega_B = \int_B \mathcal{L} \left( \psi(\varphi(\varphi)) \cdot v_B(\varphi(\varphi)) \right) \omega_B = \int_B \mathcal{L}(v_B(\varphi(\varphi)) \omega_B(\varphi(\varphi)) = \int_B \mathcal{L}(v_B(\varphi(\varphi)) \omega_B(\varphi)),$$

where $z = \varphi(x)$.

It follows from lemmas 1, 2 and 3 that the value $I_f$ depends only on the submanifold $C = f(B)$ and does not depend on the particular embedding $f$. This means that the value of the action integral (2.1) does not depend on the parametrization of $C$.

3. INTEGRABLE SUBMANIFOLDS OF $G^n(W)$

Let $\pi_1: K^n(W) \rightarrow W$ be the bundle of non-zero, simple $n$-vectors tangent to $W$. For every $v \in K^n(W)$ we have the uniquely determined element $\bar{v} = \pi_1(v) \in G^n(W)$ of the Grassmannian bundle $\pi_3: G^n(W) \rightarrow W$ of all oriented $n$-planes tangent to $W$. For every $n$-dimensional embedded submanifold $f: B \rightarrow W$ of $W$ there exists a uniquely determined submanifold $\bar{f}: B \rightarrow G^n(W)$ given by the following diagram:

$$\begin{array}{ccc}
G^n(W) & \xrightarrow{n_3} & K^n(W) \\
\downarrow f & & \downarrow f \\
W & \xrightarrow{n_1} & K^n(B)
\end{array}$$

(3.1)

(3.2)

$$\bar{f} = (\pi_3 \circ f_*) v.$$  

It is easy to see that $\bar{f}$ does not depend on the choice of $v$. We have also

$$f = \pi_3 \circ \bar{f}.  
(3.3)$$

If $C = f(B)$, we shall write $\bar{C} = \bar{f}(B)$ or $\bar{C}_f = \bar{f}(B)$.

Definition. An $n$-dimensional embedded submanifold $g: B \rightarrow G^n(W)$ is called integrable if $\pi_3 \circ g = g$.

Definition. Let $\pi: W \rightarrow B$ be a bundle over an $n$-dimensional manifold $B$. An embedded submanifold $f: B \rightarrow W$ is called $\pi$-transversal if, for every $w \in f(B)$, $\pi_* w$ is an isomorphism $T_{\pi(w)} f(B)$ onto $T_{\pi(w)} B$.

It follows from the implicit function theorem that for every $w \in f(B)$ there exists a neighbourhood $U \subset f(B)$ of $w$ in $f(B)$ such that $U$ is an
mage of some section \( f_1 \) of \( \pi \) over the set \( \pi(U) \). But \( f \) is an embedding, and so there exists a diffeomorphism \( \varphi: f^{-1}(U) \rightarrow \pi(U) \) such that \( f \circ f^{-1}(U) = f_1 \circ \varphi \). This means that if we change the parametrization, then \( f(B) \) will be locally a section of \( \pi \).

In the sequel we shall consider only those \( f \) which are globally sections of \( \pi \).

4. THE FIBRE DERIVATIVE OF A LAGRANGIAN FUNCTION

Let \( W \) be an \( r \)-dimensional manifold and let \( K^n(W) \) be a bundle of simple, non-zero \( n \)-vectors tangent to \( W \) (\( r \geq n \)). For every \( w \in W \) a fibre \( K^n_w(W) \) is a \( 1 + n(r - n) \) dimensional manifold. We have the following

**Lemma 4.** The space tangent to the manifold \( K^n_w(W) \) at a point \( v_w \in K^n_w(W) \)

is isomorphic to the subspace \( A(v_w) \) of \( \wedge T_w(W) \) spanned by the following \( n \)-vectors:

\[
\begin{align*}
&v_1 \wedge \ldots \wedge v_n, \quad v_1 \wedge \ldots \wedge v_k \wedge \ldots \wedge v_n \quad (1 \leq m \leq n, n + 1 \leq k \leq r),
\end{align*}
\]

where \( v_w = v_1 \wedge \ldots \wedge v_n \) and \( (v_k)_{k=n+1}^r \) are arbitrary linearly independent vectors which together with \( (v_j)_{j=1}^n \) form a basis of \( T_w(W) \).

**Proof.** Let \( t \rightarrow a_j^r(t), t \rightarrow \beta_k^s(t), 1 \leq s, j \leq n, n + 1 \leq k \leq r \), be smooth functions on an interval \( ]-\delta, \delta[ \) such that \( a_j^r(0) = \delta_j^r \) and \( \beta_k^s(0) = 0 \).

We have a curve in \( K^n_w(W) \):

\[
\begin{align*}
&\begin{aligned}
&\vdots t \rightarrow \frac{1}{\sum_{j=1}^n a_j^r(t) v_j + \sum_{k=n+1}^r \beta_k^s(t) v_k} \wedge \ldots \wedge \left( \sum_{j=1}^n a_j^r(t) v_j + \sum_{k=n+1}^r \beta_k^s(t) v_k \right).
\end{aligned}
\end{align*}
\]

If \( 0 < \varepsilon < \delta \) is small enough, we shall put \( \gamma_k^s(t) = \sum_{p=1}^n \alpha_{p}^r(t) \beta_k^s(t) \) and we shall have the curve

\[
\begin{align*}
&\begin{aligned}
&\vdots t \rightarrow \det \left[ a_j^r(t) \right] \cdot (v_1 + \gamma_1^r(t) v_n) \wedge \ldots \wedge (v_n + \gamma_n^r(t) v_k).
\end{aligned}
\end{align*}
\]

If we differentiate (4.2) at the point \( t = 0 \), we shall obtain a linear combination of \( n \)-vectors given in (4.1). It is easy to prove that locally every curve in \( K^n_w(W) \) passing through the point \( v_w \) is of the form (4.2) Taking all curves in \( K^n_w(W) \) passing through \( v_w \), we obtain the whole subspace \( A(v_w) \).

**Remark.** This construction does not depend on the representation of the \( n \)-vector \( v_w \) in the form \( v_1 \wedge \ldots \wedge v_n \) and does not depend on the choice of vectors \( (v_k)_{k=n+1}^r \) tangent to \( W \) at the point \( w \).

In general there does not exist a canonically defined subspace \( B(v_w) \) of \( \wedge T_w(W) \) such that \( \wedge T_w(W) = A(v_w) \oplus B(v_w) \). Therefore the dual
space
\[(4.3) \quad (A(v_\omega))^* \text{ is equal to } \bigwedge^n T^*_w(W)/(A(v_\omega))^0,\]
where \((A(v_\omega))^0\) is the subspace of \(\bigwedge^n T^*_w(W)\) which annihilates \(A(v_\omega)\).

If we want to choose a representative in the quotient space \(\bigwedge^n T^*_w(W)/(A(v_\omega))^0\), we shall have to define a section of the natural projection
\[(4.4) \quad \text{pr: } \bigwedge^n T^*_w(W) \rightarrow \bigwedge^n T^*_w(W)/(A(v_\omega))^0,\]
i.e., such a map
\[(4.5) \quad \xi: \bigwedge^n T^*_w(W)/(A(v_\omega))^0 \rightarrow \bigwedge^n T^*_w(W)\]
that
\[
\text{pr} \circ \xi = \text{id}.
\]

It is the problem of gauge in the given field theory. It can be proved that it is possible to take for every class \([\nu] \in \bigwedge^n T^*_w(W)/(A(v_\omega))^0\) such that \(\nu(v_\omega) \neq 0\) a unique simple \(n\)-covector belonging to this class. This procedure is called the Carathéodory gauge (cf. [2]).

For our purpose we shall use another gauge, which is connected with the bundle structure in \(W\). If \(\pi: W \rightarrow B\) is a bundle (\(\dim B = n\)), we shall use the following

**Definition.** An element \(v \in K^n(W)\) is called \(\pi\)-transversal if \(\pi_*\)(\(v\)) \neq 0. By \(\pi\)-tr\(K^n(W)\) (or simply tr\(K^n(W)\)) we denote the bundle of all \(\pi\)-transversal, simple, non-zero \(n\)-vectors tangent to \(W\).

In this case we choose vectors \((v_k)_{k=-n+1}^n\) in such a way that they are \(\pi\)-vertical vectors in \(T(W)\). Then there exists a subspace \(B \subset \bigwedge^n T^*_w(W)\) (independent of \(v_\omega\)) such that
\[(4.6) \quad \bigwedge^n T^*_w(W) = A(v_\omega) \oplus B.\]

\(B\) is the subspace consisting of all at least 2-vertical \(n\)-vectors and is spanned by all \(n\)-vectors of the form
\[(4.7) \quad v_{j_1} \wedge \ldots \wedge v_{k_1} \wedge \ldots \wedge v_{j_s} \wedge \ldots \wedge v_n, \quad 1 \leq j_1 < j_2 < \ldots < j_s \leq n, \quad n + 1 \leq k_1 < k_2 < \ldots < k_s \leq r, \quad s \geq 2.\]

It is easy to see that \(B\) does not depend on \(v_\omega\) and does not depend on the particular choice of \(\pi\)-vertical vectors \((v_k)_{k=-n+1}^n\). Therefore we have
\[(4.8) \quad (A(v_\omega))^* = B^0.\]

It follows from (4.7) that \(B^0\) is the subspace of all at most 1-vertical \(n\)-covectors and is spanned by \(n\)-covectors of the form
\[(4.9) \quad v^*_{j_1} \wedge \ldots \wedge v^*_n, \quad v^*_{j_1} \wedge \ldots \wedge v^*_{k_1} \wedge \ldots \wedge v^*_n, \quad 1 \leq j \leq n, \quad n + 1 \leq k \leq r, \quad j \neq k.\]
where \((v^p)_p\) form the dual basis to the basis \((v^p)_{p=1}^n\) of \(T^*_w(W)\). The subspace \(B^0\) is also independent of \(v_w\) and of the choice of \(\pi\)-vertical vectors \((v^p)_{p=1}^n\). We denote it by \(1\text{-ver} \wedge T^*_w(W)\).

**Remark.** The covectors \((v^p)_{p=1}^n\) are \(\pi\)-horizontal.

**Lemma 5.** Let \(\pi: W \to B\) be a bundle over an \(n\)-dimensional manifold \(B\). The space cotangent to the manifold \(\text{tr}K^*_w(W)\) at the point \(v_w\) is isomorphic to the space \(1\text{-ver} \wedge T^*_w(W)\).

We define, for every \(v_w \in \text{tr}K^*_w(W)\), a projection \(P_v: \wedge^n T^*_w(W) \to 1\text{-ver} \wedge T^*_w(W)\). If \(v_w = v_1 \wedge \ldots \wedge v_n\), \((v^p)_{p=1}^n\) are \(\pi\)-vertical vectors and
\[
v = A v^1 \wedge \ldots \wedge v^n + \sum_{1 \leq i < j \leq r} B^1_{k1} v^1 \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n + \sum_{1 \leq i < j \leq r} B^1_{k2} v^1 \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n + \ldots,
\]
then
\[
P_v v = A v^1 \wedge \ldots \wedge v^n + \sum_{1 \leq i < j \leq r} B^1_{k} v^1 \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n.
\]

(4.10) \(P_v v = A v^1 \wedge \ldots \wedge v^n + \sum_{1 \leq i < j \leq r} B^1_{k} v^1 \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n\).

It follows from formula (4.10) that \(P_v\) depends only on \(v_w = \pi_v(v_w) \in G^*_w(W)\).

The operator \(P_v\) gives us the so-called 1-vertical gauge (cf. [6]).

**Remark.** If \(n = 1\) or \(r = n+1\), \(A(v_w) = \wedge T^*_w(W)\) and we do not have problems with a gauge. These situations occur in mechanics and in the theory of scalar fields (cf. section 8).

Let us consider the map
\[
\text{tr}K^*_w(W) \ni v_w \to \mathcal{L}(v_w) \times \mathbb{R}.
\]

The derivative of this map is called the fibre derivative of \(\mathcal{L}\). We denote it by \(\mathcal{L}'_{\text{ver}}\). Of course, \(\mathcal{L}'_{\text{ver}}(v_w) \in 1\text{-ver} \wedge T^*_w(W)\).

**Lemma 6.**
\[
\mathcal{L}'_{\text{ver}}(v_w) = A \cdot v^1 \wedge \ldots \wedge v^n + \sum_{1 \leq m < n} B^m_{k} v^1 \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n,
\]
where \((v^p)_{p=1}^n\) and \((v^p)_{p=1}^n\) are such as in (4.9) and
\[
A = \frac{d}{dt} \bigg|_{t=0} \mathcal{L}((1+t)v_1 \wedge \ldots \wedge v_n) = \mathcal{L}(v_w)
\]
\[
= \langle v_1 \wedge \ldots \wedge v_n | \mathcal{L}'_{\text{ver}}(v_w) \rangle,
\]
\[
B^m_{k} = \frac{d}{dt} \bigg|_{t=0} \mathcal{L}(v_1 \wedge \ldots \wedge (v_m + tv_k) \wedge \ldots \wedge v_n) = \langle v_1 \wedge \ldots \wedge v_k \wedge \ldots \wedge v_n | \mathcal{L}'_{\text{ver}}(v_w) \rangle.
\]
Proof. This lemma follows from lemma 5, (4.8), (4.9) and from the positive homogeneity of $\mathcal{L}$.

From the positive homogeneity of $\mathcal{L}$ we have also

$$
\mathcal{L}_{\text{ver}}(\lambda v_w) = \mathcal{L}_{\text{ver}}(v_w) \quad \text{for } \lambda \in \mathbb{R}_+.
$$

Therefore we can define the mapping

$$
\text{tr}G^n(W) \ni \bar{v}_w \rightarrow \mathcal{L}_{\text{ver}}'(\bar{v}_w) \in \text{ver} \wedge T^*(W),
$$

where $\text{tr}G^n(W)$ is an open set in $G^n(W)$ consisting of all $\pi$-transversal oriented $n$-dimensional planes tangent to $W$ and $\bar{v}_w = \pi_2(v_w)$.

The following diagram induces a $\pi_2$-horizontal form $\psi$ on $\text{tr}G^n(W)$:

$$
\begin{array}{ccc}
\wedge T^*(\text{tr}G^n(W)) & \xrightarrow{\pi_2^*} & \wedge T^*(W) \\
\downarrow & & \downarrow \\
\text{tr}G^n(W) & \xrightarrow{\mathcal{L}_{\text{ver}}'(\cdot)} & W
\end{array}
$$

$$
\psi(\cdot) = \pi_2^* \circ \mathcal{L}_{\text{ver}}'(\cdot).
$$

Now we shall express the vertical derivative of $\mathcal{L}$ in local coordinates.

Let $(t^j, \omega^k), j = 1, \ldots, n, k = n+1, \ldots, r$, denote local coordinates in the bundle $W$.

If

$$
v_s = \alpha^j_s \frac{\partial}{\partial t^j} + \beta^k_s \frac{\partial}{\partial \omega^k}, \quad 1 \leq s \leq n,
$$

$v = v_1 \wedge \ldots \wedge v_n, v \in \text{tr}K^n(W)$, then $\det[a^j_s] \neq 0$.

Let

$$
\gamma^k_j = \sum_{m=1}^{n-1} \alpha^k_m \beta^m_j;
$$

then

$$
v = \det[a]\left(\frac{\partial}{\partial t^j} + \gamma^k_j \frac{\partial}{\partial \omega^k}\right) \wedge \ldots \wedge \left(\frac{\partial}{\partial t^m} + \gamma^k_m \frac{\partial}{\partial \omega^k}\right),
$$

where summation convention is used.

$(t^j, \omega^k, \gamma^k_j)$ form coordinates in $\text{tr}G^n(W)$ and $(t^j, \omega^k, \gamma^k_j, \lambda)$, where

$$
\lambda = \det[a], \text{ form coordinates in } \text{tr}K^n(W).
$$

In local coordinates (4.19),

$$
\mathcal{L}(v) = \mathcal{L}(t^j, \omega^k, \gamma^k_j, \lambda)
$$

and $\mathcal{L}$ is positive homogeneous in $\lambda$. 

Let
\[ v_p = \eta_p^k \frac{\partial}{\partial x^k}, \quad n + 1 \leq p \leq r, \quad \det[\eta] \neq 0; \]
then
\[ v^{*j} = \alpha^j_s \, dt^s, \quad 1 \leq j \leq n, \]
\[ v^{*p} = \eta^k_p dx^k - \eta^k_s \gamma^k_s dt^s, \quad n + 1 \leq p \leq r. \]

Using formulae (4.17), (4.22) and (4.23), we obtain
\[ v^{*1} \wedge \ldots \wedge v^{*n} = (\det[a])^{-1} dt^1 \wedge \ldots \wedge dt^n, \]
\[ v^{*1} \wedge \ldots \wedge v^{*p} \wedge \ldots \wedge v^{*n} = -\eta^k_p \beta^j_s (\det[a])^{-1} dt^1 \wedge \ldots \wedge dt^n + \]
\[ + (\det[a])^{-1} \sum_{j=1}^{n} \alpha^j_s \eta^k_p dt^1 \wedge \ldots \wedge dt^k \wedge \ldots \wedge dt^n. \]

For \( 1 \leq j \leq n, n + 1 \leq p \leq r, \) we have
\[ \frac{d}{dt} \bigg|_{t=0} \mathcal{L}(v_1 \wedge \ldots \wedge (v_j + tv_p) \wedge \ldots \wedge v_n) = \left. \frac{\partial \mathcal{L}(t^j, x^k, \gamma^j_s, \det[a])}{\partial \gamma^j_s} \right|_{t=0} \eta^k_p \alpha^j_s, \]
\[ \pi^* \circ \overline{\mathcal{L}}_{\overline{\mathcal{E}}} (v) = \psi(t^j, x^k, \gamma^j_s) \]
\[ = \frac{\partial \mathcal{L}(t^j, x^k, \gamma^j_s, 1)}{\partial \gamma^j_s} \, dt^1 \wedge \ldots \wedge dt^k \wedge \ldots \wedge dt^n - \]
\[ - \left( \frac{\partial \mathcal{L}(t^j, x^k, \gamma^j_s, 1)}{\partial \gamma^j_s} \gamma^k_s - \mathcal{L}(t^j, x^k, \gamma^j_s, 1) \right) dt^1 \wedge \ldots \wedge dt^n. \]

In local coordinates (4.19) we define
\[ \overline{\mathcal{L}}(t, x^k, \gamma^j_s) = \mathcal{L}(t^j, x^k, \gamma^j_s, 1). \]

If we change the coordinate chart:
\[ t' = t''(t^j), \quad x^k = x^k(t^j, x^k), \quad \gamma^j_s = \frac{\partial x^k}{\partial x^p} \cdot \gamma^j_s \cdot \frac{\partial t^s}{\partial t^{s'}} + \frac{\partial x^k}{\partial t^{s'}} \cdot \frac{\partial t^s}{\partial t^{s'}}, \]
then
\[ \overline{\mathcal{L}}(t'', x^k, \gamma^j_s) \det \left[ \frac{\partial t''}{\partial t'} \right] = \overline{\mathcal{L}}(t', x^k, \gamma^j_s). \]

In the same way as in lemmas 4 and 5 we can prove that the space tangent to \( \text{tr} G^\nu(W) \) at the point \( \bar{v}_\nu \) is isomorphic to the subspace of \( \wedge^n T_\nu(W) \).
spanned by $n$ vectors

$$v^1 \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n, \quad 1 \leq m \leq n, n+1 \leq k \leq r,$$

where $\bar{v}_w = \pi_\omega(v_w)$, $v_w = v^1 \wedge \ldots \wedge v^n$, and the space cotangent to $trG^m_{\omega}(W)$ at $\bar{v}_w$ is isomorphic to the subspace of $\bigwedge^n T^*_\omega(W)$ spanned by $n$-covectors

$$\omega^* \wedge \ldots \wedge \omega^{*k} \wedge \ldots \wedge \omega^{*n}, \quad 1 \leq m \leq n, n+1 \leq k \leq r.$$

5. A VARIATIONAL PRINCIPLE WITH A FIXED BOUNDARY

In this section we shall assume that $W$ is a bundle (over an $n$-dimensional manifold $B$) with a projection $\pi$. Let $\mathcal{L}$ be a Lagrangian function on $K^n(W)$, let $f: B \to W$ be a section of $\pi$ which is a diffeomorphism onto $f(B)$, let $f$ be the lift of $f$ to $trG^n(W)$ (see (3.2)) and let $\psi$ be defined by (4.15) and $I_f = \int_B L(f_* v) \omega$ (see (2.1)); then we have

**Proposition 1.**

$$I_f = \int_B f^* \psi.$$  

**Lemma 7.** Let $\omega \in f(B)$ and $(v_j)_{j=1}^r$ be a basis of $T^*_\omega (W)$ such that $(v_m)_{m=1}^n$ are tangent to $f(B)$ at the point $w$ and $(v_k)_{k=n+1}^r$ are $\pi$-vertical vectors. If $(v^{*j})_{j=1}^r$ is the dual basis, then, for $n+1 \leq k \leq r$, $f^* \circ \pi_\omega^*(v^{*k}) = 0$.

**Proof.** Vectors $(\pi_\omega v_m)_{m=1}^n$ form a basis of $T^*_\pi(B)$. We have

$$\langle (f^* \circ \pi_\omega^*) v^{*k} | \pi_\omega v_m \rangle = \langle (\pi_\omega \circ f^*) v^{*k} | \pi_\omega v_m \rangle = \langle f^* v^{*k} | \pi_\omega v_m \rangle = \langle v^{*k} | (f \circ \pi) v_m \rangle.$$

But $(f \circ \pi) v_m = id_{B^n}$ and $(v_m)_{m=1}^n$ are tangent to $f(B)$; therefore $(f \circ \pi)_* v_m = v_m$. The lemma is proved.

Proof of proposition 1. Let $(v_m)_{m=1}^n$, $(v_k)_{k=n+1}^r$ and let $(v^{*j})_{j=1}^r$ be such as in lemma 7. By lemma 6 we have

$$\psi(\bar{v}) = \pi_\omega^* \left( L(v) v^{*1} \wedge \ldots \wedge v^{*n} + \sum_{1 \leq m \leq n, \atop n+1 \leq k \leq r} B^m_k v^{*1} \wedge \ldots \wedge v^{*k} \wedge \ldots \wedge v^{*n} \right).$$

It follows from lemma 7 that

$$f^* \psi(\bar{v}) = (f^* \circ \pi_\omega^*) (\mathcal{L}(v) v^{*1} \wedge \ldots \wedge v^{*n}) = \mathcal{L}(f_* v) f^* (v^{*1} \wedge \ldots \wedge v^{*n}) = \mathcal{L}(f_* v) \omega,$$

where

$$v = v^1 \wedge \ldots \wedge v^n, \quad \bar{v} = \pi_\omega(v), \quad v(\pi(w)) = \pi_\omega(v^1 \wedge \ldots \wedge v^n), \quad \omega(\pi(w)) = f^* (v^{*1} \wedge \ldots \wedge v^{*n}).$$
Proposition 1 is proved.

Let $\pi_4: \text{ver}T(W) \to W$ be the bundle of $\pi$-vertical vectors tangent to $W$, let $\pi \circ \pi_4 \text{tr}G^n(\text{ver}T(W))$ be the bundle of all $\pi \circ \pi_4$-transversal oriented $n$-planes tangent to $\text{ver}T(W)$, and let $\pi \circ \pi_4 \text{ver}T(\text{tr}G^n(W))$ be the bundle of all $\pi \circ \pi_4$-vertical vectors tangent to $\text{tr}G^n(W)$.

**Lemma 8.** There exists an invertible mapping $\eta: \pi \circ \pi_4 \text{tr}G^n(\text{ver}T(W))$ onto $\pi \circ \pi_4 \text{ver}T(\text{tr}G^n(W))$, such that the following diagrams commute:

\[
\begin{array}{ccc}
\pi \circ \pi_4 \text{tr}G^n(\text{ver}T(W)) & \xrightarrow{\eta} & \pi \circ \pi_4 \text{ver}T(\text{tr}G^n(W)) \\
\downarrow & & \downarrow \\
W & \xrightarrow{id} & W \\
\pi \circ \pi_4 \text{tr}G^n(\text{ver}T(W)) & \xrightarrow{\pi_4 \circ \pi_4} & \text{tr}G^n(W) \\
\eta & & \pi_4 \circ \pi_4 \\
\pi \circ \pi_4 \text{ver}T(\text{tr}G^n(W)) & \xrightarrow{\pi_3} & W \\
\pi_4 & & \\
\text{ver}T(W) & \xrightarrow{\pi_4} & W
\end{array}
\]

In local coordinates $(b^*, u^k)$ on $W$ a point $(b^*, u^k, X^k, a^k_*, \beta^k_*)$ of $\pi \circ \pi_4 \text{tr}G^n(\text{ver}T(W))$ is transformed into a point $(b^*, u^k, a^k_*, X^k, \beta^k_*)$ of $\pi \circ \pi_4 \text{ver}T(\text{tr}G^n(W))$. This definition of $\eta$ does not depend on the choice of local coordinates on $W$ because of what follows:

If $(b^*, u^k, X^k, a^k_*, \beta^k_*)$ are local coordinates in $\pi \circ \pi_4 \text{tr}G^n(\text{ver}T(W))$ and $b^{k'} = b^{k'}(b^*)$, $u^{k'} = u^{k'}(b^*, u^k)$, then (cf. (4.29)):

\[
X^{k'} = X^k \frac{\partial u^{k'}}{\partial u^k}, \quad a^{k'}_* = \left( a^k_* \frac{\partial u^{k'}}{\partial u^k} + \frac{\partial u^{k'}}{\partial b^*} \right) \cdot \frac{\partial b^*}{\partial b^{k'}},
\]

\[
\beta^{k'}_* = \left( \beta^k_* \frac{\partial u^{k'}}{\partial u^k} + X^k \frac{\partial^2 u^{k'}}{\partial b^* \partial u^k} + \gamma^k_* X^p \frac{\partial^2 u^{k'}}{\partial u^p \partial u^k} \right) \cdot \frac{\partial b^*}{\partial b^{k'}}.
\]

If $(b^*, u^k, \gamma^k_*, Y^k, \lambda^k_*)$ are local coordinates in $\pi \circ \pi_4 \text{ver}T(\text{tr}G^n(W))$ and $b^{k'} = b^{k'}(b^*)$, $u^{k'} = u^{k'}(b^*, u^k)$, then (cf. (4.29)):

\[
Y^{k'} = Y^k \frac{\partial u^{k'}}{\partial u^k}, \quad \gamma^{k'}_* = \left( \gamma^k_* \frac{\partial u^{k'}}{\partial u^k} + \frac{\partial u^{k'}}{\partial b^*} \right) \cdot \frac{\partial b^*}{\partial b^{k'}},
\]

\[
\lambda^{k'}_* = \left( \lambda^k_* \frac{\partial u^{k'}}{\partial u^k} + Y^k \frac{\partial^2 u^{k'}}{\partial b^* \partial u^k} + \gamma^k_* X^p \frac{\partial^2 u^{k'}}{\partial u^p \partial u^k} \right) \cdot \frac{\partial b^*}{\partial b^{k'}}.
\]
If \( \pi_5 : \text{ver} T(W) \rightarrow B \) and \( X \) is a section of \( \pi_5 \), then we shall have the commutative diagram

\[
\begin{array}{cccccc}
\pi \circ \pi_5 \circ \text{tr} G^n (\text{ver} T(W)) & \xrightarrow{\eta} & \pi \circ \pi_5 \circ \text{ver} T(\text{tr} G^n (W)) & \xrightarrow{\pi_5} & \text{tr} G^n (W) & \\
\downarrow & & \downarrow & & \downarrow & \\
K^n (\text{ver} T(W)) & \xrightarrow{x} & \text{ver} T(W) & \xrightarrow{x} & W & \\
\uparrow K^n (B) & & \downarrow x & & \downarrow x & \\
\end{array}
\]

(5.6)

Let us define \( \bar{X} = (\eta \circ \pi_5)(X, \nu) \).

The section \( \bar{X} \) of \( \pi \circ \pi_5 \circ \text{ver} T(\text{tr} G^n (W)) \rightarrow B \) is called the canonical lift of \( X \) to \( \pi \circ \pi_5 \circ \text{ver} T(\text{tr} G^n (W)) \). If in local coordinates \( X = (b^s, u^k (b^s), X^k (b^s)) \), then

(5.7)

\[
\bar{X} = \left( b^s, u^k (b^s), \frac{\partial u^k}{\partial b^s}, X^k (b^s), \frac{\partial X^k}{\partial b^s} \right).
\]

Definition. We call a section \( Y \) of \( \pi \circ \pi_5 \circ \text{ver} T(\text{tr} G^n (W)) \rightarrow B \) integrable if there exists a section \( \bar{X} \) of \( \text{ver} T(W) \rightarrow B \) such that \( Y = \bar{X} \).

For every bundle \( \tau : V \rightarrow Z \) and every \( k \)-form \( \omega \) on \( V \) a fibre derivative of \( \omega \), is defined, \( \omega'_\text{ver} \in C^\infty (L(\text{ver} T(V), \wedge^k T^* (V))) \), where \( L(\text{ver} T(V), \wedge^k T^* (V)) \rightarrow V \) is the bundle of linear maps from \( \text{ver} T(V) \) to \( \wedge^k T^* (V) \). If \( (s', x') \) are local coordinates in \( V \),

\[
X = a^i \frac{\partial}{\partial x^i}.
\]

and

\[
\omega = \sum_{i_1 < \ldots < i_k} b_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{k-s}},
\]

then

(5.8)

\[
\omega'_\text{ver} (X) = \sum_{i_1 < \ldots < i_k} \frac{\partial}{\partial x^m} (b_{i_1 \ldots i_k}) a^m dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{k-s}}.
\]

From formula (5.8) we have the following

Lemma 9. If \( \omega \) is a \( \tau \)-horizontal form on \( V \) and \( X \) a \( \tau \)-vertical vector tangent to \( V \) at the point \( \nu \in V \), then we have

(5.9)

\[
(X \wedge d\omega) = \omega'_\text{ver} (X).
\]

We shall apply this lemma to the bundle \( \pi_5 : \text{tr} G^n (W) \rightarrow W \) and the \( \pi_5 \)-horizontal \( n \)-form \( \psi \).
For every \( \tilde{\psi} \in \text{tr}G^n(W) \) we have a mapping

\[
B_\tilde{\psi}: \pi_3^{-}\text{ver}T_{\tilde{\psi}}(\text{tr}G^n(W)) \times T_{\pi_3^-}(W)^{\circ}(X, q) \rightarrow B_{\tilde{\psi}}(X, \tilde{q}) = \langle \tilde{q}, \psi_{\text{vor}}(X) \rangle \in \mathbb{R} \quad (1).
\]

The space \( \pi_3^{-}\text{ver}T_{\tilde{\psi}}(\text{tr}G^n(W)) \) is isomorphic to a subspace of \( \bigwedge T_{\pi_3^-}(W) \) (see (4.31)), and so we have the bilinear mapping

\[
B_\tilde{\psi}: \pi_3^{-}\text{ver}T_{\tilde{\psi}}(\text{tr}G^n(W)) \times \pi_3^{-}\text{ver}T_{\tilde{\psi}}(\text{tr}G^n(W)) \rightarrow \mathbb{R}
\]
induced by (5.10). We shall assume that for every \( \tilde{\psi} \in \text{tr}G^n(W) \) the bilinear form \( B_{\tilde{\psi}} \) is non-degenerate. In section 7 we shall prove that this assumption will ensure the existence of the Legendre transformation.

In local coordinates in \( \text{tr}G^n(W) \) \((b^i, u^k, \gamma^k_\varphi)\),

\[
v = \left( \frac{\partial}{\partial b^i} + \gamma^k_\varphi \frac{\partial}{\partial u^k} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial b^n} + \gamma^k_\varphi \frac{\partial}{\partial u^k} \right),
\]

\[
\mathcal{L}(v) = \mathcal{L}^\varphi(b^i, u^k, \gamma^k_\varphi), \quad X = X^k_\varphi \frac{\partial}{\partial \gamma^k_\varphi}, \quad Y = Y^k_\varphi \frac{\partial}{\partial \gamma^k_\varphi},
\]

\[
B_{\tilde{\psi}}(X, Y) = \frac{\partial^2 \mathcal{L}^\varphi(b^i, u^k, \gamma^k_\varphi)}{\partial \gamma^k_\varphi \partial \gamma^k_\varphi} X^k_\varphi Y^k_\varphi.
\]

It is seen from formula (5.12) that \( B_{\tilde{\psi}} \) is a symmetric bilinear mapping. Its non-degeneracy is equivalent to the condition

\[
\det \left[ \frac{\partial^2 \mathcal{L}^\varphi(b^i, u^k, \gamma^k_\varphi)}{\partial \gamma^k_\varphi \partial \gamma^k_\varphi} \right] \neq 0.
\]

**Proposition 2.** If \( f: B \rightarrow W \) is a section of \( \pi_3 \), and \( \tilde{f} \) is the lift of \( f \) to \( \text{tr}G^n(W) \), then for every \( \pi_3 \)-vertical vector field \( X \) defined on \( C_{\tilde{f}} = \tilde{f}(B) \) we have \( (X \rfloor d\varphi)|_{C_{\tilde{f}}} = 0 \).

Conversely, for every \( \tilde{\psi} \in \text{tr}G^n(W) \) let the form \( B_{\tilde{\psi}} \) be non-degenerate. If \( \xi: B \rightarrow \text{tr}G^n(W) \) is a section of \( \pi_3 \pi_3 \) and, for every \( \pi_3 \)-vertical vector field \( X \) defined on \( C_\xi = \xi(B) \), \( (X \rfloor d\varphi)|_{C_\xi} = 0 \), then \( C_\xi \) is integrable, i.e., \( C_\xi = C_\varphi \), where \( \varphi = \pi_3 \circ \xi \).

We shall first prove the following

**Lemma 10.** Let \( \omega \) be a \( \pi \)-horizontal \( k \)-form on \( W \) and let \( \mathcal{L} \) be a fibre-preserving mapping from \( W \) to \( W \). Then \( \omega \) is \( \mathcal{L} \)-invariant, i.e., \( \mathcal{L}^*\omega = \omega \).

**Proof.** For every \( a \in W \) there exists a covector \( \tilde{\omega}_\pi(a) \in \bigwedge T_{\pi(a)}(B) \) such that \( \pi^* \tilde{\omega}_\pi(a) = \omega(a) \). Let \( a = \mathcal{L}(w) \). We have \( \pi^* \tilde{\omega}_\pi(\mathcal{L}(w)) = \omega(\mathcal{L}(w)) \).

Thus \( \pi^* \mathcal{L}^* \tilde{\omega}_\pi(a) = (\mathcal{L}^* \omega)(w) \) and \( \omega(w) = (\mathcal{L}^* \omega)(w) \) \( (\text{iv}) \).

---

2 \( \text{iv} \) Where \( \tilde{\mathcal{L}} \) is any lift of \( \mathcal{L} \) to \( \mathcal{T}_{\pi_3}(\text{tr}G^n(W)) \), (5.10) does not depend on the choice of \( \tilde{\mathcal{L}} \) because \( \psi_{\text{vor}}(X) \) is a \( \pi_3 \)-horizontal form.

3 \( \text{iv} \) \( \mathcal{L} \) is not the Lagrangian function.
Proof of proposition 2. Let $\bar{v} = v_1 \wedge \ldots \wedge v_n$ and $(v_i)_s^{n+1}$ are linearly independent $\pi$-vertical vectors at $w \in W$. It follows from lemma 6 that

$$\psi'_{\text{ver}}(\bar{v})(X) = \sum_{1 \leq q \leq n, n+1 \leq p, r \leq r} \frac{\partial^2 \mathcal{L}(v_1 + \lambda^i v_i) \wedge \ldots \wedge (v_n + \lambda^i v_i)}{\partial \lambda^p_q \partial \lambda^r_i} X^p_q \pi^s_{\bar{v}}(v^{s_1} \wedge \ldots \wedge v^{s_r} \wedge \ldots \wedge v^{s_n}),$$

where $X \in T_0(\text{tr} G^n(W)), X = \sum_{1 \leq q \leq n, n+1 \leq p \leq r} \frac{X^p_q v_1 \wedge \ldots \wedge v_p \wedge \ldots \wedge v_n}{k_q} (\text{cf. (4.31)}).$

If $z$ is a simple non-zero $n$-vector tangent to $C\bar{v}$ at $\bar{v}$, then $\pi^z_{\pi(z)} \bar{v}$ is a non-zero, simple $n$-vector tangent to $C_{\pi(z)} \pi(\bar{v})$. Therefore $\pi^z_{\pi(z)} \bar{v} = av$, $0 \neq a \in R$. From formula (5.14) we have $\langle z | \psi'_{\text{ver}}(\bar{v})(X) \rangle = 0$.

Conversely, let $\xi: B \rightarrow \text{tr} G^n(W)$ be a section of $\pi \circ \pi$ and let $z$ be a non-zero, simple $n$-vector tangent to $C_{\xi}$ at $\bar{v}$. Then $\pi^z_{\pi(z)} \bar{v}$ is a non-zero $\pi$-transversal $n$-vector tangent to $C_{\pi^z_{\pi(z)}} \pi(\bar{v})$. Therefore $\pi^z_{\pi(z)} \bar{v} = a(\pi^z_{\pi(z)} \pi(\bar{v})), 0 \neq a \in R$.

From formula (5.14) we have for every $X^p_q$

$$\frac{\partial^2 \mathcal{L}(\cdot)}{\partial \lambda^p_q \partial \lambda^r_i} \bigg|_{\lambda^p_q - \lambda^r_i = 0} X^p_q v^{s_1} \wedge \ldots \wedge v^{s_r} \wedge \ldots \wedge v^{s_n}(\pi^z_{\pi(z)} \pi(\bar{v}))(v_1 \wedge \ldots \wedge v_n) = 0.$$

Covectors $(v^{*j})_{j=1}^n$ are $\pi$-horizontal, and so it follows from lemma 10 that

$$(\pi^z_{\pi(z)} \pi)(v^{*j}) = v^{*j}, \quad 1 \leq j \leq n.$$  (5.16)

Formula (5.16) implies

$$\langle (\pi^z_{\pi(z)} \pi)_* v_s | v^{*j} \rangle = \langle v_s | (\pi^z_{\pi(z)} \pi)^* v^{*j} \rangle = \delta^i_s, \quad 1 \leq s, j \leq n. $$  (5.17)

Using (5.15) and (5.17) we obtain

$$\frac{\partial^2 \mathcal{L}(\cdot)}{\partial \lambda^p_q \partial \lambda^r_i} \bigg|_{\lambda^p_q - \lambda^r_i = 0} X^p_q \langle (\pi^z_{\pi(z)} \pi)_* v_s | v^{*i} \rangle = 0.$$  (5.18)

Formula (5.18) together with the non-degeneracy condition for $B_{\bar{v}}$ gives

$$\langle (\pi^z_{\pi(z)} \pi)_* v_s | v^{*i} \rangle = 0, \quad 1 \leq s \leq n, n+1 \leq i \leq r. $$  (5.19)

It follows from (5.19) that vectors $(\pi^z_{\pi(z)} \pi)_* v_s, 1 \leq s \leq n$, are linear combinations of $(v_i)_s^{n+1}$. They are also linearly independent because vectors $\pi^z_{\pi(z)} \pi v_s = \pi^z \pi v_s, 1 \leq s \leq n$, are linearly independent. We conclude that the plane tangent to $(\pi^z_{\pi(z)} B)\pi(\bar{v})$ is equal to $\bar{v}$. The proposition is proved.
In order to define a variational problem with a fixed boundary we have to introduce the notion of a one-parameter family of sections of \( \pi: W \rightarrow B \).

**Definition.** A mapping \( \delta, \delta[ \times B \times (e, b) \rightarrow f_{\epsilon}(b) \in W \) is called a one-parameter family of sections of \( \pi \) if, for every \( \epsilon \in ]-\delta, \delta[ \), \( \pi \circ f_{\epsilon}(\cdot) = id_B \). We shall consider only those mappings \( (e, b) \rightarrow f_{\epsilon}(b) \) which are at least of class \( C^2 \). We shall say that a one-parameter family of sections preserves the boundary if, for every \( \epsilon \in ]-\delta, \delta[ \), \( f_{\epsilon}(\partial B) = f_{\delta}(\partial B) \), or equivalently

\[
\delta f_{\epsilon}(\partial B) = f_{\delta}(\partial B).
\]

A one-parameter family of sections of \( \pi, f_{\epsilon}(\cdot), \epsilon \in ]-\delta, \delta[ \), defines a one-parameter family of sections of \( \pi \circ \pi_{\epsilon}, (e, b) \rightarrow \tilde{f}_{\epsilon}(b) \) \( \epsilon \in G^n(W) \). These two families of sections determine vector fields \( X \) and \( \tilde{X} \), which are defined, respectively, on \( f_{\delta}(B) \) and \( \tilde{f}_{\delta}(B) \):

\[
(5.20) \quad X(f_{\epsilon}(b)) = \frac{d}{de} \bigg|_{\epsilon=0} f_{\epsilon}(b),
\]

\[
(5.21) \quad \tilde{X}(\tilde{f}_{\epsilon}(b)) = \frac{d}{de} \bigg|_{\epsilon=0} \tilde{f}_{\epsilon}(b).
\]

**Lemma 11.** A \( \pi \circ \pi_{\epsilon} \)-vertical vector field \( \tilde{X} \) is the canonical lift (in the sense of (5.6)) of a \( \pi \)-vertical vector field \( X \). If the family \( f_{\epsilon}(\cdot) \) preserves the boundary of \( B \), then \( Xf_{\delta}(\partial B) = 0 \) and \( \tilde{X}\tilde{f}_{\delta}(\partial B) \) is \( \pi_{\epsilon} \)-vertical.

The proof follows from lemma 8 and diagram (5.6).

According to proposition 1, let

\[
(5.22) \quad I_{\tilde{f}_{\epsilon}} = \int_B \mathcal{L}(f_{\epsilon}^* \psi) \omega = \int_B \tilde{f}_{\epsilon}^* \psi, \quad \epsilon \in ]-\delta, \delta[.
\]

**Proposition 3.**

\[
(5.23) \quad \frac{d}{de} \bigg|_{\epsilon=0} I_{\tilde{f}_{\epsilon}} = \int_{\tilde{f}_{\epsilon}(\partial \Omega)} \tilde{X}\bot \tilde{d} \psi + \int_{\tilde{f}_{\epsilon}(\partial \Omega)} \tilde{d} (\tilde{X}\bot \psi),
\]

where \( \tilde{X} \) is defined by (5.21).

This proposition follows (with a slight modification) from the familiar formula for the Lie derivative:

\[
(5.24) \quad \mathcal{L}_X \psi = \tilde{X}\bot \psi + \tilde{d}(\tilde{X}\bot \psi) \quad \text{(cf. [8])}.
\]

**Proposition 4.** If the family of sections of \( \pi, f_{\epsilon}(\cdot), \epsilon \in ]-\delta, \delta[ \) is constant on the boundary of \( B \), then

\[
(5.25) \quad \int_{\tilde{f}_{\epsilon}(\partial \Omega)} \tilde{d} (\tilde{X}\bot \psi) = 0.
\]
Proof. Using the Stokes theorem, we have

\[ \int_{\delta B} d(\vec{X} \lrcorner \psi) = \int_{\delta B} \vec{X} \lrcorner \psi = \int_{\delta B} \vec{X} \lrcorner \psi. \]

But \( \psi \) is a \( \pi_{\alpha} \)-horizontal \( n \)-form and \( \vec{X} \) is \( \pi_{\alpha} \)-vertical on \( f_0(\partial B) \) (see lemma 11); therefore \( \vec{X} \lrcorner \psi = 0 \) on \( f_0(\partial B) \).

Let \( M \) be an \( (n-1) \)-dimensional embedded submanifold of \( W \) such that \( M \) is a diffeomorphism \( M \) onto \( \partial B \). The variational problem with a fixed boundary \( M \) is a triplet \( (W, \mathcal{L}, M) \), where \( \mathcal{L} \) is a Lagrangian function.

Definition. An embedded submanifold \( f: B \rightarrow W \) is called an extremum section of the variational problem \( (W, \mathcal{L}, M) \) if:

1° \( f \) is a section of \( \pi \) and \( f(\partial B) = M \),

2° for every one-parameter family \( f_\epsilon(\cdot), \epsilon \rightarrow \delta, \delta[ \), of sections of \( \pi \) fulfilling the conditions

\[ f_\epsilon(\partial B) = M, \quad \epsilon \rightarrow \delta, \delta[, \]

\[ f = f_0, \]

we have

\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} I_{f_\epsilon} = 0. \]

Theorem 1. Let \( f: B \rightarrow W \) be a section of \( \pi \) such that \( f(\partial B) = M \). If, for every \( \pi \circ \pi_{\alpha} \)-vertical integrable vector field \( \vec{X} \) on \( C_f \), \( (\vec{X} \lrcorner d\psi)|_{C_f} = 0 \), then \( f \) is an extremal section of the variational problem \( (W, \mathcal{L}, M) \).

If \( f: B \rightarrow W \) is an extremal section of the variational problem \( (W, \mathcal{L}, M) \), then for every \( \pi \circ \pi_{\alpha} \)-vertical integrable vector field \( \vec{X} \) on \( C_f \)

\[ (\vec{X} \lrcorner d\psi)|_{C_f} = 0. \]

Proof. The first statement follows from propositions 3 and 4. Let \( f: B \rightarrow W \) be an extremal section and \( \vec{X} \) a \( \pi \circ \pi_{\alpha} \)-vertical integrable vector field on \( C_f \). We have to prove that \( \vec{X} \) is generated by a one-parameter family of sections of \( \pi \). Let \( \vec{X} = \pi_{\alpha} \vec{X} \) be a \( \pi \)-vertical vector field on \( C_f \). It follows from the theorem on flows of vector fields on compact manifolds that \( X \) defines a one-parameter family of sections of \( \pi(f_\alpha) \) (cf. [1]) such that

\[ \frac{d}{d\epsilon} \bigg|_{\epsilon=0} = \vec{X}. \]

Therefore, for every \( \pi \circ \pi_{\alpha} \)-vertical integrable vector field \( \vec{X} \) on \( C_f \), we infer from (5.23) and (5.25) that

\[ \int_{C_f} (\vec{X} \lrcorner d\psi) = 0. \]
Let $\varphi$ be a function on $C_j$ and let $\varphi_1$ be the corresponding function on $C_j$ (i.e., $\varphi = \pi_1^* \varphi_1$). It is easy to see that $\varphi \cdot \overline{X - \varphi_1 X}$ is a $\pi_2$-vertical vector field on $C_j$ (cf. (5.7)). If we use (5.26') and proposition 2, we shall find that, for every $\varphi \in C^\infty(C_j)$, $(\overline{X - \varphi_1 X})|_{C_j} = 0$. Thus $(\overline{X - \varphi_1 X})|_{C_j} = 0$.

**Theorem 2.** For every $\varphi \in \text{tr} G^m(W)$, let the bilinear form $\tilde{B}_\varphi$ be non-degenerate. Then every embedded submanifold $\xi: B \to \text{tr} G^m(W)$ of $\text{tr} G^m(W)$ which is a section of $\pi \circ \pi_2$ and satisfies, for every $\pi \circ \pi_2$-vertical vector field $Y$ defined on $C_j$, the condition

$$
(\overline{X - \varphi_1 X})|_{C_j} = 0
$$

is integrable and $\pi \circ \xi$ is an extremal section of the variational problem $(W, \mathcal{L}, \pi^1 \circ \xi(\partial B))$.

If $f: B \to W$ is an extremal section of the variational problem $(W, \mathcal{L}, M)$, then for every vector field $Y$ defined on $C_j$, we have

$$
(\overline{X - \varphi_1 X})|_{C_j} = 0.
$$

**Proof.** The first part of this theorem follows from proposition 2 and theorem 1.

Let $f: B \to W$ be an extremal section of the variational problem $(W, \mathcal{L}, M)$. For every vector field $Y$ on $C_j$ there exist a vector field $Y_1$ tangent to $C_j$ and a $\pi \circ \pi_2$-vertical vector field $Y_2$ such that $Y = Y_1 + Y_2$. It is easy to see that

$$
(\overline{Y_1 - \varphi_1 Y_1})|_{C_j} = 0.
$$

Let $X = \pi_2 Y$, be a $\pi$-vertical vector field on $C_j$ and let $\overline{X}$ be the canonical lift of $X$. $\overline{X}$ is a $\pi_2$-vertical integrable vector field on $C_j$ and $\overline{X} - \overline{X}$ is a $\pi_2$-vertical vector field on $C_j$. Therefore we have

$$
(\overline{Y_1 - \varphi_1 Y_1})|_{C_j} = (\overline{X - \varphi_1 X})|_{C_j} + (\overline{Y_2 - \overline{X}})|_{C_j}.
$$

From the above formula, proposition 2 and theorem 1 we obtain

$$
(\overline{Y_2 - \varphi_1 Y_2})|_{C_j} = 0.
$$

Formula (5.30) together with (5.29) completes the proof.

**Remark.** In the second part of the proof we do not use the non-degeneracy condition for $\tilde{B}_\varphi$.

We shall consider as an example a special case of a Lagrangian function. Let $\Omega \in C^\infty(\bigwedge^n \mathcal{T}^* W)$, i.e., $\Omega$ is an $n$-form on $W$. We define a Lagrangian function which corresponds to the form $\Omega$:

$$
K(W \ast v \to \mathcal{L}(v) = \langle v | \Omega \rangle \in \mathbb{R}.
$$
If \( v = v_1 \wedge \ldots \wedge v_n \) and \((v_k)_k^{n+1}\) are linearly independent \(n\)-vertical vectors, we have by lemma 6
\[
L'_{\text{ver}}(v) = \langle v | \Omega \rangle v^* \wedge \ldots \wedge v^n +
\sum_{m=1}^n \langle v_1 \wedge \ldots \wedge v_k \wedge \ldots \wedge v_n | \Omega \rangle v^* \wedge \ldots \wedge v^k \wedge \ldots \wedge v^n = P_v \Omega,
\]
where \((v^*_k)_k^{n+1}\) is the dual basis and \(P_v\) is the projection on the subspace of \(n\)-one vertical forms (cf. formula (4.10)).

We shall use the following

**Lemma 12.** In the above notation let
\[
q = A v_1 \wedge \ldots \wedge v_n + \sum_{1 \leq m < \ell \leq n} B^\ell_m v_1 \wedge \ldots \wedge v_k \wedge \ldots \wedge v_n;
\]
then \(\langle q | P_v \Omega \rangle = \langle q | \Omega \rangle\).

**Proposition 5.** Let \(L\) be given by (5.31). A section \(f: B \rightarrow W\) is an extremal section of the variational problem \((W, L, M)\) if and only if:
1° \(f(\partial B) = M\),
2° for every \(n\)-vertical vector field \(X\) defined on \(C_f\),
\[
(X \lrcorner d\Omega) | C_f = 0.
\]

**Proof.** Let \(v \in C_f\). Locally there exist \(n\) linearly independent vector fields \(Q_1, \ldots, Q_n\) defined on a neighbourhood \(U\) of \(v\) in \(C_f\) and tangent to \(C_f\). Let \(Q_o\) be a \(\pi_o\)-\(n\)-vertical vector field defined on \(C_f\) (\(^a\)). By means of \(Q_o\) we can construct a bundle over \(U\) with 1-dimensional fibre in such a way that \(Q_o\) will be vertical. This bundle \(\tilde{D}\) is an \((n+1)\)-dimensional submanifold in \(trG^n(W)\).

We can extend the vector fields \((Q_j)_j^{n+1}\) onto \(\tilde{D}\). We denote these extensions by \(\tilde{Q}_j\). Let \(\tilde{P}_f = \pi_v \tilde{Q}_f\) and let \(D = \pi_v(\tilde{D})\). For \(1 \leq j \leq n\), \(\tilde{P}_f\) are tangent to \(C_f\) at points belonging to \(C_f\). It is clear that \(v = P_1(\pi_v v) \wedge \ldots \wedge P_n(\pi_v v)\). Now we shall use the formula for the exterior derivative (cf. [8]):
\[
\frac{1}{n+1} (\tilde{Q}_o \lrcorner dv)(\tilde{Q}_1, \ldots, \tilde{Q}_n) = d \psi(\tilde{Q}_o, \tilde{Q}_1, \ldots, \tilde{Q}_n)
\]
\[
= \frac{1}{n+1} \sum_{j=0}^n (-1)^j \tilde{Q}_j \psi(\tilde{Q}_o, \ldots, \tilde{Q}_n) +
\]
\[
+ \frac{1}{n+1} \sum_{0 \leq i < j \leq n} (-1)^{i+1} \psi([\tilde{Q}_i, \tilde{Q}_j], \tilde{Q}_o, \ldots, \tilde{Q}_n).
\]

\(^a\) And such that \(\pi_v Q_o\) is non-vanishing vector field on \(C_f\).
But \( \psi(\bar{y}) = \pi^*_x P_\bar{y} \Omega \) and, for \( i, j \neq 0 \), \( [\bar{P}_i, \bar{P}_j] \Omega \) is a linear combination of \( \bar{P}_1, \ldots, \bar{P}_n \) and \( [\bar{P}_i, \bar{P}_j] \) is a linear combination of \( \bar{P}_0, \ldots, \bar{P}_n \). Therefore we can use (5.33).

Using (5.33), we see from (5.35) that

\[
(5.36) \quad d\varphi(\bar{Q}_0, \ldots, \bar{Q}_n)|C_j = d\Omega(\bar{P}_0, \ldots, \bar{P}_n)|C_j.
\]

This ends the proof.

6. A GEOMETRICAL FORMULATION OF THE EULER–LAGRANGE EQUATIONS

Let \( (W, \mathcal{L}, M) \) be a variational problem with a fixed boundary. Let \( \mathcal{H} \) denote the set of all embedded \( n \)-dimensional submanifolds of \( W \) fulfilling the following conditions:

1° if \( C \in \mathcal{H} \), then there exists a section \( f \) of \( \pi: W \to B \) such that \( C = f(B) \),

2°

\[
(6.1) \quad \partial C = M.
\]

Let \( C^\omega(C, \pi, \text{ver} T(W)) \) denote the vector space of all smooth \( \pi \)-vertical vector fields tangent to \( W \) and defined on \( C \).

We define a functional: for every \( C \in \mathcal{H} \) and \( X \in C^\omega(C, \pi, \text{ver} T(W)) \)

\[
(6.2) \quad (C, X) \to F(C, X) = \int_X \cdot d\varphi \in \mathbb{R},
\]

where \( \varphi \) is defined by (4.15), \( \bar{C} \) is the canonical lift of \( C \) to \( \text{tr} G^n(W) \) and \( \bar{X} \) is the canonical lift of \( X \) defined by (5.6).

It follows from proposition 2 that instead of the canonical lift \( \bar{X} \) of \( X \) we can take in (6.2) any \( \pi \circ \pi \)-vertical vector field \( Y \) defined on \( \bar{C} \) such that \( \pi_* Y = X \).

In local coordinates \( (t^a, \varphi^k, \gamma^k(t^a)) \), \( C = \{ (t^a, \varphi^k(t^a)) \} \), \( \bar{C} = \{ (t^a, \varphi^k(t^a), \gamma^k(t^a) = \partial x^k / \partial t^a) \} \), \( Y = (0, X^k, Y^k_t) \).

Using (4.27), (4.28) we obtain

\[
(6.3) \quad (X \cdot d\varphi)|\bar{C} = \left[ \frac{\partial \bar{L}(t^a, \varphi^k(t^a), \gamma^k(t^a))}{\partial x^p} X^p - X^p \frac{\partial \bar{L}(t^a, \varphi^k(t^a), \gamma^k(t^a))}{\partial t^a \partial \gamma^k_m} \right. - \frac{\partial \bar{L}(t^a, \varphi^k(t^a), \gamma^k(t^a))}{\partial \gamma^k_m(t^a)} \frac{\partial \gamma^l_m(t^a)}{\partial t^a} X^p - \left. \frac{\partial \bar{L}(t^a, \varphi^k(t^a), \gamma^k(t^a))}{\partial \gamma^k m(t^a)} \frac{\partial \gamma^l_m(t^a)}{\partial t^a} X^p \right] dt^1 \wedge \ldots \wedge dt^n;
\]
where the summation convention is used. From (6.3) we obtain

\begin{equation}
(\chi \ldots \chi) \mid \tilde{C} = \left[ \begin{array}{c}
\frac{\partial \tilde{L}}{\partial \omega^p} \left( t^i, \omega^k(t^i), \gamma^k_j(t^i) \right) - \frac{\partial}{\partial t^j} \frac{\partial \tilde{L}}{\partial \gamma^p_q} \left( t^i, \omega^k(t^i), \gamma^k_j(t^i) \right)
\end{array} \right] \chi^p \chi^1 \ldots \chi^n.
\end{equation}

**Lemma 13.** If we change the coordinates in \( \text{tr} G^n(W) \) \( (t^i, \omega^k, \gamma^k_j) \rightarrow (t', \omega', \gamma'^k_j) \) (see (4.29)), we obtain

\begin{equation}
\frac{\partial \tilde{L}}{\partial \omega^p} \left( t', \omega'(t'), \gamma'^k_j(t') \right) - \frac{\partial}{\partial t'} \frac{\partial \tilde{L}}{\partial \gamma^p_q} \left( t', \omega'(t'), \gamma'^k_j(t') \right)
= \left[ \begin{array}{c}
\frac{\partial \tilde{L}}{\partial \omega^p} \left( t', \omega'(t'), \gamma'^k_j(t') \right) - \frac{\partial}{\partial t'} \frac{\partial \tilde{L}}{\partial \gamma^p_q} \left( t', \omega'(t'), \gamma'^k_j(t') \right)
\end{array} \right] \frac{\partial \omega'p}{\partial \omega^p} \cdot \det \left[ \frac{\partial t'}{\partial t^j} \right] = \frac{\partial \omega'p}{\partial \omega^p} \cdot \det \left[ \frac{\partial t'}{\partial t^j} \right].
\end{equation}

Formula (6.5) follows from (4.30) by a direct computation.

Using the transformation formula (6.5), we see that (6.4) defines on every open set \( U \subset C \) in \( C \) which is contained in a domain of a local chart \( (t^i, \omega^k) \) one form \( \xi_U \in C^\infty(U, T^*(W)) \) which fulfills the following condition: for \( f^* : B \rightarrow W \) \((C = f(B))\), volume \( n \)-form \( \omega_B \) on \( B \) and for every \( \pi \)-vertical vector field \( \chi \) on \( U \),

\begin{equation}
\tilde{f}^*(\chi) \mid \chi = (f^* \pi(U))^* (\chi \mid \chi) \cdot \omega_B \mid \chi(U).
\end{equation}

In local coordinates \( (t^i, \omega^k) \), \( \omega_B = \chi^p \chi^1 \ldots \chi^n, \chi^p > 0 \),

\begin{equation}
\omega = f(t) = (t^i, \omega^k(t^i)), \quad \gamma^k_j(t^i) = \frac{\partial \omega^k}{\partial t^j}(t^i),
\end{equation}

\begin{equation}
\xi_U(w) = (\psi(t))^\chi \left[ \left. \frac{\partial \tilde{L}}{\partial \omega^p} \left( t', \omega'(t'), \gamma'^k_j(t') \right) - \frac{\partial}{\partial t'} \frac{\partial \tilde{L}}{\partial \gamma^p_q} \left( t', \omega'(t'), \gamma'^k_j(t') \right) \right| \right.
\end{equation}

By means of a partition of unity we can construct one form \( \xi \in C^\infty(C, T^*(W)) \) such that for every \( \pi \)-vertical vector field \( \chi \) on \( C \) we have

\begin{equation}
\tilde{f}^*(\chi) \mid \chi = f^*(\chi) \omega_B.
\end{equation}

The form \( \xi \) is not uniquely determined. If \( \xi_1, \xi_2 \in C^\infty(C, T^*(W)) \) and fulfill (6.8), we have, for every \( X \in C^\infty(C, \pi-\text{ver}T(W)) \), \( \chi \mid (\xi_1 - \xi_2) = 0 \) on \( C \). Therefore \( \xi_1 - \xi_2 \in C^\infty(C, \pi-\text{hor}T^*(W)) \).

We have proved

**Theorem 3.** For every \( n \)-dimensional, \( \pi \)-transversal embedded submanifold \( C \) of \( W \) fulfilling (6.1) there exists an element \([\xi]\) of the factor space \( C^\infty(C, T^*(W)) \mid C^\infty(C, \pi-\text{hor}T^*(W)) \) such that for every \( \xi \in [\xi] \) formula (6.8) holds. \( C \) is an extremal of the variational problem \((W, \mathcal{L}, \pi)\) if and only if \([\xi] = 0\).
If we have a connection in the bundle $W$, we have the map
\[(6.9) \quad C^\infty(C, T^\ast(W)) \xrightarrow{\mathcal{L}} \text{ver} \mathcal{L} \circ C^\infty(C, \pi\text{-ver} T^\ast(W)).\]

Using (6.9), we can construct the map
\[(6.10) \quad C^\infty(C, T(W))/C^\infty(C, \pi\text{-hor} T^\ast(W)) \ni [\xi] \rightarrow \text{ver} \xi \in C^\infty(C, \pi\text{-ver} T^\ast(W)).\]

In this case $C$ is an extremal if and only if ver $\xi = 0$.
A situation like that has been investigated in the hydrodynamics of an incompressible fluid, cf. [9].

7. HAMILTONIAN FORMULATION OF VARIATIONAL PROBLEMS.
THE LEGENDRE TRANSFORMATION

In this section we shall describe how to pass from the Lagrangian to the Hamiltonian formulations of the classical field theory. We shall construct a phase space $\mathcal{P}$ of a given physical system for which we know the Lagrangian function. This construction generalizes the notion of Legendre transformation known in mechanics, cf. [1]. We shall not develop the theory of canonical fields, physical quantities, Poisson brackets etc. These notions have been investigated in [4], [6]. Recently new results concerning these problems were obtained and published in [7]. Paper [7] essentially generalizes the results of [4], [6] and gives an elegant construction of the natural symplectic structure on the space of physical states (solutions of field equations). We also leave aside the problem of construction of a phase space in theories without a Lagrangian function. This problem has recently been partially solved and the results will be published elsewhere.

Several physical examles of phase space are given in section 8.
For every $\bar{v} \in \text{tr} G^n(W)$ let the bilinear form $\tilde{B}_{\bar{v}}$ (defined by (5.11)) be non-degenerate.

DEFINITION. The Legendre transformation is the map $L$ defined by
\[(7.1) \quad \text{tr} G^n(W) \ni \bar{v} \rightarrow L(\bar{v}) = \overline{\mathcal{L}_{\text{ver}}(\bar{v})} \in T^\ast(W).\]

For a fixed $w \in W$ we have the mapping
\[(7.2) \quad \text{tr} G^n_w(W) \ni \bar{v}_w \rightarrow L(\bar{v}_w) \in T^\ast_w(W).\]

The derivative of (7.2) at the point $\bar{v}_w$ is a linear mapping
\[(7.3) \quad \pi_{\text{ver} T^\ast_w}(\text{tr} G^n(W)) \ni X \rightarrow B_{\bar{v}_w}(X, \cdot) \in T^\ast_w(W)\]
(cf. (5.10)).
It follows from the non-degeneracy of $B_{\kappa,\omega}$ that (7.3) is an injection and thus its rank is equal to $n(r-n)$. It is easy to see that the rank of $L$ is equal to $r+n(r-n)$. It follows from the rank theorem (cf. [8]) that there exists an open neighbourhood $\mathcal{V}$ of $\tilde{v}$ in $\text{tr}G^n(W)$ that $L$ maps $\mathcal{V}$ onto an $r+n(r-n)$-dimensional submanifold $\mathcal{P}$ of $\wedge^nT^*(W)$ and $\tau(\mathcal{V})$ is an open set in $W(\tau: \wedge^nT^*(W)\to W$ is the canonical projection). $L$ is a diffeomorphism of $\mathcal{V}$ onto $\mathcal{P}$. Let $\mathcal{V}$ be the maximal open set in $\text{tr}G^n(W)$ such that $L$ is a diffeomorphism $\mathcal{V}$ onto its image $\mathcal{P} = L(\mathcal{V})$. We call $\mathcal{V}$ the configuration space of a physical system and $\mathcal{P}$ the $n$-phase space of that system. $\mathcal{P}$ is a bundle over an open set $\tau(\mathcal{P}) \subset W$.

Let us consider the following diagram:

\[
\begin{array}{ccc}
\wedge^nT^*(\wedge^nT^*(W)) & \xleftarrow{r^*} & \wedge^nT^*(W) \\
\downarrow & & \downarrow \\
\wedge^nT^*(W) & \xrightarrow{\tau} & W
\end{array}
\]

(7.4)

This diagram defines the canonical $n$-form on the manifold $\wedge^nT^*(W)$.

If $x \in \wedge^nT^*(W)$, then

\[
\omega(x) = r^*(x).
\]

(7.5)

**Definition.** The form $\omega$ which is defined by (7.5) is called the canonical $n$-form on $\wedge^nT^*(W)$ and $\gamma = d\omega$ is called the canonical $(n+1)$-form on $\wedge^nT^*(W)$.

In the sequel we shall denote the pull-backs of $\omega, \gamma$ onto $\mathcal{P}$ by the same symbols.

**Definition.** An $n$-dimensional embedded submanifold $S$ of $\mathcal{P}$ which is a section of $\pi \circ \tau$ is called $\gamma$-singular if for every $\pi \circ \tau$-vertical vector field $X$ which is defined on $S$ we have

\[
(X \mid_\gamma)|S = 0.
\]

(7.6)

**Remark.** In this definition we can consider an arbitrary vector field $X$ defined on $S$. In fact, $X$ can be decomposed into a sum $X = X_1 + X_2$, where $X_1$ is tangent to $S$ and $X_2$ is $\pi \circ \tau$-vertical. But $(X_1 \mid_\gamma)|S = 0$.

**Proposition 6.** If the $n$-form $\psi$ on $\text{tr}G_n(W)$ is defined by (4.15) and $\omega$ is the canonical $n$-form on $\mathcal{P}$, then $L^*\omega = \psi$. 

Proof. We consider the diagram constructed from diagrams (4.14) and (7.4):

\[
\begin{array}{ccc}
\wedge T^* (\tau^* (W)) & \xleftarrow{\pi^*} & \wedge T^* (W) \\
\downarrow & & \downarrow \\
\tau^* (\wedge T^* (W)) & \xrightarrow{\pi^*} & \wedge T^* (\wedge T^* (W))
\end{array}
\]

(7.7)

We have

\[
L^* \left( \omega (L(\bar{\psi})) \right) = L^* \tau^* \left( L(\bar{\psi}) \right) = (\tau \circ L)^* L(\bar{\psi}) = \pi^* (L(\bar{\psi})) = \psi (\bar{\psi}).
\]

Now we can formulate the main result of this section:

**Theorem 4.** An embedded submanifold \( C \) of \( W \) which is a section of \( \pi \) is an extremal section of the variational problem \( (W, \mathcal{L}, M) \) if and only if the image \( L(\bar{C}) \) is a \( \gamma \)-singular submanifold of \( \mathcal{P} \).

**Proof.** It follows from (5.27) and (5.28) that, for every \( \pi \circ \pi_s \)-vertical vector field \( Y \) on \( \bar{C} \), \((Y, d\bar{\psi})|_{\bar{C}} = 0 \). But \( L_\ast Y \) is a \( \pi \circ \tau \)-vertical vector field defined on \( L(\bar{C}) \) and \((L^{-1})^* \psi = \omega \).

Let us notice that in Hamiltonian formulations the action integral (2.1) takes the form

\[
I_f = \int_{L(\bar{C}')} \omega.
\]

(7.8)

### 8. Examples of Classical Field Theories

1. **Classical mechanics.** We consider a \( k \)-dimensional Riemannian manifold \( M \) with a metric tensor \((g_{ij})\). Let \( W = M \times \mathbb{R}, \mathcal{L} : K^1(W) \rightarrow \mathbb{R} \) be a Lagrangian function. If \((x^i, t)\) are local coordinates on \( W \) and \( v \in \operatorname{tr} K^1(W) \), then

\[
v = \beta^i \frac{\partial}{\partial x^i} + \alpha \frac{\partial}{\partial t}, \quad \alpha \neq 0.
\]

(8.1.1)

We put

\[
\mathcal{L}(v) = \frac{m}{2\alpha} g_{ij}(x^k) \beta^i \beta^j - \alpha V(x^k, t),
\]

(8.1.2)

where \( V \in C^\infty (W) \) is a potential function.

In local coordinates

\[
\mathcal{L}_{\text{ver}}(v) = m g_{ij}(x^k) \frac{\beta^i}{\alpha} dx^j - \left( \frac{m}{2} g_{ij}(x^k) \frac{\beta^i \beta^j}{\alpha^2} + V(x^k, t) \right) dt.
\]

(8.1.3)
Let $\gamma^i = \beta^i/\alpha, 1 \leq i \leq k$, be local coordinates in a fibre of $\pi_a: \text{tr}G^1(W) \to W$. We have

$$(8.1.4) \quad \psi(\bar{v}) = mg_q(x^k)\gamma^i \, dw^i - E(t, x^k, \gamma^k) \, dt,$$

where

$$(8.1.5) \quad E(t, x^k, \gamma^k) = \frac{m}{2} g_q(x^k)\gamma^i \gamma^j + V(x^k, t)$$

is the energy. Equation $(5.27): (X \cdot d\psi)|C = 0$ for every $\pi \circ \pi_3$-vertical vector field $X$ on $C$:

$$(8.1.6) \quad X = B^k \cdot \frac{\partial}{\partial x^k} + C^k \cdot \frac{\partial}{\partial \gamma^k},$$

where

$$(8.1.7) \quad C = \{ (t, \omega^i(t), \gamma^i(t)); t \in \mathbb{R} \},$$

gives

$$(8.1.8) \quad \frac{d\omega^i}{dt} = \gamma^i,$$

$$\frac{d^2 \omega^o}{dt^2} + \Gamma^o_{ij}(x^k) \frac{d\omega^i}{dt} \cdot \frac{d\omega^j}{dt} = - \frac{1}{m} g^{oa}(x^k) \frac{\partial V(x^k, t)}{\partial \omega^a},$$

where $\Gamma^o_{ij}$ are the coefficients of the Riemannian connection.

If $V = 0$ we obtain equations of geodesic lines in $M$ cf. [8]. The condition of non-degeneracy of $\check{B}$ is here fulfilled because

$$\det \left[ \frac{\partial L(t, x^k, \gamma^k)}{\partial \gamma^i \partial \gamma^j} \right] = \det [mg_q(x^k)] \neq 0 \quad (see \ (5.13)).$$

The phase space $\mathcal{P} \subset T^*(W)$ is

$$(8.1.9) \quad \mathcal{P} = \{ \nu \in T^*(W): \nu(t, x^k) = p_j \, d\omega^j - H(t, x^k, p_k) \, dt \},$$

where

$$(8.1.10) \quad p_j = mg_h(x^k)\gamma^j, \quad H(t, x^k, p_k) = \frac{1}{2m} g^{ij}(x^k)p_ip_j + V(t, x^k).$$

The canonical 1-form on $\mathcal{P}$ is $\omega(t, x^k, p_k) = p_j \, d\omega^j - H(t, x^k, p_k) \, dt$ and the canonical 2-form is

$$(8.1.11) \quad \gamma = d\omega = dp_j \wedge d\omega^j - dH(t, x^k, p_k) \wedge dt,$$

where $H$ is given by (8.1.10).

2. Relativistic mechanics. Let $M$ be a pseudo-Riemannian manifold with a metric tensor $(g_{\mu \nu})$ with a signature $(+,-,-,-)$. $M$ is not a bundle, but we can develop constructions given in section 5 because $n = 1$.  

Let $K^1_+(M) \subset K^1(M)$ be an open cone consisting of such vectors $v$ that $(v|v) > 0$. In local coordinates $(\sigma^\mu)$ on $M$,

\begin{equation}
\sigma = \sigma^\mu \frac{\partial}{\partial \sigma^\mu}, \quad \sigma_{\mu}(\sigma^\rho) \sigma^\mu \sigma^\nu > 0.
\end{equation}

Let $\mathcal{L}: K^1_+(M) \to \mathbb{R}$ be a Lagrangian function,

\begin{equation}
\mathcal{L}(\sigma) = m \sqrt{(v|v)} + \epsilon \langle v, A \rangle,
\end{equation}

where

\begin{equation}
A = A_{\mu}(\sigma^\lambda) d\sigma^\mu
\end{equation}
is a given covector field on $M$. In local coordinates $(\sigma^\mu, \alpha^\rho)$ we have

\begin{equation}
\mathcal{L}(\sigma) = m \sqrt{\sigma_{\mu}(\sigma^\rho) \sigma^\mu \sigma^\nu} + \epsilon \sigma^\mu A_{\mu}(\sigma^\rho),
\end{equation}

\begin{equation}
\mathcal{L}_{\text{ren}}(\sigma) = \left(\sigma_{\mu}(\sigma^\rho) \sigma^\mu \sigma^\nu\right)^{-1/2} m \cdot \sigma_{\mu}(\sigma^\rho) \sigma^\mu d\sigma^\nu + \epsilon \cdot A_{\mu}(\sigma^\rho) d\sigma^\mu.
\end{equation}

Let $G^1_+(M) = \pi_1(K^1_+(M))$, where $\pi_1: K^1(M) \to G^1(M)$.

In $G^1_+(M)$ we have local coordinates

\begin{equation}
(\sigma^\mu, \gamma^\rho); \quad \gamma^\rho = \left(g_{\rho\sigma}(\sigma^\lambda) \alpha^\lambda \alpha^\nu\right)^{-1/2} \alpha^\rho,
\end{equation}

where $g_{\rho\sigma}(\sigma^\lambda) \gamma^\rho \gamma^\sigma = 1$,

\begin{equation}
\psi(\sigma^\mu, \gamma^\rho) = m \cdot g_{\rho\sigma}(\sigma^\lambda) \gamma^\rho d\sigma^\nu + \epsilon A_{\rho}(\sigma^\lambda) d\sigma^\nu,
\end{equation}

and

\begin{equation}
\tilde{\psi}(\sigma^\mu, \gamma^\rho) = m \cdot g_{\rho\sigma}(\sigma^\lambda) d\sigma^\mu \wedge d\sigma^\nu + m \frac{\partial}{\partial \sigma^\mu} \left(g_{\rho\sigma}(\sigma^\lambda) \gamma^\rho d\sigma^\nu + \gamma^\rho d\omega^\nu A d\sigma^\nu + \epsilon \frac{\partial A_{\rho}(\sigma^\lambda)}{\partial \sigma^\mu} d\sigma^\nu \wedge d\sigma^\nu\right),
\end{equation}

where

\begin{equation}
\frac{\partial}{\partial \sigma^\mu} \left(g_{\rho\sigma}(\sigma^\lambda) \gamma^\rho d\sigma^\nu + 2 g_{\rho\sigma}(\sigma^\lambda) \gamma^\rho d\gamma^\nu\right) = 0.
\end{equation}

If $X$ is tangent to $G^1_+(M)$, we have

\begin{equation}
X = P^\rho \frac{\partial}{\partial \gamma^\rho} + Q^\nu \frac{\partial}{\partial \sigma^\nu},
\end{equation}

where

\begin{equation}
\frac{\partial}{\partial \sigma^\mu} \left(g_{\rho\sigma}(\sigma^\lambda) Q^\nu \gamma^\rho \gamma^\nu + 2 g_{\rho\sigma}(\sigma^\lambda) P^\nu \gamma^\rho\right) = 0.
\end{equation}

Let $C$ be a one-dimensional submanifold of $G^1_+(W)$,

\begin{equation}
C = \{(\gamma^\rho(\tau), \gamma^\nu(\tau)) : g_{\rho\lambda} \gamma^\rho \gamma^\nu = 1, \tau \in \mathbb{R}\}.
\end{equation}
If we use (8.2.11) and (8.2.12), then equation \((X_\perp d\psi)|_{C} = 0\) will give us

\[
\frac{dx}{d\tau} = \sigma(\tau)\gamma^\mu,
\]

(8.2.13)

\[
c(\tau) \frac{d}{d\tau} \gamma^\mu + \Gamma^\mu_{\alpha\beta}(x^i) \frac{dx^\alpha}{d\tau} \cdot \frac{dx^\beta}{d\tau} = \frac{e}{m} g^{\alpha\beta} f_{\alpha\beta}(x^i(\tau)) \cdot c(\tau) \frac{dx_\mu}{d\tau},
\]

where \(\tau \rightarrow c(\tau)\) is a non-vanishing function and

\[
f_{\alpha\beta}(x^i) = \partial_\alpha A_\beta(x^i) - \partial_\beta A_\alpha(x^i).
\]

Let us introduce a new parametrization of \(C\):

\[
s = \int \sigma(\tau) d\tau, \quad \frac{d\tau}{ds} = \frac{1}{c(\tau)}.
\]

We obtain

\[
\frac{ds}{d\tau} = \gamma^\mu,
\]

(8.2.15)

\[
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(x^i(s)) \frac{dx^\alpha}{ds} \cdot \frac{dx^\beta}{ds} = \frac{e}{m} f^\mu_{\alpha\beta}(x^i(s)) \frac{dx_\alpha}{ds}.
\]

We put

\[
p_\mu = mg_{\mu\nu}(x^i) \gamma^\nu, \quad \text{where } g^{\mu\nu}(x)p_\mu p_\nu = m^2.
\]

The phase space is

\[
\mathcal{P} = \{ \nu \in T^* (M) : \nu = (p_\mu + e \cdot A_\mu(x^i)) dx^\mu, \quad g^{\mu\nu}(x)p_\mu p_\nu = m^2 \}.
\]

The canonical 1-form on \(\mathcal{P}\) is given by

\[
\omega(x^i, p_\mu) = (p_\mu + e \cdot A_\mu(x^i)) dx^\mu.
\]

3. **Theory of a scalar field.** Let \(M\) be a 4-dimensional pseudo-Riemannian manifold with the metric tensor \((g_{\mu\nu})\) which has the signature \((+, -, -)\). Let \(g = \det[g_{\mu\nu}]\). In \(W = M \times \mathbb{R}\) we have local coordinates \((x^\mu, \phi)\).

Let \(v \in \text{tr} X^4(W)\):

\[
v = v_0 \wedge v_1 \wedge v_2 \wedge v_3,
\]

(8.3.1)

\[
v_\mu = \omega_\mu + b_\mu, \quad \text{where } \omega_\mu = a_\mu^\nu \frac{\partial}{\partial x^\nu}, \quad b_\mu = \beta_\mu \frac{\partial}{\partial \phi}, \quad \mu = 0, 1, 2, 3.
\]
From the transversality of \( v \) we have \( \det[a^*_\mu] \neq 0 \). Let \( \det[a^*_\mu] > 0 \).

Let

\[
(8.3.3) \quad v^*_4 = b^*_4 = \frac{\partial}{\partial \varphi}.
\]

The dual basis consists of vectors

\[
(8.3.4) \quad v^{*\mu} = w^{*\mu}, \quad v^{*4} = b^{*4} - \beta, w^{*r}, \quad w^{*\mu} = a^*_\mu d\varphi, \quad \mu = 0, 1, 2, 3
\]

and \( b^{*4} = d\varphi \).

The Lagrangian function is given by

\[
(8.3.5) \quad \mathcal{L}(v) = \sqrt{-A} \left( \frac{1}{2} (w^{*\mu} | w^{*\nu}) \beta_\mu \beta_\nu - G(\varphi) \right),
\]

where

\[
(8.3.6) \quad A = (w_0 \wedge \ldots \wedge w_3 | w_0 \wedge \ldots \wedge w_3) = \det\{w_\mu | w_\nu\} = g \cdot (\det[a^*_\mu])^2,
\]

and \( G \) is an arbitrary smooth function of one variable;

\[
(8.3.7) \quad \mathcal{L}_{\text{var}}(v) = \sqrt{-A} \left( \frac{1}{2} (w^{*\mu} | w^{*\nu}) \beta_\mu \beta_\nu + G(\varphi) \right) w^{*0} \wedge \ldots \wedge w^{*3} + \]

\[
\sqrt{-A} \sum_{\mu=0}^{3} (w^{*\mu} | w^{*\nu}) \beta_\mu w^{*0} \wedge \ldots \wedge b^{*4} \wedge \ldots \wedge w^{*3}.
\]

If \( \gamma^*_\mu = a^*_\mu \beta_\nu, \mu = 0, 1, 2, 3 \), are local coordinates in a fibre of \( \text{tr} G^4(W) \to W \), then

\[
(8.3.5') \quad \mathcal{L}(v) = \sqrt{-g \det[a^*_\mu]} \left( \frac{1}{2} g^{\mu \nu}(x^\tau) \gamma_\mu \gamma_\nu - G(\varphi) \right),
\]

\[
(8.3.7') \quad \psi(v) = - \left( \frac{1}{2} g^{\mu \nu}(x^\tau) \gamma_\mu \gamma_\nu + G(\varphi) \right) \sqrt{-g} d\varphi \wedge \ldots \wedge d\varphi + \sum_{\nu=0}^{3} g^{\mu \nu}(x^\tau) \gamma_\mu \sqrt{-g} d\varphi \wedge \ldots \wedge d\varphi.
\]

Using (8.3.7) and (8.3.8), we obtain

\[
(8.3.7') \quad \psi(v) = - \left( \frac{1}{2} g^{\mu \nu}(x^\tau) \gamma_\mu \gamma_\nu + G(\varphi) \right) \sqrt{-g} d\varphi \wedge \ldots \wedge d\varphi + \sum_{\nu=0}^{3} g^{\mu \nu}(x^\tau) \gamma_\mu \sqrt{-g} d\varphi \wedge \ldots \wedge d\varphi.
\]

The non-degeneracy condition is here fulfilled because

\[
\det \left[ \frac{\partial^2 \mathcal{L}}{\partial \gamma_\mu \partial \gamma_\nu} \right] = \det \left[ \sqrt{-g} g_{\mu \nu} \right] \neq 0.
\]
If
\[ X = A^\mu \frac{\partial}{\partial x^\mu} + B \frac{\partial}{\partial \varphi} + C_\mu \frac{\partial}{\partial \gamma^\mu}, \]

\[ C = \{ (w^\mu, \varphi(w^\mu), \gamma^\nu(w^\mu)) : \text{tr} G^4(W) : (w^\mu) \in \mathbb{M} \}, \]

then the equation \((X \wedge d\varphi) | C = 0\) gives

\[ \frac{\partial \varphi}{\partial x^\mu} = \gamma^\mu, \]

(8.3.9)

\[ (-g)^{-1/2} \frac{\partial}{\partial x^\mu} \left( g^{\nu \rho}(x^\rho) \cdot \sqrt{-g} \frac{\partial \varphi}{\partial x^\nu} \right) + G'(\varphi) = 0. \]

Using the Laplace–Beltrami operator (cf. [3]), we can write the second equation in (8.3.9) in the form

\[ \Box \varphi + G'(\varphi) = 0. \]

Let \( \eta^\rho = g^{\mu \nu}(x^\rho) \gamma^\nu. \) The phase space is

(8.3.10) \( \mathcal{P} = \{ \nu \in \wedge^4 T^* (W) : \nu(w^\mu, \varphi) \} = \sum_{\nu=0}^3 \eta^\nu \sqrt{-g \, dx^0 \wedge \ldots \wedge dx^3 -}

\[ - H(x^0, \varphi, \eta^\nu) \sqrt{-g \, dx^0 \wedge \ldots \wedge dx^3}, \]

where

\[ (8.3.11) \]

\[ H(x^\mu, \varphi, \eta^\nu) = \frac{1}{2} (\eta \mid \eta) + G(\varphi). \]

The canonical 4-form on \( \mathcal{P} \) is equal to

(8.3.12) \[ \omega(x^\mu, \varphi, \eta^\nu) = \sum_{\nu=0}^3 \eta^\nu \sqrt{-g \, dx^0 \wedge \ldots \wedge dx^3 -}

\[ - H(x^\nu, \varphi, \eta^\nu) \sqrt{-g \cdot dx^0 \wedge \ldots \wedge dx^3}. \]

4. **Non-linear electrodynamics.** Let \( \mathbb{M} \) be as in section 3. Let \( W = T^* (M) \) with local coordinates \((w^\mu, A_\mu)\) and \( \pi : T^* (M) \rightarrow \mathbb{M} \) be the projection. For \( \nu_{\mu} \in T(W), \mu = 0, 1, 2, 3, \)

(8.4.1) \[ \nu_{\mu} = a_{\mu} \frac{\partial}{\partial x^\mu} + b_{\mu} \frac{\partial}{\partial A_\mu}, \quad \pi_{\mu} \nu_{\mu} = a_{\mu} \frac{\partial}{\partial x^\mu}, \]

and

(8.4.2) \[ \text{ver} \nu_{\mu} = (b_{\mu} - \Gamma^\nu_{\alpha \mu} a_{\nu} A_\mu) \frac{\partial}{\partial A_\mu}, \]

where \( \text{ver} \nu_{\mu} \) is the vertical component of \( \nu_{\mu} \) which is determined by the linear connection corresponding to the Riemannian structure of \( \mathbb{M} \). By \( \sim \) we denote the natural injection \( \pi \cdot \text{ver} T(T^* (M)) \) in \( T^* (M). \)
We have

\[
(8.4.3) \quad \overline{\text{ver}} v_{\mu} = (b_{\mu} - \Gamma_{\lambda}^\nu a_{\mu}^\lambda A_{\rho}) \, dx^\nu.
\]

Let

\[
(8.4.4) \quad v \in \mathcal{K}^4(W), \quad v = v_0 \wedge v_1 \wedge v_2 \wedge v_3 \quad \text{and} \quad \det [a_{\mu}^\nu] \neq 0.
\]

Let \((u_{\mu})_{\mu=0}^3\) be \(\pi\)-vertical linearly independent vectors tangent to \(T^*(M)\); e.g.,

\[
(8.4.5) \quad u_{\mu} = \frac{\partial}{\partial A_{\mu}}.
\]

Let \(w_{\mu} = \pi_* v_{\mu}\) and \((w^{*\mu})_{\mu=0}^3\) be the dual basis

\[
(8.4.6) \quad w^{*\mu} = a_{\mu}^\nu \, dw^\nu.
\]

Let

\[
(8.4.7) \quad u_{\tau}^\nu = dA_{\tau} - b_{\mu} a_{\mu}^\nu \, dw^\nu, \quad \tau = 0, 1, 2, 3.
\]

Covectors \(v^{*\mu} = \pi^* w^{*\mu}\) and \(u_{\tau}^*\) form a basis of \(T^*\{W\}\) at the given point.

We define a 2-covector \(f\) at the point \(x \in M\), where \(x = \pi(\pi_* v)\):

\[
(8.4.8) \quad f = w^{*\mu} \wedge \overline{\text{ver}} v_{\mu}.
\]

In local coordinates:

\[
(8.4.9) \quad f = \sum_{\mu<\nu} f_{\mu\nu} \, dw^\mu \wedge dw^\nu = \sum_{\mu<\nu} (a_{\mu}^\nu b_{\nu} - a_{\mu}^\nu b_{\mu}^\nu) \, dw^\mu \wedge dw^\nu = \sum_{\mu<\nu} (a_{\mu}^\nu \beta_{\nu} - a_{\mu}^\nu \beta_{\mu}) \, dw^\mu \wedge dw^\nu,
\]

where

\[
(8.4.10) \quad \beta_{\mu\nu} = b_{\mu\nu} - \Gamma_{\mu\nu}^\lambda a_{\lambda}^\rho A_{\rho}.
\]

By means of the Hodge operator \(*\) (cf. [3]) we define the dual tensor \(f^*\). In local coordinates:

\[
(8.4.11) \quad f^*_{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon_{\mu\nu\rho\sigma} f^\rho\sigma.
\]

For a physical reason we shall assume that a Lagrangian function is of the form

\[
(8.4.12) \quad \mathcal{L}(v) = \sqrt{-B} \left( \mathcal{L}_0((f|f), (f^*|f)) + \mathcal{L}_x(w^\mu, A_\mu) \right),
\]

where

\[
(8.4.13) \quad B = (w_0 \wedge \ldots \wedge w_3 | w_0 \wedge \ldots \wedge w_3) = \det (w_\mu | w_\nu) = (\det [a_{\mu}^\nu])^2 \cdot g.
\]
We have
\begin{align}
(8.4.14) \quad (f | w^\mu \wedge \tilde{w}^\nu) &= \frac{-1}{2} a_{\mu}^{\alpha} f_{\nu \alpha} (g^{\alpha \beta} g^{\beta \gamma} - g^{\gamma \epsilon} g^{\epsilon \nu}), \\
(8.4.15) \quad (\tilde{f} | w^\mu \wedge \tilde{w}^\nu) &= \frac{1}{2} \sqrt{-g} A_{\lambda (\rho} f^{\rho} a_{\sigma)} (g^{\sigma \tau} g^{\tau \lambda} - g^{\rho \sigma} g^{\sigma \rho}), \\
(8.4.16) \quad \nu^0 \wedge \ldots \wedge \nu^3 &= (\det [a_{\mu}^{\nu}])^{-1} d\nu^0 \wedge \ldots \wedge d\nu^3, \\
(8.4.17) \quad \nu^0 \wedge \ldots \wedge \nu^3 &= (\det [a_{\mu}^{\nu}])^{-1} \sum_{\lambda=0}^{3} a_{\mu}^{\lambda} d\nu^\lambda \wedge \ldots \wedge d\nu^3 - \\
&\quad - (\det [a_{\mu}^{\nu}])^{-1} b_{\mu}^{\nu} d\nu^0 \wedge \ldots \wedge d\nu^3.
\end{align}

Using (8.4.14)–(8.4.17) and lemma 6, we obtain
\begin{align}
(8.4.18) \quad \mathcal{L}_{\nu \mu} (v) &= (\mathcal{L}_0 (f | f), (f | \tilde{f})) + \mathcal{L}_f (\omega^\mu, A_{\mu}) \sqrt{-g} d\nu^0 \wedge \ldots \wedge d\nu^3 - \\
&\quad - (D_1 \mathcal{L}_0 (f | f), (f | \tilde{f})) f_{\nu \mu} + 2D_1 \mathcal{L}_0 ((f | f), (f | \tilde{f})) f_{\nu \mu} f_{\mu \nu} \times \\
&\quad \times \sqrt{-g} d\nu^0 \wedge \ldots \wedge d\nu^3 + \\
&\quad + 2D_2 \mathcal{L}_0 ((f | f), (f | \tilde{f})) \tilde{f}_{\nu \mu} \sqrt{-g} d\nu^0 \wedge \ldots \wedge d\nu^3 - \\
&\quad + 2D_3 \mathcal{L}_0 ((f | f), (f | \tilde{f})) \tilde{f}_{\nu \mu} \sqrt{-g} d\nu^0 \wedge \ldots \wedge d\nu^3.
\end{align}

Remark. Symbols $D_1, D_2$ denote partial derivatives of the function $\mathcal{L}_0$, which is an arbitrary function of 2 variables.

In local coordinates:
\begin{align}
(8.4.19) \quad (f | f) &= \frac{1}{2} f_{\mu \nu} f^{\mu \nu}, \quad (f | \tilde{f}) = \frac{1}{2} f_{\mu \nu} \tilde{f}^{\mu \nu}.
\end{align}

In what follows symbol $\partial / \partial f_{\mu \nu}$ will denote a differentiation with respect to independent components of the antisymmetric tensor $f_{\mu \nu}$.

We have from (8.4.19)
\begin{align}
(8.4.20) \quad \frac{\partial}{\partial f_{\mu \nu}} (f | f) &= 2 \cdot f_{\mu \nu}, \quad \frac{\partial}{\partial f_{\mu \nu}} (f | \tilde{f}) = 2 \tilde{f}_{\mu \nu}.
\end{align}

From (8.4.18) and (8.4.20) we obtain
\begin{align}
(8.4.21) \quad \psi (\omega^\mu, A_{\lambda}, f_{\mu \nu}) &= \frac{\partial \mathcal{L}}{\partial f_{\mu \nu}} \sqrt{-g} d\nu^0 \wedge \ldots \wedge d\nu^3 + \\
&\quad + \left(- \frac{1}{2} \frac{\partial \mathcal{L}_0}{\partial f_{\mu \nu}} f_{\mu \nu} + \mathcal{L}_0 + \mathcal{L}_1 \right) \sqrt{-g} d\nu^0 \wedge \ldots \wedge d\nu^3.
\end{align}

If $\gamma_{\mu \nu} = b_{\nu} a_{\mu}^{-1}$, then $f_{\mu \nu} = \gamma_{\mu \nu} - \gamma_{\nu \nu}$ (see (8.4.9)).
If
\[ C = \{(ω^1, A_μ(ω^1), γ_{μν}(ω^1)) : (ω^1) ∈ M\}, \]
\[ X = Q_ν \frac{∂}{∂A_ν} + P_μ \frac{∂}{∂γ_{μν}}, \]
then the equation \((X \perp dψ)|C = 0\) gives
\[ (8.4.22) \quad γ_{μν} = \partial_μ A_ν, \quad V_μ \frac{∂L_0}{∂f_μ} = \frac{∂L_I}{∂A_μ}, \]
where \(V_μ\) denotes the covariant derivative corresponding to the metric \((g_μν)\). We have also
\[ (8.4.23) \quad f_μ = \partial_μ A_ν - \partial_ν A_μ. \]
Let \(h^{μν} = -4π \partial_μ L_0/∂f_μ\); then we have
\[ (8.4.24) \quad V_μ h^{μν} = 4π \frac{∂L_I}{∂A_μ}. \]

Equations (8.4.23) and (8.4.24) form a complete set of field equations. Let us notice that \(γ_{μν}\) cannot be obtained from them (only \(f_μ\)). In Maxwell electrodynamics
\[ L_0 = -\frac{1}{16π} f_μ j^μ, \quad L_I = j^ν(ω^1) A_ν, \]
and we have
\[ h^{μν} = f^{μν}, \quad V_μ j^μ = 4π h^{ν}(ω^1). \]

The non-degeneracy condition is for non-linear electrodynamics equivalent to the condition
\[ (8.4.25) \quad \det \left[ \frac{∂^2 L_0}{∂f_μ ∂f_κ A_λ A_κ} \right]_{μ, λ, κ < 0} \neq 0. \]

If condition (8.4.25) is fulfilled, we can determine from (8.4.24)
\[ (8.4.26) \quad f_μ = f_μ (h^{μκ}). \]

The phase space is
\[ (8.4.27) \quad P = \{ν ∈ T^∗(W) : ν(ω^1, A_1) = -\frac{1}{4π} h^{μν} \sqrt{-g} \, dw^μ ∧ \ldots ∧ dA_κ ∧ \ldots ∧ dw^κ + H(ω^1, A_1, h^{κ}) \sqrt{-g} \, dw^μ ∧ \ldots ∧ dw^κ, \]
where
\[ (1) \quad H(ω^1, A_1, h^{κ}) = \frac{1}{8π} h^{μκ} f_μ (h^{κ}) + L_0 (f_μ (h^{κ})) + L_I (ω^1, A_1), \]
\[ (2) \quad h^{μκ} = -h^{κμ}. \]
The canonical 4-form on $\mathcal{P}$ is

\begin{equation}
\omega(\omega_1, A_1, h^{12}) = -\frac{1}{4\pi} k^{\mu} \sqrt{-g} d\omega^\mu \wedge \ldots \wedge dA_1 \wedge \ldots \wedge d\omega^9 + H(\omega_1, A_1, h^{12}) \sqrt{-g} d\omega^0 \wedge \ldots \wedge d\omega^9.
\end{equation}

9. INVARIANCE OF LAGRANGIAN SYSTEMS. THE NOETHER THEOREM

Let $\pi: W \to B$ be a bundle over an $n$-dimensional manifold $B$ and let $\mathcal{L}$ be a Lagrangian function on $K^n(W)$. Let $(T, F)$ be a morphism of $W$, i.e., the following diagram is commutative:

\begin{equation}
W \xrightarrow{T} W \xrightarrow{F} W
\end{equation}

\begin{equation}
B \xrightarrow{T} B \xrightarrow{F} B
\end{equation}

We assume that $T$ is a diffeomorphism of $B$.

Diagram (9.1) induces the following diagram:

\begin{equation}
\begin{array}{c}
\text{tr} K^n(W) \xrightarrow{F_*} \text{tr} K^n(W) \\
\downarrow \pi_* \downarrow \pi_* \\
W \xrightarrow{T_*} W \\
\downarrow \pi_* \downarrow \pi_* \\
B \xrightarrow{T_*} B \\
\end{array}
\end{equation}

(9.3)

It follows from (9.2) that, for $v \in \text{tr} K^n(W)$, $F_*(v) \neq 0$ and

\begin{equation}
\pi_* F_*(v) \neq 0.
\end{equation}

DEFINITION. We say that $\mathcal{L}$ is $F$-invariant if

\begin{equation}
\mathcal{L} \circ F_* = \mathcal{L}.
\end{equation}

LEMMA 13. If $\mathcal{L}$ is $F$-invariant, then, for every section of $\pi$, $f: B \to W$, $I_f = I_{FbX^{-1}}$. 

Proof.
\[ I_f = \int_{s \in B} \mathcal{L}(f \circ \nu)(z) \omega(z) = \int_{s \in B} \mathcal{L}\left((F \circ f \circ \nu)(z)\right) \omega(z) \]
\[ = \int_{s \in B} \mathcal{L}\left((F \circ f \circ T^{-1}) \circ (T \circ \nu)(z)\right) \cdot ((T^{-1})^\ast \omega)(s) = I_{F \circ f \circ T^{-1}}, \]

where \( s = T(z). \)

Lemma 14. Let \( \mathcal{L} \) be \( F \)-invariant. For every \( u, v \in \mathcal{T}(W) \) we have

\[ \langle F \ast (u \ast \mathcal{L}(v)) \rangle = \langle u \mathcal{L}(v) \rangle, \quad v \in \mathcal{T}(W). \]

Proof. This formula follows from the linearity of the mapping \( F \ast \):
\[ \mathcal{K}(W) \rightarrow \mathcal{K}(W). \]

Let \( \mathcal{F} \ast \mathcal{T}(W) \rightarrow \mathcal{T}(W) \) be the map generated by \( F \ast \mathcal{T}(W) \rightarrow \mathcal{T}(W). \)

Proposition 7. If \( \mathcal{L} \) is \( F \)-invariant, then \( (F \ast) \mathcal{L}(\cdot) = \mathcal{L}(\cdot), \) where \( \mathcal{L}(\cdot) \)
\[ = \pi^\ast \mathcal{L}(\cdot) \] (cf. (4.15))

Proof. We shall use the following diagram:

\[ \begin{array}{c}
\text{tr} \mathcal{T}(W) \\
\downarrow^{n_1} \downarrow^{n_2} \downarrow^{n_3}
\end{array} \]

\[ \begin{array}{c}
\text{K}(\text{tr} \mathcal{T}(W)) \\
\downarrow^{n_4} \downarrow^{n_5} \downarrow^{n_6}
\end{array} \]

\[ \begin{array}{c}
\text{K}(\text{tr} \mathcal{T}(W)) \\
\downarrow^{n_7} \downarrow^{n_8} \downarrow^{n_9}
\end{array} \]

\[ \begin{array}{c}
\text{tr} \mathcal{T}(W) \\
\downarrow^{n_0} \downarrow^{n_1} \downarrow^{n_2}
\end{array} \]

\[ \begin{array}{c}
\text{tr} \mathcal{T}(W) \\
\downarrow^{n_3} \downarrow^{n_4} \downarrow^{n_5}
\end{array} \]

\[ \begin{array}{c}
\text{tr} \mathcal{T}(W) \\
\downarrow^{n_6} \downarrow^{n_7} \downarrow^{n_8}
\end{array} \]

Now we shall generalize the notion of the \( F \)-invariant Lagrangian function.

Definition. We say that a Lagrangian function \( \mathcal{L} \) is \( F \)-invariant in a generalized sense if there exists a complete \( n \)-form \( \Omega \) on \( W \) (i.e., \( \Omega = \omega \)) such that

\[ (\mathcal{L} \circ F)(v) = \mathcal{L}(v) + \langle v | \Omega \rangle, \quad v \in \mathcal{T}(W). \]

In this situation we have, for \( v \in \mathcal{T}(W), \ u \in \mathcal{T}(W), \)

\[ \langle F \ast (u \mathcal{T}(v)) \rangle = \langle u \mathcal{T}(v) \rangle + \langle u | P \ast \Omega \rangle, \]
where \( P_\varphi \) is the projector on the subspace of 1-vertical forms on \( W \) (see (4.10)).

From formulae (9.10) and (5.32) we obtain

\[
(9.11) \quad ((F_\varphi)^* \psi)(\mathbf{v}) = \psi(\mathbf{v}) + \pi_* \varphi P_\varphi \Omega, \quad \mathbf{v} \in \text{tr} G^n(W).
\]

**Definition.** We say that a Lagrangian function \( \mathcal{L} \) is invariant (in the generalized sense) with respect to a one-parameter family \( (F_\varphi)_{\varphi \in \mathfrak{g}} \) of transformations of \( W \) if there exists a one-parameter family \( (\Omega_\varphi)_{\varphi \in \mathfrak{g}} \) of complete \( n \)-forms on \( W \) such that

\[
(9.12) \quad (\mathcal{L} \circ F_\varphi)(\mathbf{v}) = \mathcal{L}(\mathbf{v}) + \langle \mathbf{v} | \Omega_\varphi \rangle_{\varphi \in \mathfrak{g}} - \delta, \delta[\cdot, \cdot] - \mathbf{v} \in \text{tr} K^n(W).
\]

We assume the differentiability of mappings:

\[
\delta[\cdot, \cdot] : W \times (\mathfrak{g}, \omega) \to F_\varphi \omega \in W,
\]

\[
\delta[\cdot, \cdot] : (\mathfrak{g}, \omega) \to \Omega_\varphi \in C^\infty(W \wedge T^n(W)).
\]

We take the compact convergence topology in \( C^\infty(W \wedge T^n(W)) \).

The main result of this section is the following

**Theorem 5 (Noether).** Let \( (F_\varphi)_{\varphi \in \mathfrak{g}} \) be a one-parameter family of transformations of \( W \) and let \( (\Omega_\varphi)_{\varphi \in \mathfrak{g}} \) be a one-parameter family of complete \( n \)-forms on \( W \). Let \( \mathcal{L} \) be a Lagrangian function which is invariant with respect to the family \( (F_\varphi) \) in the sense of (9.12).

If \( f : B \to W \) is an extremal section of a variational problem \( (W, \mathcal{L}, M) \), then there exist vectors \( X \) on \( \text{tr} G^n(W) \) and an \( (n-1) \)-form \( \eta \) on \( W \) such that

\[
(9.13) \quad d(X \wedge \psi - \pi^* \varphi \Omega) | f(B) = 0.
\]

**Proof.** We known from (9.11) that

\[
(9.14) \quad ((F_\varphi)^* \psi)(\mathbf{v}) = \psi(\mathbf{v}) + \pi_* \varphi P_\varphi \Omega.
\]

If we differentiate (9.14) with respect to \( \varphi \), we shall obtain

\[
(9.15) \quad \mathbf{X} \psi(\mathbf{v}) = \pi_* (P_\varphi \chi),
\]

where \( X = \frac{d}{ds} \bigg|_{s=0} F_\varphi \) is the vector field on \( \text{tr} G^n(W) \) which generates \( (F_\varphi) \), and \( \chi = \frac{d}{ds} \bigg|_{s=0} \Omega \) is a complete \( n \)-form on \( W \) (the space of complete forms is complete in the compact convergence topology); therefore there exists an \( n-1 \) form \( \eta \) on \( W \) such that \( \chi = d\eta \). From (9.15) we obtain

\[
(9.16) \quad X \wedge d\psi + d(X \wedge \psi) = \pi^* (P_\varphi d\eta).
\]

If \( f : B \to W \) is an extremal section, then we have from (5.27)

\[
(X \wedge d\psi) | f(B) = 0.
\]
Let $\tilde{\nu} \in \tilde{\eta}(B)$ and $\nu \in K^n_\nu(\text{tr} \, G^n(W))$. We have

$$
(9.17) \quad \langle q | \xi^a \rho \hat{d} \eta \rangle = \langle \xi^a \rho q | \rho \hat{d} \eta \rangle.
$$

But by the integrability of $\tilde{\nu}(B)$, $\nu^a \cdot q = a \cdot v, 0 \neq a \in R$.

It follows from (9.17) and lemma 12 that

$$
(9.18) \quad \langle \xi^a \rho q | \rho \hat{d} \eta \rangle = \langle \xi^a \rho q | \hat{d} \eta \rangle = \langle q | \xi^a \rho \hat{d} \eta \rangle.
$$

From (9.16), (9.18) and (5.27) we obtain

$$
\hat{d}(X \cdot \psi - \xi^a \rho \hat{d} \eta) | \tilde{\eta}(B) = 0.
$$

The $(n - 1)$-form $\alpha = X \cdot \psi - \xi^a \rho \hat{d} \eta$ is called a conserved current (cf. [10]). For every extremal $\overline{\mathcal{C}} = \tilde{\eta}(B)$ we can define the physical quantity corresponding to the current $\alpha$.

Let $c_{n-1}$ be an $(n - 1)$-dimensional submanifold of $\overline{\mathcal{C}}$. A physical quantity $Q$ is a functional which assigns to every extremal $\overline{\mathcal{C}}$ the real number

$$
(9.19) \quad Q(\overline{\mathcal{C}}) = \int_{c_{n-1}} \alpha.
$$

It was proved in [6] that for every extremal $\overline{\mathcal{C}}$ we can choose a family $\mathcal{C}$ of $(n - 1)$-dimensional submanifolds of $\overline{\mathcal{C}}$ such that for every $c^i, c^i \in \mathcal{C}, \int a = \int \alpha$. This means that $Q(\overline{\mathcal{C}})$ does not depend on the choice of $c \in \overline{\mathcal{C}}$.

We shall not develop the theory of physical quantities and we refer the reader to [4], [6], [7].

**Example.** The energy-momentum tensor for a scalar field theory. We consider a scalar field theory in a flat space-time $M$ with a diagonal metric tensor ($g_{\mu\nu}$), $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$. Let $(\omega^i)$ denote affine coordinates in $M$. According to section 8.3 the Lagrangian function on $K^4(W)$ is given by

$$
\mathcal{L}(\varphi) = \lambda \left( \frac{1}{2} g^{\mu\nu} \gamma_{\mu} \gamma_{\nu} - G(\varphi) \right) \quad \text{(cf. (8.3.5'))},
$$

where $(\omega^i, \varphi)$ are coordinates in $W = M \times R$, $\lambda = \det[a^i_{\mu}]$,

$$
\varphi(\omega^i, \varphi, \gamma_{\nu}) = - \left( \frac{1}{2} g^{\mu\nu} \gamma_{\mu} \gamma_{\nu} + G(\varphi) \right) dx^0 \wedge \ldots \wedge dx^3 +
$$

$$
+ \sum_{i=0}^3 g^{\mu\nu} \gamma_{\mu} dx^0 \wedge \ldots \wedge dx^3 \wedge \omega^i.
$$

In $W$ we have a 4-parameter family of transformations which is generated by translations in $M$:

$$
F_{\omega^i, \ldots, \omega^3}(\omega^i, \varphi) = (\omega^i + \alpha^i, \varphi),
$$

$$
(F_{\omega^i, \ldots, \omega^3})^*(\omega^i, \varphi, \gamma_{\lambda}, \lambda) = (\omega^i + \alpha^i, \varphi, \gamma_{\lambda}, \lambda).
$$
The family \( \{ \mathcal{F}, \ldots, \gamma \} \) generates four vector fields on \( \text{tr} G^4(W) \):

\[
X_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3.
\]

Let \( \mathcal{C} = \{ \mathcal{C}^{(x)}, \gamma(x) \} \) be a solution of field equations (8.3.9),

\[
\gamma = \partial \varphi, \quad \square \varphi + G'(\varphi) = 0; \quad \text{then}
\]

\[
(X_\mu \gamma) \mathcal{C} = \sum_{r=0}^{3} (-1)^{r+1} \left\{ \gamma^r \eta_\mu \gamma - \delta_\mu^r (\frac{1}{2} \eta^4 \eta_\lambda - G(\varphi)) \right\} d\omega^0 \wedge \ldots \wedge d\omega^3.
\]

The energy momentum tensor is equal to

\[
T^\mu_\nu = \eta^\mu \eta^\nu - g^\mu_\nu (\frac{1}{2} \eta^4 \eta_\lambda - G(\varphi)).
\]

The 4-energy momentum vector of the system is

\[
P^\mu = \int T^\mu_\nu(x) dS_\nu,
\]

where \( \sigma \) is any space-like surface in \( M \) and \( dS_\nu \) is its surface element (dim \( \sigma = 3 \) (cf. [6], [7]).

\[
10. \ \text{A VARIATIONAL PROBLEM WITH CONSTRAINTS}
\]

Let \( \mathcal{L}_0 \) be a positive homogeneous function on \( K^n(W) \). The equation

\[
(10.1) \quad \mathcal{L}_0(v) = 0 \quad \text{defines a subset } \mathcal{N} \text{ in } K^n(W)
\]

and a subset \( \mathcal{N} \) in \( G^n(W) \). Let

\[
(10.2) \quad v_0 \in \mathcal{N} \quad \text{and} \quad \mathcal{L}_0'(v_0) \neq 0.
\]

It follows from the rank theorem that (10.1) defines locally a submanifold \( \mathcal{N} \) in \( G^n(W) \) such that \( \pi_0(\mathcal{N}) \) is an open set in \( W \). In local coordinates \( (t^a, \sigma^k, \gamma^k) \) in \( \text{tr} G^n(W) \) (10.2) is equivalent to the condition: there exists a \( (p, q) \) such that

\[
(10.3) \quad \frac{\partial \mathcal{L}(t^a, \sigma^k, \gamma^k)}{\partial \gamma^p_\gamma} \neq 0.
\]

We shall consider only those sections \( f: (\pi \circ \pi_0)(\mathcal{N}) \to W \) for which

\[
(10.4) \quad f(\pi \circ \pi_0(\mathcal{N})) \subset \mathcal{N} \subset \text{tr} G^n(W).
\]

DEFINITION. A variational problem with constraints is a system \( (W, \mathcal{L}, M, \mathcal{L}_0, \mathcal{N}) \), where \( \mathcal{L}_0 \) and \( \mathcal{L} \) are Lagrangian functions, \( \mathcal{N} \) is defined by (10.1), and \( M \) is an \((n-1)\)-dimensional compact submanifold of \( W \) which is a section of \( \pi \) over some \((n-1)\)-dimensional compact submanifold \( B_1 \) of \( B \) contained in

\[
(10.5) \quad \pi \circ \pi_0(\mathcal{N}).
\]
DEFINITION. A section $f$ of $\pi$ which fulfils (10.4) is called an extremal section of the variational problem (10.5) if, for every one-parameter family $(f_t)$ of sections of $\pi$ over $B_1$ fulfilling the conditions

1° $f_t(\partial B_1) = M, f_0 = f,$
2° $\ddot{f}_t(B_1) \subset \mathcal{N}$,

we have

$$
\frac{d}{ds} \bigg|_{s=0} \left( \int_{B_1} \mathcal{L}(f_*, \nu_{B_1}) \omega_{B_1} \right) = 0.
$$

Now we introduce a bundle $W_1$ and consider a variational problem in $W_1$. Let $W_1 = W \oplus (\mathbb{R} \times B)$, i.e., fibre of $W_1$ over $w \in B$ is equal to $W_x \times \mathbb{R}$. Let $\varrho : W_1 \to B, \text{pr}_1 : W_1 \to W, \text{pr}_2 : W_1 \to \mathbb{R} \times B$ be natural projections. These projections generate the maps

$$
\text{pr}_*: K^n(W_1) \to K^n(W), \quad \text{pr}_0*: G^n(W_1) \to G^n(W).
$$

Using (10.7), we can extend functions $\mathcal{L}_\varrho$ and $\mathcal{L}$ onto $K^n(W_1)$. We denote these extensions by $\mathcal{L}_\varrho^\omega$ and $\mathcal{L}^\omega$. Let $\alpha$ be a function on $W_1$ defined by $\alpha(w, \lambda) = \lambda$, for $w \in W, \lambda \in \mathbb{R}$. Let

$$
\tilde{\alpha} = \varrho^* \alpha = \alpha \circ \varrho, \quad \text{where} \ \varrho: G^n(W_1) \to W_1.
$$

Let

$$
\psi_\varrho(\cdot) = \varrho^* \mathcal{L}_{\text{var}}(\cdot), \quad \psi^\omega(\cdot) = \varrho^* \mathcal{L}^\omega(\cdot).
$$

**Lemma 15.** If $C$ is an integrable submanifold of $\text{tr} \ G^n(W)$ which is a section of $\pi \circ \pi_0$ over $B_1$ and $\psi_\varrho|\mathcal{C} = 0$, then $C \subset \mathcal{N}$.

**Proof.** For an integrable submanifold $C$ the condition $\psi_\varrho|\mathcal{C} = 0$ is equivalent to the condition $\mathcal{L}_\varrho(f\nu) = 0$ (where $\mathcal{C} = \bar{f}(B_1)$). This fact follows from proposition 1.

We shall use lemma 15 for the bundle $\varrho : W_1 \to B_1$ and the Lagrangian function $\mathcal{L}_\varrho^\omega$. It is easy to see that

$$
\psi_\varrho = (\text{pr}_1)^* (\psi_\varrho), \quad \psi^\omega = (\text{pr}_2)^* (\psi).
$$

We consider the variational problem with constraints $(W_1, B_1, \mathcal{L}, M^\varrho, \mathcal{L}_\varrho^\omega, \mathcal{N})$, where $M^\varrho$ is an $(n-1)$-dimensional submanifold of $W_1$ contained in $\varrho(\mathcal{N})$ such that $\pi \circ \varrho(M^\varrho) = \partial B_1$.

**Theorem 6.** If there exists an integrable section $\tilde{g} : B_1 \to G^n(W_1)$ such that $\text{pr}_1(g(\partial B_1)) = M$ and, for every $\varrho \circ \varrho_2$-vertical vector field $Y$ on $\mathcal{C}^\varrho = \tilde{g}(B_1)$, $(Y \cdot \ddot{\tilde{g}}(\varrho_0))|\mathcal{C}^\varrho = 0$, then the section $f = \text{pr}_1 \circ g$ is an extremal section of a variational problem with constraints (10.5).

**Proof.** If $Y = \partial / \partial \lambda$, we obtain $\psi_\varrho|\mathcal{C}(B_1) = 0$. It follows from (10.10) and from lemma 15 that $\psi_\varrho|\tilde{f}(B_1) = 0$ and $\tilde{f}|\mathcal{C} \subset \mathcal{N}$.
Let \((g_*)\) be a one-parameter family of sections of \(\varrho \colon W_1 \to B_1\) such that \(\text{pr}_1 \circ g_*(\partial B_1) = M; \bar{g}_* \colon B_1 \to \mathbb{N}^e\). Let \(f_* = \text{pr}_1 \circ g_*\) and let \(\bar{f} : B_1 \to \mathbb{N}\) be the canonical lift of \(f\). From (10.10) and lemma 15 we have

\[
(10.11) \quad \int_{B_1} \bar{f}_*^* \varphi = \int_{B_2} (\text{pr}_1 \circ \bar{g}_*)^* \varphi = \int_{B_1} \bar{g}_*^* (\varrho^e - \alpha \varrho^e_0).
\]

If we differentiate (10.11) we obtain

\[
(10.12) \quad \int_{\partial B_1} (\bar{X} \cdot d(\varrho^e - \alpha \varrho^e_0)) + \int_{\partial B_1} (\bar{X} \cdot d(\varrho^e - \alpha \varrho^e_0)) = \frac{d}{d\varepsilon} \int_{B_1} \bar{f}_*^* \varphi \big|_{\varepsilon = 0},
\]

where

\[
\bar{X} = \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} \bar{g}_*.
\]

The boundary term in (10.12) vanishes because \(\bar{X}\) is \(\varrho_0\)-vertical on \(\partial B_1\) and \(\varrho^e - \alpha \varrho^e_0\) is \(\varrho_0\)-horizontal. If we use the assumption of the theorem, we shall see that

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} \int_{B_1} \bar{f}_*^* \varphi = 0.
\]

**Example.** We consider the hydrodynamics of an incompressible fluid with a constant density \(\varrho\) on an \(n\)-dimensional Riemannian manifold \(M\). We shall use here some results published in [9]. We have

\[
W = M_1 \times [t_0, t_1] \times M_2, \quad B = M_1 \times [t_0, t_1], \quad M_1 \cong M \cong M_2,
\]

\[
\text{pr}_1 : W \to M_1, \quad \text{pr}_2 : W \to [t_0, t_1], \quad \text{pr}_3 : W \to M_2.
\]

We denote by \((y^i)\) local coordinates in \(M_1\) and by \((\omega^i)\) local coordinates in \(M_2\); \(t\) is a coordinate in \([t_0, t_1]\). If \((y^i_0, t, \omega^i, (y^i_0)_{i=1}^{n}, (y^i_{n+1})_{i=1}^{n+1})\) denote local coordinates in \(\text{tr} \varrho^{n+1} (W)\) we have

\[
\bar{\varphi}(y^i, t, \omega^i, y^i_0, y^i_{n+1}) = \frac{\varrho}{2} g_{pq}(\omega) \gamma^2_{n+1} \gamma^2_{n+1} - \varrho V(t, \omega).
\]

The function \(\varphi_0\) on \(K^{n+1}(W)\) is given by the \((n+1)\)-form \(\eta\) on \(W\);

\[
\varphi_0 (\nu) = \langle \nu | \eta \rangle, \quad \nu \in K^{n+1}(W);
\]

\[
(10.13) \quad \eta = (V \det g_{ij}(\omega) d\omega_1 \wedge \ldots \wedge d\omega^n - V \det g_{ij}(y) dy_1 \wedge \ldots \wedge dy^n);
\]

\[
\bar{\varphi}_0 (y^i, t, \omega^i, y^i_0, y^i_{n+1}) = (\det [y^i_0] \sqrt{\det g_{ij}(\omega)} - \sqrt{\det g_{ij}(y)});
\]

\[
\varphi (y^i, t, \omega^i, y^i_0, y^i_{n+1}) = \gamma^2_{n+1} \varrho \det g_{ij}(\omega) \sqrt{\det g_{ij}(y)} dy_1 \wedge \ldots \wedge dy^n \wedge d\omega^i -
\]

\[
- \left(\frac{\varrho}{2} g_{pq}(\omega) \gamma^2_{n+1} \gamma^2_{n+1} + V(t, \omega)\right) \sqrt{\det g_{ij}(y)} dy_1 \wedge \ldots \wedge dy^n \wedge dt;
\]
\[(10.14) \quad \psi_\theta(y', t, \omega', \gamma'_j, \gamma'_{n+1}) = - \left( (n-1) \det [\gamma'_j] \sqrt{\det g_{ij}(\omega)} + \sqrt{\det g_{ij}(y)} \cdot dy^1 \wedge \ldots \wedge dy^n \wedge dt + \right. \]
\[+ \left. \sum_{k=1}^{n-1} \gamma'_k \det [\gamma'_j] \sqrt{\det g_{ij}(\omega)} \frac{\partial \omega'}{\partial \omega} \wedge \ldots \wedge \frac{\partial \omega'}{\partial \omega} \wedge dt. \right\]

We construct the bundle \( W_1 \). \( Y_1 = M_1 \times [t_0, t_1] \times M_2 \times \mathbb{R} \) with local coordinates \((y', t, \omega', \lambda)\). The forms \( \psi_0^\theta, \psi^\theta \) are given also by formulae (10.13) and (10.14). The equations of motion are
\[(10.15) \quad \{ Y \_ d(\psi - \lambda \psi_0) \} |_{\overline{C}} = 0, \]
where \( \overline{C} \) is an integrable submanifold of \( W_1 \) given by
\[\omega' = \omega'(y', t), \quad \lambda = \lambda(y', t), \quad \gamma'_j = \frac{\partial \omega'}{\partial y_j}, \quad i, j = 1, \ldots, n, \]
\[\gamma'_{n+1} = \frac{\partial \omega'}{\partial t}. \]

If \( Y = \partial / \partial \lambda \), we obtain \( \psi_0^\theta |_{\overline{C}} = 0 \), i.e.,
\[(10.16) \quad \det [\gamma'_j] \sqrt{\det g_{ij}(\omega)} = \sqrt{\det g_{ij}(y)}. \]

If \( Y = \lambda^k \partial / \partial \omega^k \), then we obtain
\[(10.17) \quad \frac{\partial^2 \omega^p}{\partial t^2} + \Gamma^p_{mp}(\omega) \frac{\partial \omega'}{\partial t} \frac{\partial \omega^p}{\partial t} = g^{np}(\omega) \left( \frac{\partial V(t, \omega)}{\partial x^p} + \frac{1}{\epsilon} \frac{\partial \lambda(t, y(t, \omega))}{\partial x^p} \right). \]

Therefore \( \lambda = -P \), where \( P(t, \omega) \) is a pressure at the point \( \omega \) and time \( t \). Equations (10.16) and (10.17) are the equations of motion in the Lagrange form, cf. [9].

References

Lagrangian formalism in the field theory


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