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Lagrangian formalism in the classical field theory

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Abstract. A variational problem with a fixed boundary in the geometric form is considered. Equations of extremals are formulated in the language of differential forms on the Grassmannian manifold. A discussion on the transformation properties of the classical form of the Euler–Lagrange equations is presented and a new interpretation of these equations is proposed. The Legendre transformation in the field theory is defined and several physical examples of it are given. Invariance problems are considered and the Noether theorem is proved. Problems with constraints are formulated geometrically.

It is well known that a variational principle is an appropriate tool for the formulation of physical laws. Equations of mechanics, scalar field theory, electrodynamics can be derived from "the least action principle". In mechanics this approach allows us to obtain two formulations of the theory: the Lagrangian and the canonical one. A mapping which gives an isomorphism between these formulations is called the *Legendre transformation* [1]. In the present paper we consider the Lagrangian approach to the classical field theory based on the geometrical theory of the calculus of variations. This theory was given by P. Dedecker [2] in the fifties. Recently H. Goldschmidt and S. Sternberg have given a modern exposition of the geometrical theory of the calculus of variations [5]. Their results are equivalent to ours, but are given in another formulation. The starting point of our considerations is a bundle W over an n-dimensional manifold B. In relativistic field theories B is the space-time. We construct, for a given Lagrangian function $\mathscr{L}$, an n-form ψ on the Grassmannian bundle $G^n(W)$. Equations of motions take the form $(X \lrcorner d\psi)|C = 0$, where C is an n-dimensional submanifold of $G^n(W)$ and X is any vector field defined on C, tangent to $G^n(W)$. This approach is in a 1–1 correspondence (when some conditions of regularity of $\mathscr{L}$ are fulfilled) with the canonical formulation. In the canonical formulation (cf. [4], [6]) a subspace $\mathscr{P} \subset \overset{n}{\bigwedge} T^*(W)$ and the canonical n-form ω on $\overset{n}{\bigwedge} T^*(W)$ are given. Equations of motion are $(Y \lrcorner d\omega)|S = 0$, where S is an n-dimensional submanifold of $\mathscr{P}$ and Y is any vector field defined on S, tangent to $\mathscr{P}$. A diffeomorphism

$L: G^n(W) \to \mathscr{P}$ such that $L^*\omega = \psi$ is called the *Legendre transformation*. We present several examples of Lagrangian functions and Legendre transformations.

In section 6 we study a new approach to the Euler–Lagrange equations. The analysis of geometrical properties of the classical form of the Euler–Lagrange equations in the language of differential forms is given. Especially simple are in our approach problems connected with the invariance of the Lagrangian function with respect to an m-parameter family of transformations of W. In section 9 we present an elegant proof of the Noether theorem. The results of that section are equivalent to the results of A. Trautman [10], who has given an exposition in the language of jet-bundles.

In section 10 we consider a variational principle with constraints in the "velocity space". Such problems appear for instance in the hydrodynamics of an incompressible fluid, cf. [9].

We do not investigate the canonical structure of field theories. Results concerning that problem were published in papers of J. Kijowski and K. Gawędzki [6], [4]. Completely new results on the canonical structure of the classical field theory were recently obtained by J. Kijowski and the author [7]. Paper [7] gives the natural symplectic structure in an infinite dimensional manifold of solutions of the given field equations. This paper is an essential generalization of [4], [6] and provides a simple and elegant definition of physical quantities and Poisson brackets.

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1. NOTATION

In this paper we shall use notions of modern differential geometry. All these notions can be found in [8]. Let us recall some definitions.

If V_1, V_2 are smooth manifolds and f is a smooth mapping from V_1 into V_2, then f_* denotes the tangent map to f. For any $x \in V_1$, f_* is a linear map from $T_x(V_1)$ into $T_{f(x)}(V_2)$, where $T_x(V_1)$ ($T_{f(x)}(V_2)$) denotes the space tangent to $V_1(V_2)$ at the point $x\big(f(x)\big)$. If $T_x^*(V_1)$, $T_{f(x)}^*(V_2)$ are the corresponding cotangent spaces, then f^* is a linear map from $T_{f(x)}^*(V_2)$ into $T_x^*(V_1)$. The maps f_*, f^* can be extended onto exterior products:

$$f_*: \bigwedge^n T_x(V_1) \to \bigwedge^n T_{f(x)}(V_2), \tag{1.1}$$

$$f_*: \bigwedge^n T_{f(x)}^*(V_2) \to \bigwedge^n T_x^*(V_1). \tag{1.2}$$

Let $K^n(V_i)$ denote the bundle of all simple, non-vanishing n-vectors tangent to V_i ($i = 1, 2$). There exist natural projections

$$K^n(V_i) \to G^n(V_i), \quad i = 1, 2, \tag{1.3}$$

where $G^n(V_i)$ denotes the Grassmannian bundle of all oriented n-planes tangent to V_i. There exists a map generated by (1.1)

$$f_*\colon G^n_x(V_1) \to G^n_{f(x)}(V_2). \tag{1.4}$$

For every $f\colon V_1 \to V_2$ the map (1.2) induces a linear map f^* from $C^\infty(\overset{n}{\wedge} T^*(V_2))$ into $C^\infty(\overset{n}{\wedge} T^*(V_1))$, where $C^\infty(\overset{n}{\wedge} T^*(V_i))$ are the vector spaces of sections of the bundle $\overset{n}{\wedge} T^*(V_i) \to V_i$, $i = 1, 2$ (the "pull-back" of n-forms).

If f is an injection, it generates a linear map f^* from $C^\infty(\overset{n}{\wedge} T(V_1))$ into $C^\infty(\overset{n}{\wedge} T(V_2)|f(V_1))$. For

$$\begin{aligned} v &= v_1 \wedge \ldots \wedge v_n, & v_i &\in T_x(V), & i &= 1, \ldots, n, \\ u^* &= u^{*1} \wedge \ldots \wedge u^{*n}, & u^{*j} &\in T^*_x(V), & j &= 1, \ldots, n, \end{aligned}$$

we define a bilinear form:

$$\begin{aligned} \langle v | u^* \rangle &= \langle v_1 \wedge \ldots \wedge v_n | u^{*1} \wedge \ldots \wedge u^{*n} \rangle = \det \langle v_i | u^{*j} \rangle \\ &= n! \cdot u^{*1} \wedge \ldots \wedge u^{*n}(v_1 \wedge \ldots \wedge v_n). \end{aligned} \tag{1.5}$$

We shall use the following definition of the interior product:

$$\langle v_2 \wedge \ldots \wedge v_n | v_1 \lrcorner u^* \rangle = \langle v_1 \wedge \ldots \wedge v_n \lrcorner u^* \rangle, \tag{1.6}$$

$$v_1 \lrcorner u^* \in \overset{n-1}{\wedge} T^*(V).$$

Let $C \subset V$ be an embedded submanifold of V and let $i\colon C \to V$ be the natural injection. For every n-form ω on V we shall write

$$\omega | C := i^*(\omega). \tag{1.7}$$

Let $\pi\colon W \to B$ be a bundle and $v \in T_w(W)$, $w \in W$. We call v a *π-vertical vector* if $\pi^* v = 0$. The subspace of all π-vertical vectors tangent to W at w we denote $\pi\text{-ver}\, T_w(W)$.

By $\pi\text{-hor}\, T^*_w(W)$ we denote a subspace of $T^*_w(W)$ which anihilates $\pi\text{-ver}\, T_w(W)$. Elements of $\pi\text{-hor}\, T^*_w(W)$ are called *π-horizontal covectors*.

In this paper we shall use the summation convention in formulae which contain sums with respect to upper and lower indexes.

2. THE ACTION INTEGRAL

Let W be an r-dimensional, smooth manifold (with boundary) and let B be an n-dimensional, smooth, compact, orientable and connected manifold with boundary ($n \leqslant r$). Let $K^n(W)$ be a bundle of all simple, non-zero n-vectors tangent to W. Let ω_B be an n-form giving an orientation of B (volume n-form) and let ν_B be the dual field of n-vectors on B, i.e., $\langle \nu_B | \omega_B \rangle \equiv 1$.

DEFINITION. A Lagrangian function $\mathscr{L}$ is a *positive homogeneous function* on $K^n(W)$, i.e., $\mathscr{L}: K^n(W) \to \boldsymbol{R}$, for $\lambda \in \boldsymbol{R}_+$, $v \in K^n(W)$, $\lambda \cdot \mathscr{L}(v) = \mathscr{L}(\lambda v)$.

We shall assume that $\mathscr{L}$ is at least of class C^2.

For every smooth embedding $f: B \to W$ (f is a diffeomorphism B onto $f(B)$) we can define the action integral

$$I_f = \int_B \mathscr{L}(f_* \nu_B)\, \omega_B, \tag{2.1}$$

where $f_*: \overset{n}{\wedge} T(B) \to K^n(W)$ is induced by the following diagram:

$$\begin{tikzcd} \overset{n}{\wedge} T(B) \arrow[r, "f_*"] \arrow[d] & K^n(W) \arrow[d] \\ B \arrow[r, "f"] & W \end{tikzcd} \tag{2.2}$$

LEMMA 1. *If ω_B belongs to a given orientation class of B, then I_f defined by* (2.1) *does not depend on the particular choice of ω_B.*

Proof. If $'\omega_B = \psi \cdot \omega_B$, $\psi \in C^\infty(B)$, $\psi > 0$, then from the positive homogeneity of $\mathscr{L}$ we have

$$\int_B \mathscr{L}(f_* {}'\nu_B)\, {}'\omega_B = \int_B \mathscr{L}(f_* \nu_B)\, \omega_B.$$

LEMMA 2. *Let B_1 be a manifold diffeomorphic to B and let $\lambda: B_1 \to B$ be an orientation-preserving diffeomorphism. Let $f_1: B_1 \to W$ be an embedding such that $f_1 = f \circ \lambda$. Then for any ω_{B_1}, ω_B which belong to the orientation classes corresponding to one another by λ we have*

$$\int_B \mathscr{L}(f_* \nu_B)\, \omega_B = \int_{B_1} \mathscr{L}(f_{1*} \nu_{B_1})\, \nu_{B_1}.$$

Proof. If $\omega_{B_1} = \lambda^* \omega_B$, $\nu_{B_1} = (\lambda_*)^{-1} \nu_B$ we have

$$\int_{B_1} \mathscr{L}(f_{1*} \nu_{B_1})\, \omega_{B_1} = \int_{B_1} \mathscr{L}\big((f_1 \circ \lambda^{-1})_* \nu_B\big)\, \lambda^* \omega_B = \int_B \mathscr{L}(f_* \nu_B)\, \omega_B.$$

For arbitrary ω_{B_1}, ω_B the result follows from lemma 1.

LEMMA 3. *If φ is an orientation-preserving diffeomorphism $f(B)$ onto $f(B)$, then*

$$\int_B \mathscr{L}\big((\varphi \circ f)_* \nu_B\big)\, \omega_B = \int_B \mathscr{L}(f_* \nu_B)\, \omega_B.$$

Proof. For simplicity we assume that $W = B$, $f = \mathrm{id}_B$, $\varphi \in \mathrm{Dif}(B)$. Then

$$\varphi_*\big(\nu_B(x)\big) = \psi\big(\varphi(x)\big)\cdot\nu_B\big(\varphi(x)\big), \qquad x \in B,\ 0 < \psi \in C^\infty(B),$$

$$\psi\big(\varphi(x)\big)\cdot(\varphi^{-1})^*\big(\omega_B(x)\big) = \omega_B\big(\varphi(x)\big),$$

$$\int_B \mathscr{L}(\varphi_*\nu_B)\,\omega_B = \int_{x\in B} \mathscr{L}\Big(\psi\big(\varphi(x)\big)\cdot\nu_B\big(\varphi(x)\big)\Big)\omega_B(x)$$

$$= \int_{z\in B} \psi(z)\mathscr{L}\big(\nu_B(z)\big)(\varphi^{-1})^*\Big(\omega_B\big(\varphi^{-1}(z)\big)\Big) = \int_{z\in B} \mathscr{L}\big(\nu_B(z)\big)\,\omega_B(z),$$

where $z = \varphi(x)$.

It follows from lemmas 1, 2 and 3 that the value I_f depends only on the submanifold $C = f(B)$ and does not depend on the particular embedding f. This means that the value of the action integral (2.1) does not depend on the parametrization of C.

3. INTEGRABLE SUBMANIFOLDS OF $G^n(W)$

Let $\pi_1\colon K^n(W)\to W$ be the bundle of non-zero, simple n-vectors tangent to W. For every $v \in K^n(W)$ we have the uniquely determined element $\bar{v} = \pi_2(v) \in G^n(W)$ of the Grassmannian bundle $\pi_3\colon G^n(W)\to W$ of all oriented n-planes tangent to W. For every n-dimensional embedded submanifold $f\colon B\to W$ of W there exists a uniquely determined submanifold $\bar{f}\colon B \to G^n(W)$ given by the following diagram:

(3.1)
$$\begin{tikzcd}
 & G^n(W) \arrow[dl, "\pi_3"'] & \\
W & & K^n(W) \arrow[ll, "\pi_1"'] \arrow[ul, "\pi_2"'] \\
B \arrow[u, "f"] \arrow[rr, "\nu"] & & K^n(B) \arrow[u, "f_*"']
\end{tikzcd}$$

(3.2)
$$\bar{f} := (\pi_2\circ f_*)\nu.$$

It is easy to see that $\bar{f}$ does not depend on the choice of ν. We have also

(3.3)
$$f = \pi_3\circ\bar{f}.$$

If $C = f(B)$, we shall write $\bar{C} = \bar{f}(B)$ or $C_{\bar{f}} = \bar{f}(B)$.

DEFINITION. An n-dimensional embedded submanifold $g\colon B\to G^n(W)$ is called *integrable* if $\overline{\pi_3\circ g} = g$.

DEFINITION. Let $\pi\colon W\to B$ be a bundle over an n-dimensional manifold B. An embedded submanifold $f\colon B \to W$ is called *π-transversal* if, for every $w \in f(B)$, π_* is an isomorphism $T_w\big(f(B)\big)$ onto $T_{\pi(w)}(B)$.

It follows from the implicit function theorem that for every $w \in f(B)$ there exists a neighbourhood $U \subset f(B)$ of w in $f(B)$ such that U is an

mage of some section f_1 of π over the set $\pi(U)$. But f is an embedding, and so there exists a diffeomorphism $\psi: f^{-1}(U) \to \pi(U)$ such that $f|f^{-1}(U) = f_1 \circ \psi$. This means that if we change the parametrization, then $f(B)$ will be locally a section of π.

In the sequel we shall consider only those f which are globally sections of π.

4. THE FIBRE DERIVATIVE OF A LAGRANGIAN FUNCTION

Let W be an r-dimensional manifold and let $K^n(W)$ be a bundle of simple, non-zero n-vectors tangent to W $(r \geqslant n)$. For every $w \in W$ a fibre $K^n_w(W)$ is a $1+n(r-n)$ dimensional manifold. We have the following

LEMMA 4. *The space tangent to the manifold $K^n_w(W)$ at a point $v_w \in K^n_w(W)$ is isomorphic to the subspace $A(v_w)$ of $\overset{n}{\wedge} T_w(W)$ spanned by the following n-vectors:*

$$(4.1) \qquad v_1 \wedge \ldots \wedge v_n, \qquad v_1 \wedge \ldots \wedge \underbrace{v_k}_{m} \wedge \ldots \wedge v_n \qquad (1 \leqslant m \leqslant n,\ n+1 \leqslant k \leqslant r),$$

where $v_w = v_1 \wedge \ldots \wedge v_n$ and $(v_k)_{k=n+1}^r$ are arbitrary linearly independent vectors which together with $(v_j)_{j=1}^n$ form a basis of $T_w(W)$.

Proof. Let $t \to a^j_s(t)$, $t \to \beta^k_s(t)$, $1 \leqslant s, j \leqslant n$, $n+1 \leqslant k \leqslant r$, be smooth functions on an interval $]-\delta, \delta[$ such that $a^j_s(0) = \delta^j_s$ and $\beta^k_s(0) = 0$. We have a curve in $K^n_w(W)$:

$$]-\delta, \delta[\ni t \to \Big(\sum_{j=1}^n a^j_1(t) v_j + \sum_{k=n+1}^r \beta^k_1(t) v_k\Big) \wedge \ldots \wedge \Big(\sum_{j=1}^n a^j_n(t) v_j + \sum_{k=n+1}^r \beta^k_n(t) v_k\Big).$$

If $0 < \varepsilon < \delta$ is small enough, we shall put $\gamma^k_s(t) = \sum_{p=1}^n a^{p\,-1}_s(t) \beta^k_p(t)$ and we shall have the curve

$$(4.2) \qquad]-\varepsilon, \varepsilon[\ni t \to \det[a^j_s(t)] \cdot \big(v_1 + \gamma^k_1(t) v_k\big) \wedge \ldots \wedge \big(v_n + \gamma^k_n(t) v_k\big).$$

If we differentiate (4.2) at the point $t = 0$, we shall obtain a lineay combination of n-vectors given in (4.1). It is easy to prove that locall. every curve in $K^n_w(W)$ passing through the point v_w is of the form (4.2) Taking all curves in $K^n_w(W)$ passing through v_w we obtain the whole subspace $A(v_w)$.

Remark. This construction does not depend on the representation of the n-vector v_w in the form $v_1 \wedge \ldots \wedge v_n$ and does not depend on the choice of vectors $(v_k)_{k=n+1}^r$ tangent to W at the point w.

In general there does not exist a canonically defined subspace $B(v_w)$ of $\overset{n}{\wedge} T_w(W)$ such that $\overset{n}{\wedge} T_w(W) = A(v_w) \oplus B(v_w)$. Therefore the dual

space

$$(4.3)\qquad \big(A(v_w)\big)^* \text{ is equal to } \overset{n}{\wedge} T_w^*(W)/\big(A(v_w)\big)^0,$$

where $\big(A(v_w)\big)^0$ is the subspace of $\overset{n}{\wedge} T_w^*(W)$ which anihilates $A(v_w)$.

If we want to choose a representative in the quotient space $\overset{n}{\wedge} T_w^*(W)/\big(A(v_w)\big)^0$, we shall have to define a section of the natural projection

$$(4.4)\qquad \mathrm{pr}\colon \overset{n}{\wedge} T_w^*(W) \to \overset{n}{\wedge} T_w^*(W)/\big(A(v_w)\big)^0,$$

i.e., such a map

$$(4.5)\qquad \xi\colon \overset{n}{\wedge} T_w^*(W)/\big(A(v_w)\big)^0 \to \overset{n}{\wedge} T_w^*(W)$$

that

$$\mathrm{pr}\circ\xi = \mathrm{id}.$$

It is the problem of gauge in the given field theory. It can be proved that it is possible to take for every class $[\nu] \in \overset{n}{\wedge} T_w^*(W)/\big(A(v_w)\big)^0$ such that $\nu(v_w) \neq 0$ a unique simple n-covector belonging to this class. This procedure is called the *Carathéodory gauge* (cf. [2]).

For our purpose we shall use another gauge, which is connected with the bundle structure in W. If $\pi\colon W \to B$ is a bundle ($\dim B = n$), we shall use the following

DEFINITION. An element $v \in K^n(W)$ is called *π-transversal* if $\pi_*(v) \neq 0$. By $\pi\text{-tr}K^n(W)$ (or simply $\mathrm{tr}K^n(W)$) we denote the bundle of all π-transversal, simple, non-zero n-vectors tangent to W.

In this case we choose vectors $(v_k)_{k=n+1}^r$ in such a way that they are π-vertical vectors in $T(W)$. Then there exists a subspace $B \subset \overset{n}{\wedge} T_w(W)$ (independent of v_w) such that

$$(4.6)\qquad \overset{n}{\wedge} T_w(W) = A(v_w) \oplus B.$$

B is the subspace consisting of all at least 2-vertical n-vectors and is spanned by all n-vectors of the form

$$(4.7)\qquad v_1 \wedge \ldots \wedge \underbrace{v_{k_1}}_{j_1} \wedge \ldots \wedge \underbrace{v_{k_s}}_{j_s} \wedge \ldots \wedge v_n, \quad 1 \leqslant j_1 < j_2 < \ldots < j_s \leqslant n,$$
$$n+1 \leqslant k_1 < k_2 < \ldots < k_s \leqslant r, \quad s \geqslant 2.$$

It is easy to see that B does not depend on v_w and does not depend on the particular choice of π-vertical vectors $(v_k)_{k=n+1}^r$. Therefore we have

$$(4.8)\qquad \big(A(v_w)\big)^* = B^0.$$

It follows from (4.7) that B^0 is the subspace of all at most 1-vertical n-covectors and is spanned by n-covectors of the form

$$(4.9)\qquad v^{*1} \wedge \ldots \wedge v^{*n}, \quad v^{*1} \wedge \ldots \wedge \underbrace{v^{*k}}_{j} \wedge \ldots \wedge v^{*n}, \quad 1 \leqslant j \leqslant n,\ n+1 \leqslant k \leqslant r,$$

where $(v^{*p})_{p=1}^r$ form the dual basis to the basis $(v_p)_{p=1}^r$ of $T_w(W)$. The subspace B^0 is also independent of v_w and of the choice of π-vertical vectors $(v_k)_{k=n+1}^r$. We denote it by 1-ver$\overset{n}{\bigwedge} T_w^*(W)$.

Remark. The covectors $(v^{*j})_{j=1}^n$ are π-horizontal.

LEMMA 5. *Let $\pi\colon W \to B$ be a bundle over an n-dimensional manifold B. The space cotangent to the manifold* $\operatorname{tr} K_w^n(W)$ *at the point v_w is isomorphic to the space* 1-ver$\overset{n}{\bigwedge} T_w^*(W)$.

We define, for every $v_w \in \operatorname{tr} K_w^n(W)$, a projection $P_v\colon \overset{n}{\bigwedge} T_n^*(W) \to$ 1-ver $\overset{n}{\bigwedge} T_w^*(W)$. If $v_w = v_1 \wedge \ldots \wedge v_n$, $(v_k)_{k=n+1}^r$ are π-vertical vectors and

$$v = Av^{*1} \wedge \ldots \wedge v^{*n} + \sum_{\substack{1 \leqslant j \leqslant n \\ n+1 \leqslant k \leqslant r}} B_k^j v^{*1} \wedge \ldots \wedge \underbrace{v^{*k}}_{j} \wedge \ldots \wedge v^{*n} +$$
$$+ \sum_{\substack{1 \leqslant j_1 < j_2 \leqslant n \\ n+1 \leqslant k_1 < k_2 \leqslant r}} B_{k_1 k_2}^{j_1 j_2} v^{*1} \wedge \ldots \wedge \underbrace{v^{*k_1}}_{j_1} \wedge \ldots \wedge \underbrace{v^{*k_2}}_{j_2} \wedge \ldots \wedge v^{*n} + \ldots,$$

then

$$P_v v = Av^{*1} \wedge \ldots \wedge v^{*n} + \sum_{\substack{1 \leqslant j \leqslant n \\ n+1 \leqslant k \leqslant r}} B_k^j v^{*1} \wedge \ldots \wedge \underbrace{v^{*k}}_{j} \wedge \ldots \wedge v^{*n}. \tag{4.10}$$

It follows from formula (4.10) that P_v depends only on $\bar{v}_w = \pi_2(v_w) \in G_w^n(W)$.

The operator P_v gives us the so-called 1-*vertical gauge* (cf. [6]).

Remark. If $n = 1$ or $r = n+1$, $A(v_w) = \overset{n}{\bigwedge} T_w(W)$ and we do not have problems with a gauge. These situations occur in mechanics and in the theory of scalar fields (cf. section 8).

Let us consider the map

$$\operatorname{tr} K_w^n(W) \ni v_w \to \mathscr{L}(v_w) \in \boldsymbol{R}. \tag{4.11}$$

The derivative of this map is called the *fibre derivative* of $\mathscr{L}$. We denote it by $\mathscr{L}'_{\text{ver}}$. Of course, $\mathscr{L}'_{\text{ver}}(v_w) \in$ 1-ver$\overset{n}{\bigwedge} T_w^*(W)$.

LEMMA 6.

$$\mathscr{L}'_{\text{ver}}(v_w) = A \cdot v^{*1} \wedge \ldots \wedge v^{*n} + \sum_{\substack{1 \leqslant m \leqslant n \\ n+1 \leqslant k \leqslant r}} B_k^m v^{*1} \wedge \ldots \wedge \underbrace{v^{*k}}_{m} \wedge \ldots \wedge v^{*n},$$

where $(v_p)_{p=1}^r$ and $(v_p^*)_{p=1}^r$ are such as in (4.9) *and*

$$A = \frac{d}{dt}\Big|_{t=0} \mathscr{L}\big((1+t)v_1 \wedge \ldots \wedge v_n\big) = \mathscr{L}(v_w)$$
$$= \langle v_1 \wedge \ldots \wedge v_n | \mathscr{L}'_{\text{ver}}(v_w) \rangle,$$
$$B_k^m = \frac{d}{dt}\Big|_{t=0} \mathscr{L}\big(v_1 \wedge \ldots \wedge (v_m + tv_k) \wedge \ldots \wedge v_n\big)$$
$$= \langle v_1 \wedge \ldots \wedge \underbrace{v_k}_{m} \wedge \ldots \wedge v_n | \mathscr{L}'_{\text{ver}}(v_w) \rangle.$$

Proof. This lemma follows from lemma 5, (4.8), (4.9) and from the positive homogeneity of $\mathscr{L}$.

From the positive homogeneity of $\mathscr{L}$ we have also

$$\mathscr{L}'_{\mathrm{ver}}(\lambda v_w) = \mathscr{L}'_{\mathrm{ver}}(v_w) \quad \text{for } \lambda \in \boldsymbol{R}_+. \tag{4.12}$$

Therefore we can define the mapping

$$\mathrm{tr}G^n_w(W) \ni \bar{v}_w \to \overline{\mathscr{L}'_{\mathrm{ver}}}(\bar{v}_w) \in 1\text{-ver}\bigwedge^n T^*_w(W), \tag{4.13}$$

where $\mathrm{tr}G^n(W)$ is an open set in $G^n(W)$ consisting of all π-transversal oriented n-dimensional planes tangent to W and $\bar{v}_w = \pi_2(v_w)$.

The following diagram induces a π_3-horizontal form ψ on $\mathrm{tr}G^n(W)$:

$$\begin{tikzcd}
\bigwedge^n T^*(\mathrm{tr}G^n(W)) \arrow[d] & & \bigwedge^n T^*(W) \arrow[ll, "\pi_3^*"'] \arrow[d] \arrow[dll, "\overline{\mathscr{L}'_{\mathrm{ver}}}(\cdot)"] \\
\mathrm{tr}G^n(W) \arrow[rr, "\pi_3"] & & W
\end{tikzcd} \tag{4.14}$$

$$\psi(\cdot) = \pi_3^* \circ \overline{\mathscr{L}'_{\mathrm{ver}}}(\cdot). \tag{4.15}$$

Now we shall express the vertical derivative of $\mathscr{L}$ in local coordinates. Let (t^j, w^k), $j = 1, \ldots, n$, $k = n+1, \ldots, r$, denote local coordinates in the bundle W.

If

$$v_s = a^j_s \frac{\partial}{\partial t^j} + \beta^k_s \frac{\partial}{\partial w^k}, \quad 1 \leqslant s \leqslant n, \tag{4.16}$$

$v = v_1 \wedge \ldots \wedge v_n$, $v \in \mathrm{tr}K^n(W)$, then $\det[a^j_s] \neq 0$.

Let

$$\gamma^k_j = \sum_{m=1}^{n} \overset{-1}{a}{}^m_j \beta^k_m; \tag{4.17}$$

then

$$v = \det[a] \cdot \left(\frac{\partial}{\partial t^1} + \gamma^k_1 \frac{\partial}{\partial w^k}\right) \wedge \ldots \wedge \left(\frac{\partial}{\partial t^n} + \gamma^k_n \frac{\partial}{\partial w^k}\right), \tag{4.18}$$

where summation convention is used.

(t^j, w^k, γ^k_j) form coordinates in $\mathrm{tr}G^n(W)$ and $(t^j, w^k, \gamma^k_j, \lambda)$, where

$$\lambda = \det[a], \text{ form coordinates in } \mathrm{tr}K^n(W). \tag{4.19}$$

In local coordinates (4.19),

$$\mathscr{L}(v) = \mathscr{L}(t^j, w^k, \gamma^k_j, \lambda) \tag{4.20}$$

and $\mathscr{L}$ is positive homogeneous in λ.

Let

$$v_p = \eta_p^k \frac{\partial}{\partial x^k}, \quad n+1 \leqslant p \leqslant r, \quad \det[\eta] \neq 0; \tag{4.21}$$

then

$$v^{*j} = \overset{-1}{a}{}_s^j \, dt^s, \quad 1 \leqslant j \leqslant n, \tag{4.22}$$

$$v^{*p} = \overset{-1}{\eta}{}_k^p dx^k - \overset{-1}{\eta}{}_k^p \gamma_s^k dt^s, \quad n+1 \leqslant p \leqslant r. \tag{4.23}$$

Using formulae (4.17), (4.22) and (4.23), we obtain

$$v^{*1} \wedge \ldots \wedge v^{*n} = (\det[a])^{-1} dt^1 \wedge \ldots \wedge dt^n, \tag{4.24}$$

$$v^{*1} \wedge \ldots \wedge \underbrace{v^{*p}}_{j} \wedge \ldots \wedge v^{*n} = -\overset{-1}{\eta}{}_k^p \cdot \beta_j^k (\det[a])^{-1} dt^1 \wedge \ldots \wedge dt^n + \tag{4.25}$$

$$+ (\det[a])^{-1} \sum_{s=1}^{n} a_j^s \overset{-1}{\eta}{}_k^p dt^1 \wedge \ldots \wedge \underbrace{dx^k}_{s} \wedge \ldots \wedge dt^n.$$

For $1 \leqslant j \leqslant n$, $n+1 \leqslant p \leqslant r$, we have

$$\frac{d}{dt}\bigg|_{t=0} \mathscr{L}\big(v_1 \wedge \ldots \wedge (v_j + t v_p) \wedge \ldots \wedge v_n\big) = \frac{\partial \mathscr{L}(t^j, x^k, \gamma_j^k, \det[a])}{\partial \gamma_s^q} \overset{-1}{\eta}{}_p^q a_s^j, \tag{4.26}$$

$$\begin{aligned} \pi_3^* \circ \overline{\mathscr{L}'_{\mathrm{ver}}}(\bar{v}) &= \psi(t^j, x^k, \gamma_j^k) \\ &= \frac{\partial \mathscr{L}(t^j, x^k, \gamma_j^k, 1)}{\partial \gamma_s^k} dt^1 \wedge \ldots \wedge \underbrace{dx^k}_{s} \wedge \ldots \wedge dt^n - \\ &\quad - \left(\frac{\partial \mathscr{L}(t^j, x^k, \gamma_j^k, 1)}{\partial \gamma_s^k} \gamma_s^k - \mathscr{L}(t^j, x^k, \gamma_j^k, 1) \right) dt^1 \wedge \ldots \wedge dt^n. \end{aligned} \tag{4.27}$$

In local coordinates (4.19) we define

$$\overline{\mathscr{L}}(t, x^k, \gamma_j^k) = \mathscr{L}(t^j, x^k, \gamma_s^k, 1). \tag{4.28}$$

If we change the coordinate chart:

$$t^{j'} = t^{j'}(t^j), \quad x^{k'} = x^{k'}(t^j, x^k), \quad \gamma_{s'}^{k'} = \frac{\partial x^{k'}}{\partial x^p} \cdot \gamma_s^p \cdot \frac{\partial t^s}{\partial t^{s'}} + \frac{\partial x^{k'}}{\partial t^s} \cdot \frac{\partial t^s}{\partial t^{s'}}, \tag{4.29}$$

then

$$\overline{\mathscr{L}}(t^{j'}, x^{k'}, \gamma_{j'}^{k'}) \det\left[\frac{\partial t^{j'}}{\partial t^j}\right] = \overline{\mathscr{L}}(t^j, x^k, \gamma_j^k). \tag{4.30}$$

In the same way as in lemmas 4 and 5 we can prove that the space tangent to $\operatorname{tr} G_w^n(W)$ at the point $\bar{v}_w$ is isomorphic to the subspace of $\overset{n}{\bigwedge} T_w(W)$

spanned by n vectors

$$v_1\wedge\ldots\wedge\underbrace{v_k}_{m}\wedge\ldots\wedge v_n,\quad 1\leqslant m\leqslant n,\ n+1\leqslant k\leqslant r, \tag{4.31}$$

where $\bar{v}_w = \pi_2(v_w)$, $v_w = v_1\wedge\ldots\wedge v_n$, and the space cotangent to $\mathrm{tr}G^n_w(W)$ at $\bar{v}_w$ is isomorphic to the subspace of $\bigwedge^n T^*_w(W)$ spanned by n-covectors

$$v^{*1}\wedge\ldots\wedge\underbrace{v^{*k}}_{m}\wedge\ldots\wedge v^{*n},\quad 1\leqslant m\leqslant n,\ n+1\leqslant k\leqslant r. \tag{4.32}$$

5. A VARIATIONAL PRINCIPLE WITH A FIXED BOUNDARY

In this section we shall assume that W is a bundle (over an n-dimensional manifold B) with a projection π. Let $\mathscr{L}$ be a Lagrangian function on $K^n(W)$, let $f\colon B\to W$ be a section of π which is a diffeomorphism onto $f(B)$, let $\bar{f}$ be the lift of f to $\mathrm{tr}G^n(W)$ (see (3.2)) and let ψ be defined by (4.15) and $I_f = \int_B \mathscr{L}(f_*\nu)\omega$ (see (2.1)); then we have

PROPOSITION 1.

$$I_f = \int_B \bar{f}^*\psi. \tag{5.1}$$

LEMMA 7. *Let $w\in f(B)$ and $(v_j)_{j=1}^r$ be a basis of $T_w(W)$ such that $(v_m)_{m=1}^n$ are tangent to $f(B)$ at the point w and $(v_k)_{k=n+1}^r$ are π-vertical vectors. If $(v^{*j})_{j=1}^r$ is the dual basis, then, for $n+1\leqslant k\leqslant r$, $\bar{f}^*\circ\pi_3^*(v^{*k}) = 0$.*

Proof. Vectors $(\pi_* v_m)_{m=1}^n$ form a basis of $T_{\pi(w)}(B)$. We have

$$\begin{aligned}\langle(\bar{f}^*\circ\pi_3^*)v^{*k}\,|\,\pi_* v_m\rangle &= \langle(\pi_3\circ\bar{f})^* v^{*k}\,|\,\pi_* v_m\rangle\\ &= \langle f^* v^{*k}\,|\,\pi_* v_m\rangle = \langle v^{*k}\,|\,(f\circ\pi)_* v_m\rangle.\end{aligned}$$

But $(f\circ\pi)|f(B) = \mathrm{id}_{f(B)}$ and $(v_m)_{m=1}^n$ are tangent to $f(B)$; therefore $(f\circ\pi)_* v_m = v_m$. The lemma is proved.

Proof of proposition 1. Let $(v_m)_{m=1}^n$, $(v_k)_{k=n+1}^r$ and let $(v^{*j})_{j=1}^r$ be such as in lemma 7. By lemma 6 we have

$$\psi(\bar{v}) = \pi_3^*\Big(\mathscr{L}(v)v^{*1}\wedge\ldots\wedge v^{*m} + \sum_{\substack{1\leqslant m\leqslant n\\ n+1\leqslant k\leqslant r}} B_k^m v^{*1}\wedge\ldots\wedge\underbrace{v^{*k}}_{m}\wedge\ldots\wedge v^{*n}\Big).$$

It follows from lemma 7 that

$$\begin{aligned}\bar{f}^*\big(\psi(\bar{v})\big) &= (\bar{f}^*\circ\pi_3^*)\big(\mathscr{L}(v)v^{*1}\wedge\ldots\wedge v^{*n}\big)\\ &= \mathscr{L}(f_*\nu)f^*(v^{*1}\wedge\ldots\wedge v^{*n}) = \mathscr{L}(f_*\nu)\omega,\end{aligned}$$

where

$$v = v_1\wedge\ldots\wedge v_n,\qquad \bar{v} = \pi_2(v),$$

$$\nu\big(\pi(w)\big) = \pi_*(v_1\wedge\ldots\wedge v_n),\qquad \omega\big(\pi(w)\big) = f^*(v^{*1}\wedge\ldots\wedge v^{*n}).$$

Proposition 1 is proved.

Let $\pi_4\colon \operatorname{ver}T(W)\to W$ be the bundle of π-vertical vectors tangent to W, let $\pi\circ\pi_4\text{-}\operatorname{tr}G^n\big(\operatorname{ver}T(W)\big)$ be the bundle of all $\pi\circ\pi_4$-transversal oriented n-planes tangent to $\operatorname{ver}T(W)$, and let $\pi\circ\pi_3\text{-}\operatorname{ver}T\big(\operatorname{tr}G^n(W)\big)$ be the bundle of all $\pi\circ\pi_3$-vertical vectors tangent to $\operatorname{tr}G^n(W)$.

LEMMA 8. *There exists an invertible mapping η, $\pi\circ\pi_4\text{-}\operatorname{tr}G^n\big(\operatorname{ver}T(W)\big)$ onto $\pi\circ\pi_3\text{-}\operatorname{ver}T\big(\operatorname{tr}G^n(W)\big)$, such that the following diagrams commute:*

(5.2)
$$\begin{tikzcd}
\pi\circ\pi_4\text{-}\operatorname{tr}G^n\big(\operatorname{ver}T(W)\big) \arrow[r, "\eta"] \arrow[d] & \pi\circ\pi_3\text{-}\operatorname{ver}T\big(\operatorname{tr}G^n(W)\big) \arrow[d] \\
W \arrow[r, "\mathrm{id}"] & W
\end{tikzcd}$$

(5.3)
$$\begin{tikzcd}
\pi\circ\pi_4\text{-}\operatorname{tr}G^n\big(\operatorname{ver}T(W)\big) \arrow[rr, "\pi_{4*}"] \arrow[dr, "\eta"] \arrow[dd] & & \operatorname{tr}G^n(W) \arrow[dd, "\pi_3"] \\
& \pi\circ\pi_3\text{-}\operatorname{ver}T\big(\operatorname{tr}G^n(W)\big) \arrow[ur] \arrow[dl, "\pi_{3*}"'] \arrow[dr] & \\
\operatorname{ver}T(W) \arrow[rr, "\pi_4"] & & W
\end{tikzcd}$$

In local coordinates (b^s, u^k) on W a point $(b^s, u^k, X^k, a^k_s, \beta^k_s)$ of $\pi\circ\pi_4\text{-}\operatorname{tr}G^n\big(\operatorname{ver}T(W)\big)$ is transformed into a point $(b^s, u^k, a^k_s, X^k, \beta^k_s)$ of $\pi\circ\pi_3\text{-}\operatorname{ver}T\big(\operatorname{tr}G^n(W)\big)$. This definition of η does not depend on the choice of local coordinates on W because of what follows:

If $(b^s, u^k, X^k, a^k_s, \beta^k_s)$ are local coordinates in $\pi\circ\pi_4\text{-}\operatorname{tr}G^n\big(\operatorname{ver}T(W)\big)$ and $b^{s'} = b^{s'}(b^s)$, $u^{k'} = u^{k'}(b^s, u^k)$, then (cf. (4.29)):

$$\begin{aligned}
&X^{k'} = X^k\frac{\partial u^{k'}}{\partial u^k}, \qquad a^{k'}_{s'} = \left(a^k_s\frac{\partial u^{k'}}{\partial u^k} + \frac{\partial u^{k'}}{\partial b^s}\right)\cdot\frac{\partial b^s}{\partial b^{s'}},\\
&\beta^{k'}_{s'} = \left(\beta^k_s\frac{\partial u^{k'}}{\partial u^k} + X^k\cdot\frac{\partial^2 u^{k'}}{\partial b^s\partial u^k} + a^k_s X^p\frac{\partial^2 u^{k'}}{\partial u^p\partial u^k}\right)\cdot\frac{\partial b^s}{\partial b^{s'}}.
\end{aligned}\tag{5.4}$$

If $(b^s, u^k, \gamma^k_s, Y^k, \lambda^k_s)$ are local coordinates in $\pi\circ\pi_3\text{-}\operatorname{ver}T\big(\operatorname{tr}G^n(W)\big)$ and $b^{s'} = b^{s'}(b^s)$, $u^{k'} = u^{k'}(b^s, u^k)$, then (cf. (4.29)):

$$\begin{aligned}
&Y^{k'} = Y^k\cdot\frac{\partial u^{k'}}{\partial u^k}, \qquad \gamma^k_s = \left(\gamma^k_s\frac{\partial u^{k'}}{\partial u^k} + \frac{\partial u^{k'}}{\partial b^s}\right)\cdot\frac{\partial b^s}{\partial b^{s'}},\\
&\lambda^{k'}_{s'} = \left(\lambda^k_s\frac{\partial u^{k'}}{\partial u^k} + Y^k\frac{\partial^2 u^{k'}}{\partial b^s\partial u^k} + \gamma^k_s Y^p\frac{\partial^2 u^{k'}}{\partial u^k\partial u^p}\right)\cdot\frac{\partial b^{s'}}{\partial b^s}.
\end{aligned}\tag{5.5}$$

If $\pi_5\colon \operatorname{ver}T(W)\to B$ and X is a section of π_5, then we shall have the commutative diagram

$$\begin{tikzcd}
 & \pi\circ\pi_4\text{-}\operatorname{tr}G^n(\operatorname{ver}T(W)) \arrow[r, "\eta"] \arrow[d] & \pi\circ\pi_3\text{-}\operatorname{ver}T(\operatorname{tr}G^n(W)) \arrow[d] \\
 & & \operatorname{tr}G^n(W) \arrow[d, "\pi_3"] \\
K^n(\operatorname{ver}T(W)) \arrow[ur, "\pi_6"] \arrow[r] & \operatorname{ver}T(W) \arrow[r] & W \arrow[dll, "\pi"] \\
K^n(B) \arrow[u, "X_*"] & B \arrow[l, "\nu"] \arrow[u, "X"] &
\end{tikzcd}$$

(5.6)

Let us define $\bar{X} = (\eta\circ\pi_6)(X_*\nu)$.

The section $\bar{X}$ of $\pi\circ\pi_3\text{-}\operatorname{ver}T(\operatorname{tr}G^n(W))\to B$ is called the *canonical lift of* X *to* $\pi\circ\pi_3\text{-}\operatorname{ver}T(\operatorname{tr}G^n(W))$. If in local coordinates $X = (b^s, u^k(b^s), X^k(b^s))$, then

$$\bar{X} = \left(b^s, u^k(b^s), \frac{\partial u^k}{\partial b^s}, X^k(b^s), \frac{\partial X^k}{\partial b^s}\right). \tag{5.7}$$

DEFINITION. We call a section Y of $\pi\circ\pi_3\text{-}\operatorname{ver}T(\operatorname{tr}G^n(W))\to B$ *integrable* if there exists a section X of $\operatorname{ver}T(W)\to B$ such that $Y = \bar{X}$.

For every bundle $\tau\colon V\to Z$ and every k-form ω on V a fibre derivative of ω, is defined, $\omega'_{\mathrm{ver}}\in C^\infty\big(L(\operatorname{ver}T(V), \bigwedge^k T^*(V))\big)$, where $L(\operatorname{ver}T(V), \bigwedge^k T^*(V))\to V$ is the bundle of linear maps from $\operatorname{ver}T(V)$ to $\bigwedge^k T^*(V)$. If (z^i, x^j) are local coordinates in V,

$$X = a^i\frac{\partial}{\partial x^i}$$

and

$$\omega = \sum_{\substack{i_1<\ldots<i_s\\ j_1<\ldots<j_{k-s}}} b_{i_1\ldots i_s j_1\ldots j_{k-s}}\, dz^{i_1}\wedge\ldots\wedge dz^{i_s}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-s}},$$

then

$$\omega'_{\mathrm{ver}}(X) = \sum_{\substack{i_1<\ldots<i_s\\ j_1<\ldots<j_{k-s}}} \frac{\partial}{\partial x^m}(b_{i_1\ldots i_s j_1\ldots j_{k-s}})\, a^m\, dz^{i_1}\wedge\ldots\wedge dz^{i_s}\wedge \wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-s}}. \tag{5.8}$$

From formula (5.8) we have the following

LEMMA 9. *If ω is a τ-horizontal form on V and X a τ-vertical vector tangent to V at the point $v\in V$, then we have*

$$(X\lrcorner\, d\omega) = \omega'_{\mathrm{ver}}(X). \tag{5.9}$$

We shall apply this lemma to the bundle $\pi_3\colon \operatorname{tr}G^n(W)\to W$ and the π_3-horizontal n-form ψ.

For every $\bar{v} \in \operatorname{tr} G^n(W)$ we have a mapping

$$(5.10)\qquad B_{\bar{v}}\colon \pi_3\text{-}\operatorname{ver} T_{\bar{v}}\big(\operatorname{tr} G^n(W)\big) \times \overset{n}{\bigwedge} T_{\pi_3\bar{v}}(W) \ni (X, q) \to B_{\bar{v}}(X, \bar{q}) = \langle \bar{q} \,|\, \psi'_{\mathrm{ver}}(X)\rangle \in \boldsymbol{R}\ (^1).$$

The space $\pi_3\text{-}\operatorname{ver} T_{\bar{v}}\big(\operatorname{tr} G^n(W)\big)$ is isomorphic to a subspace of $\overset{n}{\bigwedge} T_{\pi_3\bar{v}}(W)$ (see (4.31)), and so we have the bilinear mapping

$$(5.11)\qquad B_{\bar{v}}\colon \pi_3\text{-}\operatorname{ver} T_{\bar{v}}\big(\operatorname{tr} G^n(W)\big) \times \pi_3\text{-}\operatorname{ver} T_{\bar{v}}\big(\operatorname{tr} G^n(W)\big) \to \boldsymbol{R}$$

induced by (5.10). We shall assume that for every $\bar{v} \in \operatorname{tr} G^n(W)$ the bilinear form $B_{\bar{v}}$ is non-degenerate. In section 7 we shall prove that this assumption will ensure the existence of the Legendre transformation.

In local coordinates in $\operatorname{tr} G^n(W)$ (b^s, u^k, γ^k_s),

$$v = \left(\frac{\partial}{\partial b^1} + \gamma^k_1 \frac{\partial}{\partial u^k}\right) \wedge \ldots \wedge \left(\frac{\partial}{\partial b^n} + \gamma^k_n \frac{\partial}{\partial u^k}\right),$$

$$(5.12)\qquad \mathscr{L}(v) = \overline{\mathscr{L}}(b^s, u^k, \gamma^k_s), \qquad X = X^k_s \frac{\partial}{\partial \gamma^k_s}, \qquad Y = Y^k_s \frac{\partial}{\partial \gamma^k_s},$$

$$B_{\bar{v}}(X, Y) = \frac{\partial^2 \overline{\mathscr{L}}(b^s, u^k, \gamma^k_s)}{\partial \gamma^k_s \partial \gamma^p_q} X^k_s Y^p_q.$$

It is seen from formula (5.12) that $\bar{B}_{\bar{v}}$ is a symmetric bilinear mapping. Its non-degeneracy is equivalent to the condition

$$(5.13)\qquad \det\left[\frac{\partial^2 \overline{\mathscr{L}}(b^s, u^k, \gamma^k_s)}{\partial \gamma^k_s \partial \gamma^p_q}\right] \neq 0.$$

PROPOSITION 2. *If $f\colon B \to W$ is a section of π, and $\bar{f}$ is the lift of f to $\operatorname{tr} G^n(W)$, then for every π_3-vertical vector field X defined on $C_{\bar{f}} = \bar{f}(B)$ we have $(X \lrcorner\, d\psi)\,|\,C_{\bar{f}} = 0$.*

Conversely, for every $\bar{v} \in \operatorname{tr} G^n(W)$ let the form $\bar{B}_{\bar{v}}$ be non-degenerate. If $\xi\colon B \to \operatorname{tr} G^n(W)$ is a section of $\pi \circ \pi_3$ and, for every π_3-vertical vector field X defined on $C_\xi = \xi(B)$, $(X \lrcorner\, d\psi)\,|\,C_\xi = 0$, then C_ξ is integrable, i.e., $C_\xi = C_{\bar{\varphi}}$, where $\varphi = \pi_3 \circ \xi$.

We shall first prove the following

LEMMA 10. *Let ω be a π-horizontal k-form on W and let $\mathscr{L}$ be a fibre-preserving mapping from W to W. Then ω is $\mathscr{L}$-invariant, i.e., $\mathscr{L}^*\omega = \omega$.*

Proof. For every $a \in W$ there exists a covector $\tilde{\omega}_{\pi(a)} \in \overset{k}{\bigwedge} T^*_{\pi(a)}(B)$ such that $\pi^*\tilde{\omega}_{\pi(a)} = \omega(a)$. Let $a = \mathscr{L}(w)$. We have $\pi^*\tilde{\omega}_{\pi(\mathscr{L}(w))} = \omega\big(\mathscr{L}(w)\big)$. Thus $(\pi \circ \mathscr{L})^*\tilde{\omega}_{\pi(a)} = (\mathscr{L}^*\omega)(w)$ and $\omega(w) = (\mathscr{L}^*\omega)(w)$ (2).

---

(1) Where $\bar{q}$ is any lift of $q \in \overset{n}{\bigwedge} T_{\pi_3\bar{v}}(W)$ to $\overset{n}{\bigwedge} T_{\bar{v}}(\operatorname{tr} G^n(W))$. (5.10) does not depend on the choice $\bar{q} \in \overset{n}{\bigwedge} T_{\bar{v}}(\operatorname{tr} G(W))$ because $\psi'_{\mathrm{ver}}(X)$ is a π_3-horizontal form.

(2) $\mathscr{L}$ is not the Lagrangian function.

Proof of proposition 2. Let $\bar{v} = \overline{v_1 \wedge \ldots \wedge v_n}$ and $(v_k)_{k=n+1}^r$ are linearly independent π-vertical vectors at $w \in W$. It follows from lemma 6 that

$$(5.14)\qquad \psi'_{\mathrm{ver}}(\bar{v})(X) = \sum_{\substack{1\leqslant q,s\leqslant n\\ n+1\leqslant p,i\leqslant r}} \frac{\partial^2 \mathscr{L}\big((v_1+\lambda_1^k v_k)\wedge\ldots\wedge(v_n+\lambda_n^k v_k)\big)}{\partial\lambda_q^p\partial\lambda_s^i}\bigg|_{\lambda_q^p=\lambda_s^i=0} X_q^p \, \pi_3^*(v^{*1}\wedge\underbrace{\ldots}_{s}\wedge v^{*i}\wedge\ldots\wedge v^{*n}),$$

where $X \in \pi_3\text{-ver}\, T_{\bar{v}}\big(\mathrm{tr}G^n(W)\big)$, $X = \sum_{\substack{1\leqslant q\leqslant n\\ n+1\leqslant p\leqslant r}} X_q^p v_1\wedge\ldots\wedge\underbrace{v_p}_{q}\wedge\ldots\wedge v_n$ (cf. (4.31)).

If $\overset{n}{z}$ is a simple non-zero n-vector tangent to Cv at $\bar{v}$, then $\pi_{3*}\overset{n}{z}$ is a non-zero, simple n-vector tangent to C_φ at $\pi_3(\bar{v})$. Therefore $\pi_{3*}\overset{n}{z} = av$, $0 \neq a \in \boldsymbol{R}$. From formula (5.14) we have $\langle \overset{n}{z} | \psi'_{\mathrm{ver}}(\bar{v})(X)\rangle = 0$.

Conversely, let $\xi\colon B \to \mathrm{tr}G^n(W)$ be a section of $\pi\circ\pi_3$ and let $\overset{n}{z}$ be a non-zero, simple n-vector tangent to C_ξ at $\bar{v}$. Then $\pi_{3*}\overset{n}{z}$ is a non-zero π-transversal n-vector tangent to $C_{\pi_3\circ\xi}$ at $\pi_3(\bar{v})$. Therefore $\pi_{3*}\overset{n}{z} = a(\pi_3\circ\xi)_*\big(\pi_*(v)\big)$, $0 \neq a \in \boldsymbol{R}$.

From formula (5.14) we have for every X_q^p

$$(5.15)\qquad \frac{\partial^2\mathscr{L}(\cdot)}{\partial\lambda_q^p\partial\lambda_s^i}\bigg|_{\lambda_q^p=\lambda_s^i=0} X_q^p v^{*1}\wedge\ldots\wedge v^{*i}\wedge\ldots\wedge v^{*n}(\pi_3\circ\xi\circ\pi)_*(v_1\wedge\ldots\wedge v_n) = 0.$$

Covectors $(v^{*j})_{j=1}^n$ are π-horizontal, and so it follows from lemma 10 that

$$(5.16)\qquad (\pi_3\circ\xi\circ\pi)^* v^{*j} = v^{*j}, \quad 1\leqslant j\leqslant n.$$

Formula (5.16) implies

$$(5.17)\qquad \langle(\pi_3\circ\xi\circ\pi)_* v_s | v^{*j}\rangle = \langle v_s | (\pi_3\circ\xi\circ\pi)^* v^{*j}\rangle = \delta_s^j, \quad 1\leqslant s,j\leqslant n.$$

Using (5.15) and (5.17) we obtain

$$(5.18)\qquad \frac{\partial^2\mathscr{L}(\cdot)}{\partial\lambda_q^p\partial\lambda_s^i}\bigg|_{\lambda_q^p=\lambda_s^i=0} X_q^p\langle(\pi_3\circ\xi\circ\pi)_* v_s | v^{*i}\rangle = 0.$$

Formula (5.18) together with the non-degeneracy condition for $\bar{B}_{\bar{v}}$ gives

$$(5.19)\qquad \langle(\pi_3\circ\xi\circ\pi)_* v_s | v^{*i}\rangle = 0, \quad 1\leqslant s\leqslant n,\ n+1\leqslant i\leqslant r.$$

It follows from (5.19) that vectors $(\pi_3\circ\xi\circ\pi)_* v_s$, $1\leqslant s\leqslant n$, are linear combinations of $(v_j)_{j=1}^n$. They are also linearly independent because vectors $\pi_*(\pi_3\circ\xi\circ\pi)_* v_s = \pi_* v_s$, $1\leqslant s\leqslant n$, are linearly independent. We conclude that the plane tangent to $(\pi_3\circ\xi)(B)$ at the point $\pi_3(\bar{v})$ is equal to $\bar{v}$. The proposition is proved.

In order to define a variational problem with a fixed boundary we have to indroduce the notion of a one-parameter family of sections of $\pi\colon W \to B$.

DEFINITION. A mapping $]-\delta, \delta[\times B \ni (\varepsilon, b) \to f_\varepsilon(b) \in W$ is called a *one-parameter family of sections of π if, for every $\varepsilon \in]-\delta, \delta[$, $\pi \circ f_\varepsilon(\cdot) = \mathrm{id}_B$.* We shall consider only those mappings $(\varepsilon, b) \to f_\varepsilon(b)$ which are at least of class C^2. We shall say that a one-parameter family of sections preserves the boundary if, for every $\varepsilon \in]-\delta, \delta[$, $f_\varepsilon(b) = f_0(b)$, $b \in \partial B$, or equivalently

$$f_\varepsilon(\partial B) = f_0(\partial B).$$

A one-parameter family of sections of π, $f_\varepsilon(\cdot)$, $\varepsilon \in]-\delta, \delta[$, defines a one-parameter family of sections of $\pi \circ \pi_3$, $(\varepsilon, b) \to \bar{f}_\varepsilon(b) \in \mathrm{tr}G^n(W)$. These two families of sections determine vector fields X and $\bar{X}$, which are defined, respectively, on $f_0(B)$ and $\bar{f}_0(B)$:

$$X\big(f_0(b)\big) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f_\varepsilon(b), \tag{5.20}$$

$$\bar{X}\big(\bar{f}_0(b)\big) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \bar{f}_\varepsilon(b). \tag{5.21}$$

LEMMA 11. *A $\pi \circ \pi_3$-vertical vector field $\bar{X}$ is the canonical lift (in the sense of* (5.6)) *of a. π-vertical vector field X. If the family $f_\varepsilon(\cdot)$ preserves the boundary of B, then $X|f_0(\partial B) = 0$ and $\bar{X}|\bar{f}_0(\partial B)$ is π_3-vertical.*

The proof follows from lemma 8 and diagram (5.6).

According to proposition 1, let

$$I_{f_\varepsilon} = \int_B \mathscr{L}(f_{\varepsilon *}\nu)\omega = \int_B \bar{f}_\varepsilon^* \psi, \qquad \varepsilon \in]-\delta, \delta[. \tag{5.22}$$

PROPOSITION 3.

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} I_{f_\varepsilon} = \int_{\bar{f}_0(B)} (\bar{X} \lrcorner d\psi) + \int_{\bar{f}_0(B)} d(\bar{X} \lrcorner \psi), \tag{5.23}$$

where $\bar{X}$ is defined by (5.21).

This proposition follows (with a slight modification) from the familiar formula for the Lie derivative:

$$\mathscr{L}_{\bar{X}}\psi = \bar{X} \lrcorner \psi + d(\bar{X} \lrcorner \psi) \quad \text{(cf. [8])}. \tag{5.24}$$

PROPOSITION 4. *If the family of sections of π, $f_\varepsilon(\cdot)$, $\varepsilon \in]-\delta, \delta[$ is constant on the boundary of B, then*

$$\int_{\bar{f}_0(B)} d(\bar{X} \lrcorner \psi) = 0. \tag{5.25}$$

Proof. Using the Stokes theorem, we have

$$\int_{\bar{f}_0(B)} d(\bar{X}\lrcorner\psi) = \int_{\partial\bar{f}_0(B)} \bar{X}\lrcorner\psi = \int_{\bar{f}_0(\partial B)} \bar{X}\lrcorner\psi.$$

But ψ is a π_3-horizontal n-form and $\bar{X}$ is π_3-vertical on $\bar{f}_0(\partial B)$ (see lemma 11); therefore $\bar{X}\lrcorner\psi = 0$ on $\bar{f}_0(\partial B)$.

Let M be an $(n-1)$-dimensional embedded submanifold of W such that M is a diffeomorphism M onto ∂B. The variational problem with a fixed boundary M is a triplet $(W, \mathscr{L}, M)$, where $\mathscr{L}$ is a Lagrangian function.

DEFINITION. An embedded submanifold $f\colon B \to W$ is called an *extrema section of the variational problem* $(W, \mathscr{L}, M)$ if:

1° f is a section of π and $f(\partial B) = M$,

2° for every one-parameter family $f_\varepsilon(\cdot)$, $\varepsilon \in]-\delta, \delta[$, of sections of π fulfilling the conditions

$$f_\varepsilon(\partial B) = M, \quad \varepsilon \in]-\delta, \delta[,$$
$$f = f_0,$$

we have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} I_{f_\varepsilon} = 0.$$

THEOREM 1. *Let $f\colon B \to W$ be a section of π such that $f(\partial B) = M$. If, for every $\pi\circ\pi_3$-vertical integrable vector field $\bar{X}$ on $C_{\bar{f}}$, $(\bar{X}\lrcorner d\psi)|C_{\bar{f}} = 0$, then f is an extremal section of the variational problem $(W, \mathscr{L}, M)$.*

If $f\colon B \to W$ is an extremal section of the variational problem $(W, \mathscr{L}, M)$, then for every $\pi\circ\pi_3$-vertical integrable vector field $\bar{X}$ on $C_{\bar{f}}$

$$(\bar{X}\lrcorner d\psi)|C_{\bar{f}} = 0. \tag{5.26}$$

Proof. The first statement follows from propositions 3 and 4. Let $f\colon B \to W$ be an extremal section and $\bar{X}$ a $\pi\circ\pi_3$-vertical integrable vector field on $C_{\bar{f}}$. We have to prove that $\bar{X}$ is generated by a one-parameter family of sections of π. Let $X = \pi_{3*}\bar{X}$ be a π-vertical vector field on C_f. It follows from the theorem on flows of vector fields on compact manifolds that X defines a one-parameter family of sections of $\pi(f_\varepsilon)$ (cf. [1]) such that $\dfrac{d\bar{f}_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = \bar{X}$.

Therefore, for every $\pi\circ\pi_3$-vertical integrable vector field $\bar{X}$ on $C_{\bar{f}}$, we infer from (5.23) and (5.25) that

$$\int_{C_{\bar{f}}} (\bar{X}\lrcorner d\psi) = 0. \tag{5.26'}$$

Let φ be a function on $C_{\bar{f}}$ and let φ_1 be the corresponding function on C_f (i.e., $\varphi = \pi_3^* \varphi_1$). It is easy to see that $\varphi \cdot \bar{X} - \overline{\varphi_1 X}$ is a π_3-vertical vector field on $C_{\bar{f}}$ (cf. (5.7)). If we use (5.26′) and proposition 2, we shall find that, for every $\varphi \in C^\infty(C_{\bar{f}})$, $\int_{C_{\bar{f}}} \varphi(\bar{X} \lrcorner d\psi) = 0$. Thus $(\bar{X} \lrcorner d\psi) | C_{\bar{f}} = 0$.

THEOREM 2. *For every $\bar{v} \in \operatorname{tr} G^n(W)$, let the bilinear form $\bar{B}_{\bar{v}}$ be non-degenerate. Then every embedded submanifold $\xi\colon B \to \operatorname{tr} G^n(W)$ of $\operatorname{tr} G^n(W)$ which is a section of $\pi \circ \pi_3$ and satisfies, for every $\pi \circ \pi_3$-vertical vector field Y defined on C_ξ, the condition*

$$(Y \lrcorner d\psi) | C_\xi = 0 \tag{5.27}$$

is integrable and $\pi \circ \xi$ is an extremal section of the variational problem $\big(W, \mathscr{L}, (\pi_3 \circ \xi)(\partial B)\big)$.

If $f\colon B \to W$ is an extremal section of the variational problem $(W, \mathscr{L}, M)$, then for every vector field Y defined on $C_{\bar{f}}$ we have

$$(Y \lrcorner d\psi) | C_{\bar{f}} = 0. \tag{5.28}$$

Proof. The first part of this theorem follows from proposition 2 and theorem 1.

Let $f\colon B \to W$ be an extremal section of the variational problem $(W, \mathscr{L}, M)$. For every vector field Y on $C_{\bar{f}}$ there exist a vector field Y_1 tangent to $C_{\bar{f}}$ and a $\pi \circ \pi_3$-vertical vector field Y_2 such that $Y = Y_1 + Y_2$. It is easy to see that

$$(Y_1 \lrcorner d\psi | C_{\bar{f}} = 0. \tag{5.29}$$

Let $X = \pi_{3*} Y_2$ be a π-vertical vector field on C_f and let $\bar{X}$ be the canonical lift of X. $\bar{X}$ is a $\pi \circ \pi_3$-vertical integrable vector field on $C_{\bar{f}}$ and $Y_2 - \bar{X}$ is a π_3-vertical vector field on $C_{\bar{f}}$. Therefore we have

$$(Y_2 \lrcorner d\psi) | C_{\bar{f}} = (\bar{X} \lrcorner d\psi) | C_{\bar{f}} + \big((Y_2 - \bar{X}) \lrcorner d\psi\big) | C_{\bar{f}}.$$

From the above formula, proposition 2 and theorem 1 we obtain

$$(Y_2 \lrcorner d\psi) | C_{\bar{f}} = 0. \tag{5.30}$$

Formula (5.30) together with (5.29) completes the proof.

Remark. In the second part of the proof we do not use the non-degeneracy condition for $\bar{B}_{\bar{v}}$.

We shall consider as an example a special case of a Lagrangian function. Let $\Omega \in C^\infty\big(\bigwedge^n T^*(W)\big)$, i.e., Ω is an n-form on W. We define a Lagrangian function which corresponds to the form Ω:

$$K(W) \ni v \to \mathscr{L}(v) = \langle v | \Omega \rangle \in \boldsymbol{R}. \tag{5.31}$$

If $v = v_1 \wedge \ldots \wedge v_n$ and $(v_k)_{k=n+1}^r$ are linearly independent π-vertical vectors, we have by lemma 6

$$(5.32)\qquad \mathscr{L}'_{\mathrm{ver}}(v) = \langle v \,|\, \Omega\rangle v^{*1} \wedge \ldots \wedge v^{*n} + \\ + \langle v_1 \wedge \ldots \wedge \underbrace{v_k}_{m} \wedge \ldots \wedge v_n \,|\, \Omega\rangle v^{*1} \wedge \ldots \wedge \underbrace{v^{*k}}_{m} \wedge \ldots \wedge v^{*n} = P_{\bar{v}}\Omega,$$

where $(v^{*j})_{j=1}^r$ is the dual basis and $P_{\bar{v}}$ is the projection on the subspace of π-one vertical forms (cf. formula (4.10)).

We shall use the following

LEMMA 12. *In the above notation let*

$$(5.33)\qquad q = Av_1 \wedge \ldots \wedge v_n + \sum_{\substack{1 \leqslant m \leqslant n \\ n+1 \leqslant k \leqslant r}} B_m^k v_1 \wedge \ldots \wedge \underbrace{v_k}_{m} \wedge \ldots \wedge v_n;$$

then $\langle q \,|\, P_{\bar{v}}\Omega\rangle = \langle q \,|\, \Omega\rangle$.

PROPOSITION 5. *Let $\mathscr{L}$ be given by* (5.31). *A section $f\colon B \to W$ is an extremal section of the variational problem $(W, \mathscr{L}, M)$ if and only if:*

1° $f(\partial B) = M$,

2° *for every π-vertical vector field X defined on C_f,*

$$(5.34)\qquad (X \lrcorner\, d\Omega) \,|\, C_f = 0.$$

Proof. Let $v \in C_{\bar{f}}$. Locally there exist n linearly independent vector fields $Q_1, \ldots, Q_n$ defined on a neighbourhood U of $\bar{v}$ in $C_{\bar{f}}$ and tangent to $C_{\bar{f}}$. Let Q_0 be a $\pi \circ \pi_3$-vertical vector field defined on $C_{\bar{f}}$ ([3]). By means of Q_0 we can construct a bundle over U with 1-dimensional fibre in such a way that Q_0 will be vertical. This bundle $\bar{D}$ is an $(n+1)$-dimensional submanifold in $\mathrm{tr}G^n(W)$.

We can extend the vector fields $(Q_j)_{j=0}^n$ onto $\bar{D}$. We denote these extensions by $\tilde{Q}_j$. Let $\tilde{P}_j = \pi_{3*}\tilde{Q}_j$ and let $D = \pi_3(\bar{D})$. For $1 \leqslant j \leqslant n$, $\tilde{P}_j$ are tangent to C_f at points belonging to C_f. It is clear that $\bar{v} = \overline{P_1(\pi_3\bar{v}) \wedge \ldots \wedge P_n(\pi_3\bar{v})}$. Now we shall use the formula for the exterior derivative (cf. [8]):

$$(5.35)\qquad \frac{1}{n+1}(\tilde{Q}_0 \lrcorner\, d\psi)(\tilde{Q}_1, \ldots, \tilde{Q}_n) = d\psi(\tilde{Q}_0, \tilde{Q}_1, \ldots, \tilde{Q}_n) \\ = \frac{1}{n+1}\sum_{j=0}^{n}(-1)^j \tilde{Q}_j \psi(\tilde{Q}_0, \ldots \underset{\hat{j}}{\ldots}, \tilde{Q}_n) + \\ + \frac{1}{n+1}\sum_{0 \leqslant i < j \leqslant n}(-1)^{i+1}\psi([\tilde{Q}_i, \tilde{Q}_j], \tilde{Q}_0, \ldots \underset{\hat{i}}{\ldots} \underset{\hat{j}}{\ldots} \ldots, \tilde{Q}_n).$$

([3]) And such that $\pi_{3*}Q_0$ is non-vanishing vector field on C_f.

But $\psi(\bar{v}) = \pi_3^* P_{\bar{v}} \Omega$ and, for $i, j \neq 0$, $[\tilde{P}_i, \tilde{P}_j] | C_f$ is a linear combination of $\tilde{P}_1, \ldots, \tilde{P}_n$ and $[\tilde{P}_0, \tilde{P}_j]$ is a linear combination of $\tilde{P}_0, \ldots, \tilde{P}_n$. Therefore we can use (5.33).

Using (5.33), we see from (5.35) that

$$d\psi(\tilde{Q}_0, \ldots, \tilde{Q}_n) | C_f = d\Omega(\tilde{P}_0, \ldots, \tilde{P}_n) | C_f. \tag{5.36}$$

This ends the proof.

6. A GEOMETRICAL FORMULATION OF THE EULER-LAGRANGE EQUATIONS

Let $(W, \mathscr{L}, M)$ be a variational problem with a fixed boundary. Let $\mathscr{H}$ denote the set of all embedded n-dimensional submanifolds of W fulfilling the following conditions:

1° if $C \in \mathscr{H}$, then there exists a section f of π: $W \to B$ such that $C = f(B)$,

2°

$$\partial C = M. \tag{6.1}$$

Let $C^\infty(C, \pi\text{-ver}\, T(W))$ denote the vector space of all smooth π-vertical vector fields tangent to W and defined on C.

We define a functional: for every $C \in \mathscr{H}$ and $X \in C^\infty(C, \pi\text{-ver}\, T(W))$

$$(C, X) \to F(C, X) = \int_{\bar{C}} (\bar{X} \lrcorner d\psi) \in \boldsymbol{R}, \tag{6.2}$$

where ψ is defined by (4.15), $\bar{C}$ is the canonical lift of C to $\mathrm{tr}G^n(W)$ and $\bar{X}$ is the canonical lift of X defined by (5.6).

It follows from proposition 2 that instead of the canonical lift $\bar{X}$ of X we can take in (6.2) any $\pi \circ \pi_3$-vertical vector field Y defined on $\bar{C}$ such that $\pi_{3*} Y = X$.

In local coordinates (t^s, x^k, γ_s^k), $C = \{(t^s, x^k(t^s))\}$, $\bar{C} = \left\{\left(t^s, x^k(t^s), \gamma_j^k(t^s) = \frac{\partial x^k}{\partial t^j}\right)\right\}$, $Y = (0, X^k, Y_j^k)$.

Using (4.27), (4.28) we obtain

$$\begin{aligned}(Y \lrcorner d\psi) | \bar{C} = \Bigg[& \frac{\partial \bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial x^p} X^p - X^p \frac{\partial \bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial t^q \partial \gamma_q^p} - \\ & - \frac{\partial^2 \bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial \gamma_q^p \partial \gamma_m^i} \cdot \frac{\partial \gamma_m^i(t^s)}{\partial t^q} X^p - \\ & - \frac{\partial^2 \bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial x^i \partial \gamma_q^p} \frac{\partial x^i(t^s)}{\partial t^q} X^p \Bigg] dt^1 \wedge \ldots \wedge dt^n,\end{aligned} \tag{6.3}$$

where the summation convention is used. From (6.3) we obtain

$$(6.4)\qquad (Y \lrcorner d\psi)|\bar{C}$$
$$= \left[\frac{\partial\bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial x^p} - \frac{\partial}{\partial t^q}\frac{\partial\bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial\gamma_q^p}\right] X^p dt^1\wedge\ldots\wedge dt^n.$$

LEMMA 13. *If we change the coordinates in* $\operatorname{tr}G^n(W)$ $(t^s, x^k, \gamma_s^k)\to(t^{s'}, x^{k'}, \gamma_{s'}^{k'})$ (see (4.29)), *we obtain*

$$(6.5)\qquad \frac{\partial\bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial x^p} - \frac{\partial}{\partial t^q}\frac{\partial\bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial\gamma_q^p}$$
$$= \left[\frac{\partial\bar{\mathscr{L}}(t^{s'}, x^{k'}(t^{s'}), \gamma_{j'}^{k'}(t^{s'}))}{\partial x^{p'}} - \frac{\partial}{\partial t^{q'}}\frac{\partial\bar{\mathscr{L}}(t^{s'}, x^{k'}(t^{s'}), \gamma_{j'}^{k'}(t^{s'}))}{\partial\gamma_{q'}^{p'}}\right]\frac{\partial x^{p'}}{\partial x^p}\cdot\det\left[\frac{\partial t^{s'}}{\partial t^s}\right].$$

Formula (6.5) follows from (4.30) by a direct computation.

Using the transformation formula (6.5), we see that (6.4) defines on every open set $U\subset C$ in C which is contained in a domain of a local chart (t^s, x^k) one form $\xi_U\in C^\infty(U, T^*(W))$ which fulfils the following condition: for $f\colon B\to W$ $(C = f(B))$, volume n-form ω_B on B and for every π-vertical vector field X on U,

$$(6.6)\qquad (\bar{f}|\pi(U))^*(\bar{X}\lrcorner d\psi) = (f|\pi(U))^*(X\lrcorner\xi_U)\cdot\omega_B|\pi(U).$$

In local coordinates (t^s, x^k), $\omega_B = \psi\cdot dt^1\wedge\ldots\wedge dt^n$, $\psi>0$,

$$w = f(t) = (t^s, x^k(t^s)),\qquad \gamma_j^k(t^s) = \frac{\partial x^k}{\partial t^j}(t^s),$$

$$(6.7)\qquad \xi_{pU}(w) = (\psi(t^s))^{-1}\left[\frac{\partial\bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial x^p} - \frac{\partial}{\partial t^q}\frac{\partial\bar{\mathscr{L}}(t^s, x^k(t^s), \gamma_j^k(t^s))}{\partial\gamma_q^p}\right],$$
$$\xi_U(w) = \xi_{pU}(w)\cdot dx^p.$$

By means of a partition of unity we can construct one form $\xi\in C^\infty(C, T^*(W))$ such that for every π-vertical vector field X on C we have

$$(6.8)\qquad \bar{f}^*(\bar{X}\lrcorner d\psi) = f^*(X\lrcorner\xi)\omega_B.$$

The form ξ is not uniquely determined. If $\xi_1, \xi_2\in C^\infty(C, T^*(W))$ and fulfil (6.8), we have, for every $X\in C^\infty(C, \pi\text{-ver}\,T(W))$, $X\lrcorner(\xi_1-\xi_2) = 0$ on C. Therefore $\xi_1-\xi_2\in C^\infty(C, \pi\text{-hor}\,T^*(W))$.

We have proved

THEOREM 3. *For every n-dimensional, π-transversal embedded submanifold C of W fulfilling* (6.1) *there exists an element* $[\xi]$ *of the factor space* $C^\infty(C, T^*(W))/C^\infty(C, \pi\text{-hor}\,T^*(W))$ *such that for every* $\xi\in[\xi]$ *formula* (6.8) *holds. C is an extremal of the variational problem* $(W, \mathscr{L}, M)$ *if and only if* $[\xi] = 0$.

If we have a connection in the bundle W, we have the map

$$C^\infty\big(C, T^*(W)\big) \in \mathscr{L} \to \operatorname{ver}\mathscr{L} \in C^\infty\big(C, \pi\text{-}\operatorname{ver}T^*(W)\big). \tag{6.9}$$

Using (6.9), we can construct the map

$$C^\infty\big(C, T(W)\big)/C^\infty\big(C, \pi\text{-}\operatorname{hor}T^*(W)\big) \ni [\xi] \to \operatorname{ver}\xi \in C^\infty\big(C, \pi\text{-}\operatorname{ver}T^*(W)\big). \tag{6.10}$$

In this case C is an extremal if and only if $\operatorname{ver}\xi = 0$.

A situation like that has been investigated in the hydrodynamics of an incompressible fluid, cf. [9].

7. HAMILTONIAN FORMULATION OF VARIATIONAL PROBLEMS. THE LEGENDRE TRANSFORMATION

In this section we shall describe how to pass from the Lagrangian to the Hamiltonian formulations of the classical field theory. We shall construct a phase space $\mathscr{P}$ of a given physical system for which we know the Lagrangian function. This construction generalizes the notion of Legendre transformation known in mechanics, cf. [1]. We shall not develop the theory of canonical fields, physical quantities, Poisson brackets etc. These notions have been investigated in [4], [6]. Recently new results concerning these problems were obtained and published in [7]. Paper [7] essentially generalizes the results of [4], [6] and gives an elegant construction of the natural symplectic structure on the space of physical states (solutions of field equations). We also leave aside the problem of construction of a phase space in theories without a Lagrangian function. This problem has recently been partially solved and the results will be published elsewhere.

Several physical examles of phase space are given in section 8.

For every $\bar{v} \in \operatorname{tr}G^n(W)$ let the bilinear form $\bar{B}_{\bar{v}}$ (defined by (5.11)) be non-degenerate.

DEFINITION. The Legendre transformation is the map L defined by

$$\operatorname{tr}G^n(W) \ni \bar{v} \to L(\bar{v}) = \overline{\mathscr{L}'_{\operatorname{ver}}}(\bar{v}) \in \bigwedge^n T^*(W). \tag{7.1}$$

For a fixed $w \in W$ we have the mapping

$$\operatorname{tr}G^n_w(W) \ni \bar{v}_w \to L(\bar{v}_w) \in \bigwedge^n T^*_w(W). \tag{7.2}$$

The derivative of (7.2) at the point $\bar{v}_w$ is a linear mapping

$$\pi_3\text{-}\operatorname{ver}T_{\bar{v}_w}\big(\operatorname{tr}G^n(W)\big) \ni X \to B_{\bar{v}_w}(X, \cdot) \in \bigwedge^n T^*_w(W) \tag{7.3}$$

(cf. (5.10)).

It follows from the non-degeneracy of $B_{\bar{v}_w}$ that (7.3) is an injection and thus its rank is equal to $n(r-n)$. It is easy to see that the rank of L is equal to $r+n(r-n)$. It follows from the rank theorem (cf. [8]) that there exists an open neighbourhood $\mathscr{C}$ of $\bar{v}$ in $\mathrm{tr}G^n(W)$ that L maps $\mathscr{C}$ onto an $r+n(r-n)$-dimensional submanifold $\mathscr{P}$ of $\overset{n}{\wedge}T^*(W)$ and $\tau(\mathscr{C})$ is an open set in W (τ: $\overset{n}{\wedge}T^*(W)\to W$ is the canonical projection). L is a diffeomorphism of $\mathscr{C}$ onto $\mathscr{P}$. Let $\mathscr{C}$ be the maximal open set in $\mathrm{tr}G^n(W)$ such that L is a diffeomorphism $\mathscr{C}$ onto its image $\mathscr{P} = L(\mathscr{C})$. We call $\mathscr{C}$ the *configuration space* of a physical system and $\mathscr{P}$ the n-phase space of that system. $\mathscr{P}$ is a bundle over an open set $\tau(\mathscr{P}) \subset W$.

Let us consider the following diagram:

$$\begin{tikzcd} \overset{n}{\wedge}T^*(\overset{n}{\wedge}T^*(W)) \arrow[d] & \overset{n}{\wedge}T^*(W) \arrow[l, "\tau^*"'] \arrow[d, "\tau"] \\ \overset{n}{\wedge}T^*(W) \arrow[r, "\tau"] & W \end{tikzcd} \tag{7.4}$$

This diagram defines the canonical n-form on the manifold $\overset{n}{\wedge}T^*(W)$. If $z\in\overset{n}{\wedge}T^*(W)$, then

$$\omega(z) = \tau^*(z). \tag{7.5}$$

DEFINITION. The form ω which is defined by (7.5) is called the *canonical n-form* on $\overset{n}{\wedge}T^*(W)$ and $\gamma = d\omega$ is called the *canonical $(n+1)$-form* on $\overset{n}{\wedge}T^*(W)$.

In the sequel we shall denote the pull-backs of ω, γ onto $\mathscr{P}$ by the same symbols.

DEFINITION. An n-dimensional embedded submanifold S of $\mathscr{P}$ which is a section of $\pi\circ\tau$ is called *γ-singular* if for every $\pi\circ\tau$-vertical vector field X which is defined on S we have

$$(X \lrcorner \gamma)|S = 0. \tag{7.6}$$

Remark. In this definition we can consider an arbitrary vector field X defined on S. In fact, X can be decomposed into a sum $X = X_1+X_2$, where X_1 is tangent to S and X_2 is $\pi\circ\tau$-vertical. But $(X_1 \lrcorner \gamma)|S = 0$.

PROPOSITION 6. *If the n-form ψ on $\mathrm{tr}G_n(W)$ is defined by* (4.15) *and ω is the canonical n-form on $\mathscr{P}$, then $L^*\omega = \psi$.*

Proof. We consider the diagram constructed from diagrams (4.14) and (7.4):

$$
\begin{tikzcd}
\overset{n}{\wedge} T^*(\mathrm{tr}G^n(W)) \arrow[d] & \overset{n}{\wedge} T^*(W) \arrow[l, "\pi_3^*"'] \arrow[d, "\tau"] & \overset{n}{\wedge} T^*(\overset{n}{\wedge} T^*(W)) \arrow[l] \\
\mathrm{tr}G^n(W) \arrow[r, "\pi_3"] \arrow[ur, "L"] \arrow[ur, phantom, shift right=3ex, "l"] & W & \overset{n}{\wedge} T^*(W) \arrow[l, "\tau"'] \arrow[u, "\tau^*"']
\end{tikzcd}
\tag{7.7}
$$

We have

$$
\begin{aligned}
L^*\big(\omega(L(\bar{v}))\big) &= L^*\tau^*\big(L(\bar{v})\big) = (\tau\circ L)^* L(\bar{v}) \\
&= \pi_3^*\big(L(\bar{v})\big) = \psi(\bar{v}).
\end{aligned}
$$

Now we can formulate the main result of this section:

THEOREM 4. *An embedded submanifold C of W which is a section of π is an extremal section of the variational problem $(W, \mathscr{L}, M)$ if and only if the image $L(\bar{C})$ is a γ-singular submanifold of $\mathscr{P}$.*

Proof. It follows from (5.27) and (5.28) that, for every $\pi\circ\pi_3$-vertical vector field Y on $\bar{C}$, $(Y \lrcorner d\psi)|\bar{C} = 0$. But $L_* Y$ is a $\pi\circ\tau$-vertical vector field defined on $L(\bar{C})$ and $(L^{-1})^*\psi = \omega$.

Let us notice that in Hamiltonian formulations the action integral (2.1) takes the form

$$
I_f = \int_{L(\bar{f}(B))} \omega. \tag{7.8}
$$

8. EXAMPLES OF CLASSICAL FIELD THEORIES

1. Classical mechanics. We consider a k-dimensional Riemannian manifold M with a metric tensor (g_{ij}). Let $W = M\times \boldsymbol{R}$, $\mathscr{L}$: $K^1(W) \to \boldsymbol{R}$ be a Lagrangian function. If (x^i, t) are local coordinates on W and $v\in \mathrm{tr}K^1(W)$, then

$$
v = \beta^i\frac{\partial}{\partial x^i} + a\frac{\partial}{\partial t}, \qquad a \neq 0. \tag{8.1.1}
$$

We put

$$
\mathscr{L}(v) = \frac{m}{2a} g_{ij}(x^k)\beta^i\beta^j - aV(x^k, t), \tag{8.1.2}
$$

where $V\in C^\infty(W)$ is a potential function.

In local coordinates

$$
\mathscr{L}'_{\mathrm{ver}}(v) = mg_{ij}(x^k)\frac{\beta}{a}dx^j - \left(\frac{m}{2}\cdot g_{ij}(x^k)\frac{\beta^i\beta^j}{a^2} + V(x^k, t)\right)dt. \tag{8.1.3}
$$

Let $\gamma^i = \beta^i/a$, $1 \leqslant i \leqslant k$, be local coordinates in a fibre of π_3: $\mathrm{tr}G^1(W) \to W$. We have

$$\psi(\bar{v}) = mg_{ij}(x^k)\gamma^i dx^j - E(t, x^k, \gamma^k)\,dt, \tag{8.1.4}$$

where

$$E(t, x^k, \gamma^k) = \frac{m}{2} g_{ij}(x^k)\gamma^i\gamma^j + V(x^k, t) \tag{8.1.5}$$

is the energy. Equation (5.27): $(X \lrcorner d\psi)|C = 0$ for every $\pi\circ\pi_3$-vertical vector field X on C:

$$X = B^k \cdot \frac{\partial}{\partial x^k} + C^k \frac{\partial}{\partial \gamma^k}, \tag{8.1.6}$$

where

$$C = \{(t, x^i(t) \;\; \gamma^i(t)); t \in \boldsymbol{R}\}, \tag{8.1.7}$$

gives

$$\begin{gathered} \frac{dx^j}{dt} = \gamma^j, \\ \frac{d^2x^p}{dt^2} + \Gamma^p_{ij}(x^k)\frac{dx^i}{dt}\cdot\frac{dx^j}{dt} = -\frac{1}{m} g^{pa}(x^k)\frac{\partial V(x^k, t)}{\partial x^a}, \end{gathered} \tag{8.1.8}$$

where Γ^p_{ij} are the coefficients of the Riemannian connection.

If $V = 0$ we obtain equations of geodesic lines in M cf. [8]. The condition of non-degeneracy of $\bar{B}_{\bar{v}}$ is here fulfilled because

$$\det\left[\frac{\partial\bar{\mathscr{L}}(t, x^k, \gamma^k)}{\partial\gamma^i\partial\gamma^j}\right] = \det[mg_{ij}(x^k)] \neq 0 \qquad \text{(see (5.13))}.$$

The phase space $\mathscr{P} \subset T^*(W)$ is

$$\mathscr{P} = \{\nu \in T^*(W)\colon\ \nu(t, x^k) = p_j dx^j - H(t, x^k, p_k)\,dt\}, \tag{8.1.9}$$

where

$$p_j = mg_{ji}(x^k)\gamma^i, \qquad H(t, x^k, p_k) = \frac{1}{2m} g^{ij}(x^k)p_i p_j + V(t, x^k). \tag{8.1.10}$$

The canonical 1-form on $\mathscr{P}$ is $\omega(t, x^k, p_k) = p_j dx^j - H(t, x^k, p_k)\,dt$ and the canonical 2-form is

$$\gamma = d\omega = dp_j \wedge dx^j - dH(t, x^k, p_k) \wedge dt, \tag{8.1.11}$$

where H is given by (8.1.10).

2. Relativistic mechanics. Let M be a pseudoriemannian manifold with a metric tensor $(g_{\mu\nu})$ with a signature $(+, -, -, -)$. M is not a bundle, but we can develop constructions given in section 5 because $n = 1$.

Let $K^1_+(M) \subset K^1(M)$ be an open cone consisting of such vectors v that $(v|v) > 0$. In local coordinates (x^μ) on M,

$$v = a^\mu \frac{\partial}{\partial x^\mu}, \quad g_{\mu\nu}(x^\lambda) a^\mu a^\nu > 0. \tag{8.2.1}$$

Let $\mathscr{L}\colon K^1_+(M) \to \boldsymbol{R}$ be a Lagrangian function,

$$\mathscr{L}(v) = m\sqrt{(v|v)} + e\langle v|A\rangle, \tag{8.2.2}$$

where

$$A = A_\mu(x^\lambda) dx^\mu \tag{8.2.3}$$

is a given covector field on M. In local coordinates (x^μ, a^μ) we have

$$\mathscr{L}(v) = m\sqrt{g_{\mu\nu}(x^\lambda) a^\mu a^\nu} + e a^\mu A_\mu(x^\lambda), \tag{8.2.4}$$

$$\mathscr{L}'_{\mathrm{ver}}(v) = \big(g_{\mu\nu}(x^\lambda) a^\mu a^\nu\big)^{-1/2} m \cdot g_{\mu\nu}(x^\lambda) a^\mu dx^\nu + e \cdot A_\mu(x^\lambda) dx^\mu. \tag{8.2.5}$$

Let $G^1_+(M) = \pi_2\big(K^1_+(M)\big)$, where $\pi_2\colon K^1(M) \to G^1(M)$. In $G^1_+(M)$ we have local coordinates

$$(x^\mu, \gamma^\mu);\ \gamma^\mu = \big(g_{\tau\nu}(x^\lambda) a^\tau a^\nu\big)^{-1/2} a^\lambda, \tag{8.2.6}$$

where $g_{\mu\nu}(x^\lambda)\gamma^\mu\gamma^\nu = 1$,

$$\psi(x^\mu, \gamma^\mu) = m \cdot g_{\mu\nu}(x^\lambda)\gamma^\mu dx^\nu + e A_\mu(x^\lambda) dx^\mu, \tag{8.2.7}$$

and

$$d\psi(x^\mu, \gamma^\mu) = m \cdot g_{\mu\nu}(x^\lambda) d\gamma^\mu \wedge dx^\nu + m \frac{\partial}{\partial x^\tau}\big(g_{\mu\nu}(x^\lambda)\big) + \gamma^\mu dx^\tau A dx^\nu) + \\ + e \frac{\partial A_\mu(x)}{\partial x^\nu} dx^\nu \wedge dx^\mu, \tag{8.2.8}$$

where

$$\frac{\partial}{\partial x^\tau}\big(g_{\mu\nu}(x^\lambda)\big)\gamma^\mu\gamma^\nu dx^\tau + 2 g_{\mu\nu}(x^\lambda)\gamma^\mu d\gamma^\nu = 0. \tag{8.2.9}$$

If X is tangent to $G^1_+(M)$, we have

$$X = P^\mu \frac{\partial}{\partial \gamma^\mu} + Q^\nu \frac{\partial}{\partial x^\nu}, \tag{8.2.10}$$

where

$$\frac{\partial}{\partial x^\tau}\big(g_{\mu\nu}(x^\lambda)\big) Q^\tau \gamma^\mu\gamma^\nu + 2 \cdot g_{\mu\nu}(x^\lambda) P^\nu \gamma^\mu = 0. \tag{8.2.11}$$

Let C be a one-dimensional submanifold of $G^1_+(W)$,

$$C = \big\{\big(x^\mu(\tau), \gamma^\mu(\tau)\big)\colon g_{\mu\nu}\gamma^\mu\gamma^\nu = 1,\ \tau \in \boldsymbol{R}\big\}. \tag{8.2.12}$$

If we use (8.2.11) and (8.2.12), then equation $(X \lrcorner d\psi)|C = 0$ will give us

$$\frac{dx}{d\tau} = c(\tau)\gamma^{\mu},$$

(8.2.13)

$$c(\tau)\frac{d}{d\tau}\gamma^{\mu} + \Gamma^{\mu}_{\alpha\beta}(x^{\lambda})\frac{dx^{\alpha}}{d\tau}\cdot\frac{dx^{\beta}}{d\tau} = \frac{e}{m}g^{\mu\alpha}f_{\alpha\beta}\big(x^{\lambda}(\tau)\big)\cdot c(\tau)\frac{dx_{\beta}}{d\tau},$$

where $\tau \to c(\tau)$ is a non-vanishing function and

$$f_{\alpha\beta}(x^{\lambda}) = \partial_{\alpha}A_{\beta}(x^{\lambda}) - \partial_{\beta}A_{\alpha}(x^{\lambda}).$$

Let us introduce a new parametrization of C:

(8.2.14) $$s = \int c(\tau)\,d\tau, \qquad \frac{d\tau}{ds} = \frac{1}{c(\tau)}.$$

We obtain

$$\frac{dx^{\mu}}{ds} = \gamma^{\mu},$$

(8.2.15)

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\big(x^{\lambda}(s)\big)\frac{dx^{\alpha}}{ds}\cdot\frac{dx^{\beta}}{ds} = \frac{e}{m}f^{\mu}{}_{\beta}\big(x^{\lambda}(s)\big)\frac{dx_{\beta}}{ds}.$$

We put

(8.2.16) $$p_{\mu} = mg_{\mu\nu}(x^{\lambda})\gamma^{\nu}, \qquad \text{where } g^{\mu\nu}(x)p_{\mu}p_{\nu} = m^2.$$

The phase space is

(8.2.17) $$\mathscr{P} = \big\{\nu \in T^*(M)\colon\ \nu = \big(p_{\mu} + e\cdot A_{\mu}(x^{\lambda})\big)dx^{\mu},\ g^{\mu\nu}(x^{\lambda})p_{\mu}p_{\nu} = m^2\big\}.$$

The canonical 1-form on $\mathscr{P}$ is given by

(8.2.18) $$\omega(x^{\mu}, p_{\mu}) = \big(p_{\nu} + e\cdot A_{\nu}(x^{\lambda})\big)dx^{\nu}.$$

3. Theory of a scalar field. Let M be a 4-dimensional pseudoriemannian manifold with the metric tensor $(g_{\mu\nu})$ which has the signature $(+, -, -, -)$. Let $g = \det[g_{\mu\nu}]$. In $W = M \times \boldsymbol{R}$ we have local coordinates (x^{μ}, φ).

Let $v \in \operatorname{tr} K^4(W)$:

(8.3.1) $$v = v_0 \wedge v_1 \wedge v_2 \wedge v_3,$$

(8.3.2) $$v_{\mu} = w_{\mu} + b_{\mu}, \qquad \text{where } w_{\mu} = a^{\nu}_{\mu}\frac{\partial}{\partial x^{\nu}}, \qquad b_{\mu} = \beta_{\mu}\frac{\partial}{\partial\varphi},$$

$$\mu = 0, 1, 2, 3.$$

From the transversality of v we have $\det[a^\nu_\mu] \neq 0$. Let $\det[a^\nu_\mu] > 0$. Let

$$v_4 = b_4 = \frac{\partial}{\partial\varphi}. \tag{8.3.3}$$

The dual basis consists of vectors

$$v^{*\mu} = w^{*\mu}, \quad v^{*4} = b^{*4} - \beta_\nu w^{*\nu}, \quad w^{*\mu} = \overset{-1}{a}{}^\mu_\nu dx^\nu, \quad \mu = 0, 1, 2, 3 \text{ and } b^{*4} = d\varphi. \tag{8.3.4}$$

The Lagrangian function is given by

$$\mathscr{L}(v) = \sqrt{-A}\left(\tfrac{1}{2}(w^{*\mu}|w^{*\nu})\beta_\mu\beta_\nu - G(\varphi)\right), \tag{8.3.5}$$

where

$$A = (w_0\wedge\ldots\wedge w_3|w_0\wedge\ldots\wedge w_3) = \det(w_\mu|w_\nu) = g\cdot(\det[a^\nu_\mu])^2, \tag{8.3.6}$$

and G is an arbitrary smooth function of one variable;

$$\mathscr{L}'_{\mathrm{ver}}(v) = \sqrt{-A}\left(\tfrac{1}{2}(w^{*\mu}|w^{*\nu})\beta_\mu\beta_\nu + G(\varphi)\right)w^{*0}\wedge\ldots\wedge w^{*3} + \sqrt{-A}\sum_{\nu=0}^{3}(w^{*\mu}|w^{*\nu})\beta_\mu w^{*0}\wedge\ldots\wedge\underbrace{b^{*4}}_{\nu}\wedge\ldots\wedge w^{*3}. \tag{8.3.7}$$

If $\gamma_\mu = \overset{-1}{a}{}^\nu_\mu\beta_\nu$, $\mu = 0, 1, 2, 3$, are local coordinates in a fibre of $\mathrm{tr}G^4(W)\to W$, then

$$\mathscr{L}(v) = \sqrt{-g}\det[a^\nu_\mu]\left(\tfrac{1}{2}g^{\mu\nu}(x^\lambda)\gamma_\mu\gamma_\nu - G(\varphi)\right), \tag{8.3.5'}$$

$$\begin{aligned} w^{*0}\wedge\ldots\wedge\underbrace{b^{*4}}_{\nu}\wedge\ldots\wedge w^{*3} &= (\det[a^\nu_\mu])^{-1}\sum_{\mu=0}^{3}a^\mu_\nu dx^0\wedge\ldots\wedge\underbrace{d\varphi}_{\mu}\wedge\ldots\wedge dx^3, \\ w^{*0}\wedge\ldots\wedge w^{*3} &= (\det[a^\nu_\mu])^{-1}dx^0\wedge\ldots\wedge dx^3. \end{aligned} \tag{8.3.8}$$

Using (8.3.7) and (8.3.8), we obtain

$$\psi(v) = -\left(\tfrac{1}{2}g^{\mu\nu}(x^\lambda)\gamma_\mu\gamma_\nu + G(\varphi)\right)\sqrt{-g}\,dx^0\wedge\ldots\wedge dx^3 + \sum_{\nu=0}^{3}g^{\mu\nu}(x^\lambda)\gamma_\mu\sqrt{-g}\,dx^0\wedge\ldots\wedge\underbrace{d\varphi}_{\nu}\wedge\ldots\wedge dx^3. \tag{8.3.7'}$$

The non-degeneracy condition is here fulfilled because

$$\det\left[\frac{\partial^2\mathscr{L}}{\partial\gamma_\mu\partial\gamma_\nu}\right] = \det[\sqrt{-g}\,g_{\mu\nu}] \neq 0.$$

If

$$X = A^\mu \frac{\partial}{\partial x^\mu} + B\frac{\partial}{\partial \varphi} + C_\mu \frac{\partial}{\partial \gamma_\mu},$$

$$C = \{(x^\mu, \varphi(x^\mu), \gamma_\nu(x^\mu)) \in \operatorname{tr} G^4(W) \colon (x^\mu) \in M\},$$

then the equation $(X \lrcorner d\psi)|C = 0$ gives

$$\frac{\partial \varphi}{\partial x^\mu} = \gamma_\mu,$$

(8.3.9)

$$(-g)^{-1/2} \frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(x^\lambda)\cdot\sqrt{-g}\,\frac{\partial\varphi}{\partial x^\nu}\right) + G'(\varphi) = 0.$$

Using the Laplace–Beltrami operator (cf. [3]), we can write the second equation in (8.3.9) in the form

$$\Box\varphi + G'(\varphi) = 0.$$

Let $\eta^\mu = g^{\mu\nu}(x^\lambda)\gamma_\nu$. The phase space is

(8.3.10)
$$\mathscr{P} = \{\nu \in \bigwedge^4 T^*(W) \colon \nu(x^\mu, \varphi)\} = \sum_{\mu=0}^{3} \eta^\mu \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{d\varphi}_{\mu} \wedge \ldots \wedge dx^3 - H(x^\lambda, \varphi, \eta^\lambda)\sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3,$$

where

(8.3.11)
$$H(x^\lambda, \varphi, \eta^\lambda) = \tfrac{1}{2}(\eta|\eta) + G(\varphi).$$

The canonical 4-form on $\mathscr{P}$ is equal to

(8.3.12)
$$\omega(x^\lambda, \varphi, \eta^\lambda) = \sum_{\mu=0}^{3} \eta^\mu \sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{d\varphi}_{\mu} \wedge \ldots \wedge dx_3 - H(x^\lambda, \varphi, \eta^\lambda)\sqrt{-g}\cdot dx^0 \wedge \ldots \wedge dx^3.$$

4. Non-linear electrodynamics. Let M be as in section 3. Let $W = T^*(M)$ with local coordinates (x^μ, A_μ) and $\pi\colon T^*(M) \to M$ be the projection. For $v_\mu \in T(W)$, $\mu = 0, 1, 2, 3$,

(8.4.1)
$$v_\mu = a^\nu_\mu \frac{\partial}{\partial x^\nu} + b_{\mu\nu}\frac{\partial}{\partial A_\nu}, \qquad \pi_* v_\mu = a^\nu_\mu \frac{\partial}{\partial x^\nu},$$

and

(8.4.2)
$$\operatorname{ver} v_\mu = (b_{\mu\nu} - \Gamma^\varrho_{\nu\lambda} a^\lambda_\mu \cdot A_\varrho)\frac{\partial}{\partial A_\nu},$$

where $\operatorname{ver} v_\mu$ is the vertical component of v_μ which is determined by the linear connection corresponding to the Riemannian structure of M. By $\sim$ we denote the natural injection $\pi\text{-}\operatorname{ver} T(T^*(M))$ in $T^*(M)$.

We have

$$\overline{\mathrm{ver}\, v_\mu} = (b_{\mu\nu} - \Gamma^{\varrho}_{\nu\lambda} a^{\lambda}_{\mu} A_{\varrho}) \cdot dx^{\nu}. \tag{8.4.3}$$

Let

$$v \in \mathrm{tr}\, K^4(W), \quad v = v_0 \wedge v_1 \wedge v_2 \wedge v_3 \quad \text{and} \quad \det[a^{\nu}_{\mu}] \neq 0. \tag{8.4.4}$$

Let $(u_\mu)^3_{\mu=0}$ be π-vertical linearly independent vectors tangent to $T^*(M)$; e.g.,

$$u_\mu = \frac{\partial}{\partial A_\mu}. \tag{8.4.5}$$

Let $w_\mu = \pi_* v_\mu$ and $(w^{*\mu})^3_{\mu=0}$ be the dual basis

$$w^{*\mu} = \overset{-1}{a}{}^{\mu}_{\nu} dx^{\nu}. \tag{8.4.6}$$

Let

$$u^{\lambda}_{\tau} = dA_\tau - b_{\mu\tau} \overset{-1}{a}{}^{\mu}_{\nu} dx^{\nu}, \quad \tau = 0, 1, 2, 3. \tag{8.4.7}$$

Covectors $v^{*\mu} = \pi^* w^{*\mu}$ and u^*_τ form a basis of $T^*(W)$ at the given point.

We define a 2-covector f at the point $x \in M$, where $x = \pi(\pi_* v)$:

$$f = w^{*\mu} \wedge \overline{\mathrm{ver}\, v_\mu}. \tag{8.4.8}$$

In local coordinates:

$$\begin{aligned} f &= \sum_{\mu<\nu} f_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = \sum_{\mu<\nu} (\overset{-1}{a}{}^{\tau}_{\mu} b_{\tau\nu} - \overset{-1}{a}{}^{\tau}_{\nu} b_{\tau\mu}) dx^{\mu} \wedge dx^{\nu} \\ &= \sum_{\mu<\nu} (\overset{-1}{a}{}^{\tau}_{\mu} \beta_{\tau\nu} - \overset{-1}{a}{}^{\tau}_{\nu} \beta_{\tau\mu}) dx^{\mu} \wedge dx^{\nu}, \end{aligned} \tag{8.4.9}$$

where

$$\beta_{\tau\mu} = b_{\tau\mu} - \Gamma^{\varrho}_{\mu\lambda} a^{\lambda}_{\tau} A_{\varrho}. \tag{8.4.10}$$

By means of the Hodge operator $*$ (cf. [3]) we define the dual tensor $\check{f} = *f$. In local coordinates:

$$\check{f}_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\tau\varrho} f^{\tau\varrho}. \tag{8.4.11}$$

For a physical reason we shall assume that a Lagrangian function is of the form

$$\mathscr{L}(v) = \sqrt{-B}\big(\mathscr{L}_0\big((f|f), (f|\check{f})\big) + \mathscr{L}_I(x^{\mu}, A_\mu)\big), \tag{8.4.12}$$

where

$$\begin{aligned} B &= (w_0 \wedge \ldots \wedge w_3 | w_0 \wedge \ldots \wedge w_3) = \det(w_\mu | w_\nu) \\ &= (\det[a^{\nu}_{\mu}])^2 \cdot g. \end{aligned} \tag{8.4.13}$$

We have

$$(8.4.14)\qquad (f|w^{*\mu}\wedge\tilde{u}^{\nu}) = \tfrac{1}{2}\overset{-1}{a}{}^{\mu}_{\lambda}f_{\tau\varrho}(g^{\nu\varrho}g^{\lambda\tau}-g^{\lambda\varrho}g^{\nu\tau}),$$

$$(8.4.15)\qquad (\check{f}|w^{*\mu}\wedge\tilde{u}^{\nu}) = \tfrac{1}{4}\sqrt{-g}\,\varepsilon_{\lambda\varrho\tau\xi}f^{\tau\xi}\overset{-1}{a}{}^{\mu}_{\alpha}(g^{\varrho\nu}g^{\lambda\alpha}-g^{\lambda\nu}g^{\varrho\alpha}),$$

$$(8.4.16)\qquad v^{*0}\wedge\ldots\wedge v^{*3} = (\det[a^{\nu}_{\mu}])^{-1}dx^0\wedge\ldots\wedge dx^3,$$

$$(8.4.17)\qquad v^{*0}\wedge\ldots\wedge\underbrace{u^{*}_{\nu}}_{\mu}\wedge\ldots\wedge v^{*3} = (\det[a^{\nu}_{\mu}])^{-1}\sum_{\lambda=0}^{3}a^{\lambda}_{\mu}dx^0\wedge\ldots\wedge\underbrace{dA_{\nu}}_{\lambda}\wedge\ldots\wedge dx^3-$$
$$-(\det[a^{\nu}_{\mu}])^{-1}b_{\mu\nu}dx^0\wedge\ldots\wedge dx^3.$$

Using (8.4.14)–(8.4.17) and lemma 6, we obtain

$$(8.4.18)\qquad \mathscr{L}'_{\mathrm{ver}}(v) = \big(\mathscr{L}_0\big((f|f),(f|\check{f})\big)+\mathscr{L}_I(x^{\mu},A_{\mu})\big)\sqrt{-g}\,dx^0\wedge\ldots\wedge dx^3-$$
$$-\big(D_1\mathscr{L}_0\big((f|f),(f|\check{f})\big)f^{\mu\nu}f_{\mu\nu}+D_2\mathscr{L}_0\big((f|f),(f|\check{f})\big)\check{f}_{\mu\nu}f^{\mu\nu}\big)\times$$
$$\times\sqrt{-g}\,dx^0\wedge\ldots\wedge dx^3+$$
$$+2D_1\mathscr{L}_0\big((f|f),(f|\check{f})\big)f^{\nu\lambda}\sqrt{-g}\cdot dx^0\wedge\ldots\wedge\underbrace{dA_{\lambda}}_{\nu}\wedge\ldots\wedge dx^3+$$
$$+2D_2\mathscr{L}_0\big((f|f),(f|\check{f})\big)\check{f}^{\nu\lambda}\sqrt{-g}\cdot dx^0\wedge\underbrace{\ldots}_{\nu}\wedge dA_{\lambda}\wedge\ldots\wedge dx^3.$$

Remark. Symbols D_1, D_2 denote partial derivatives of the function $\mathscr{L}_0$, which is an arbitrary function of 2 variables.

In local coordinates:

$$(8.4.19)\qquad (f|f) = \tfrac{1}{2}f_{\mu\nu}f^{\mu\nu},\qquad (f|\check{f}) = \tfrac{1}{2}f_{\mu\nu}\check{f}^{\mu\nu}.$$

In what follows symbol $\partial/\partial f_{\mu\nu}$ will denote a differentiation with respect to independent components of the antysymmetric tensor $f_{\mu\nu}$.

We have from (8.4.19)

$$(8.4.20)\qquad \frac{\partial}{\partial f_{\mu\nu}}(f|f) = 2\cdot f^{\mu\nu},\qquad \frac{\partial}{\partial f_{\mu\nu}}(f|\check{f}) = 2\check{f}^{\mu\nu}.$$

From (8.4.18) and (8.4.20) we obtain

$$(8.4.21)\qquad \psi(x^{\lambda},A_{\lambda},f_{\lambda\varrho}) = \frac{\partial\mathscr{L}_0}{\partial f_{\mu\nu}}\sqrt{-g}\,dx^0\wedge\ldots\wedge\underbrace{dA_{\nu}}_{\mu}\wedge\ldots\wedge dx^3+$$
$$+\left(-\frac{1}{2}\frac{\partial\mathscr{L}_0}{\partial f_{\mu\nu}}f_{\mu\nu}+\mathscr{L}_0+\mathscr{L}_I\right)\sqrt{-g}\,dx^0\wedge\ldots\wedge dx^3.$$

If $\gamma_{\mu\nu} = b_{\lambda\nu}\overset{-1}{a}{}^{\lambda}_{\mu}$, then $f_{\mu\nu} = \gamma_{\mu\nu}-\gamma_{\nu\mu}$ (see (8.4.9)).

If

$$C = \{(\varpi^\lambda, A_\mu(\varpi^\lambda), \gamma_{\mu\nu}(\varpi^\lambda)) : (\varpi^\lambda) \in M\},$$

$$X = Q_\nu \frac{\partial}{\partial A_\nu} + P_{\mu\nu} \frac{\partial}{\partial \gamma_{\mu\nu}},$$

then the equation $(X \lrcorner d\psi)|C = 0$ gives

$$\gamma_{\mu\nu} = \partial_\mu A_\nu, \quad \nabla_\mu \frac{\partial \mathscr{L}_0}{\partial f_{\mu\nu}} = \frac{\partial \mathscr{L}_I}{\partial A_\nu}, \tag{8.4.22}$$

where ∇ denotes the covariant derivative corresponding to the metric $(g_{\mu\nu})$. We have also

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{8.4.23}$$

Let $h^{\mu\nu} = -4\pi\, \partial\mathscr{L}_0/\partial f_{\mu\nu}$; then we have

$$\nabla_\mu h^{\nu\mu} = 4\pi \frac{\partial \mathscr{L}_I}{\partial A_\nu}. \tag{8.4.24}$$

Equations (8.4.23) and (8.4.24) form a complete set of field equations. Let us notice that $\gamma_{\mu\nu}$ cannot be obtained from them (only $f_{\mu\nu}$). In Maxwell electrodynamics

$$\mathscr{L}_0 = -\frac{1}{16\pi} f_{\mu\nu} f^{\mu\nu}, \quad \mathscr{L}_I = j^\nu(\varpi^\lambda) A_\nu,$$

and we have

$$h^{\mu\nu} = f^{\mu\nu}, \quad \nabla_\mu f^{\nu\mu} = 4\pi h j^\nu(\varpi^\lambda).$$

The non-degeneracy condition is for non-linear electrodynamics equivalent to the condition

$$\det\left[\frac{\partial^2 \mathscr{L}_0}{\partial f_{\mu\nu} \partial f_{\lambda\varrho}}\right]_{\substack{\mu<\nu \\ \lambda<\varrho}} \neq 0. \tag{8.4.25}$$

If condition (8.4.25) is fulfilled, we can determine from (8.4.24)

$$f_{\mu\nu} = f_{\mu\nu}(h^{\lambda\varrho}). \tag{8.4.26}$$

The phase space is

$$\mathscr{P} = \{\nu \in \overset{4}{\bigwedge} T^*(W) : \nu(\varpi^\lambda, A_\lambda) \tag{8.4.27}$$

$$= -\frac{1}{4\pi} h^{\mu\nu} \sqrt{-g}\, d\varpi^0 \wedge \ldots \wedge \underbrace{dA_\nu}_{\mu} \wedge \ldots \wedge d\varpi^3 + H(\varpi^\lambda, A_\lambda, h^{\lambda\varrho}) \sqrt{-g}\, d\varpi^0 \wedge \ldots \wedge d\varpi^3,$$

where

$$\begin{aligned} &(1)\ H(\varpi^\lambda, A_\lambda, h^{\lambda\varrho}) = \frac{1}{8\pi} h^{\mu\nu} f_{\mu\nu}(h^{\lambda\varrho}) + \mathscr{L}_0\big(f_{\mu\nu}(h^{\lambda\varrho})\big) + \mathscr{L}_I(\varpi^\lambda, A_\lambda), \\ &(2)\ h^{\mu\nu} = -h^{\nu\mu}. \end{aligned} \tag{8.4.28}$$

The canonical 4-form on $\mathscr{P}$ is

$$\omega(x^\lambda, A_\lambda, h^{\lambda\varrho}) = -\frac{1}{4\pi} h^{\mu\nu}\sqrt{-g}\, dx^0 \wedge \ldots \wedge \underbrace{dA_\nu}_{\mu} \wedge \ldots \wedge dx^3 + \tag{8.4.29}$$

$$+ H(x^\lambda, A_\lambda, h^{\lambda\varrho})\sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3.$$

9. INVARIANCE OF LAGRANGIAN SYSTEMS. THE NOETHER THEOREM

Let $\pi\colon W \to B$ be a bundle over an n-dimensional manifold B and let $\mathscr{L}$ be a Lagrangian function on $K^n(W)$. Let (T, F) be a morphism of W, i.e., the following diagram is commutative:

$$\begin{tikzcd} W \arrow[r, "F"] \arrow[d, "\pi"] & W \arrow[d, "\pi"] \\ B \arrow[r, "T"] & B \end{tikzcd} \tag{9.1, 9.2}$$

We assume that T is a diffeomorphism of B.

Diagram (9.1) induces the following diagram:

$$\begin{tikzcd} \operatorname{tr} K^n(W) \arrow[rr, "F_*"] \arrow[d, "\pi_1"] \arrow[ddd, bend right=60, "\pi_*"'] & & \operatorname{tr} K^n(W) \arrow[d, "\pi_1"'] \arrow[ddd, bend left=60, "\pi_*"] \\ W \arrow[rr, "F"] \arrow[d, "\pi"] & & W \arrow[d, "\pi"] \\ B \arrow[rr, "T"] & & B \\ K^n(B) \arrow[uur] \arrow[rrr, "T_*"] & & & K^n(B) \arrow[uul] \end{tikzcd} \tag{9.3}$$

It follows from (9.2) that, for $v \in \operatorname{tr} K^n(W)$, $F_*(v) \neq 0$ and

$$\pi_* F_*(v) \neq 0. \tag{9.4}$$

DEFINITION. We say that $\mathscr{L}$ is *F-invariant* if

$$\mathscr{L} \circ F_* = \mathscr{L}. \tag{9.5}$$

LEMMA 13. *If $\mathscr{L}$ is F-invariant, then, for every section of π, $f\colon B \to W$, $I_f = I_{F\circ f\circ T^{-1}}$.*

Proof.

$$I_f = \int_{z\in B} \mathscr{L}\big(f_*\nu(z)\big)\omega(z) = \int_{z\in B} \mathscr{L}\big((F_*\circ f_*)\nu(z)\big)\omega(z)$$
$$= \int_{x\in B} \mathscr{L}\big((F\circ f\circ T^{-1})_*(T_*\nu)(x)\big)\cdot\big((T^{-1})^*\omega\big)(x) = I_{F\circ f\circ T^-},$$

where $x = T(z)$.

LEMMA 14. *Let $\mathscr{L}$ be F-invariant. For every $u_w \in \overset{n}{\bigwedge} T_w(W)$ we have*

$$\langle F_*(u_w) \,|\, \mathscr{L}'_{\mathrm{ver}}\big(F_*(v_w)\big)\rangle = \langle u_w \,|\, \mathscr{L}'_{\mathrm{ver}}(v_w)\rangle, \quad v_w \in \mathrm{tr}\,K^n_w(W). \tag{9.6}$$

Proof. This formula follows from the linearity of the mapping F_*: $\mathrm{tr}\,K^n_w(W) \to \mathrm{tr}\,K^n_{F(w)}(W)$.

Let $\bar{F}_*$: $\mathrm{tr}\,G^n(W) \to \mathrm{tr}\,G^n(W)$ be the map generated by F_*: $\mathrm{tr}\,K_n(W) \to \mathrm{tr}\,K^n(W)$.

PROPOSITION 7. *If $\mathscr{L}$ is F-invariant, then $(\bar{F}_*)^*\psi = \psi$, where $\psi(\cdot) = \pi_3^*\circ\bar{\mathscr{L}}'_{\mathrm{ver}}(\cdot)$ (cf. (4.15))*

Proof. We shall use the following diagram:

$$\begin{tikzcd}
\mathrm{tr}\,G^n(W) \arrow[d, "\pi_3"'] & K^n\big(\mathrm{tr}\,G^n(W)\big) \arrow[l] \arrow[r, "(\bar{F}_*)_*"] \arrow[d, "\pi_{3*}"] & K^n\big(\mathrm{tr}\,G^n(W)\big) \arrow[r] \arrow[d, "\pi_{3*}"] & \mathrm{tr}\,G^n(W) \arrow[d, "\pi_3"] \\
W & \mathrm{tr}\,K^n(W) \arrow[l, "\pi_1"'] \arrow[r, "F_*"] \arrow[d, "\pi_3"] & \mathrm{tr}\,K^n(W) \arrow[r, "\pi_1"] \arrow[d, "\pi_3"] & W \\
 & \mathrm{tr}\,G^n(W) \arrow[ul, "\pi_3"] \arrow[r, "F"] & \mathrm{tr}\,G^n(W) \arrow[ur, "\pi_3"'] &
\end{tikzcd} \tag{9.7}$$

Let $\bar{v} \in \mathrm{tr}\,G^n(W)$ and $q \in K^n_{\bar{v}}\big(\mathrm{tr}\,G^n(W)\big)$. We have from the diagram

$$\begin{aligned}\langle F_*\pi_{3*}q \,|\, \bar{\mathscr{L}}'_{\mathrm{ver}}\big(\bar{F}_*(v)\big)\rangle &= \langle \pi_{3*}(\bar{F}_*)_*q \,|\, \bar{\mathscr{L}}'_{\mathrm{ver}}\big(\bar{F}_*(\bar{v})\big)\rangle \\ &= \langle (\bar{F}_*)_*q \,|\, \pi_3^*\circ\bar{\mathscr{L}}'_{\mathrm{ver}}\big(\bar{F}_*(\bar{v})\big)\rangle = \langle q \,|\, \big((\bar{F}_*)^*\psi\big)(\bar{v})\rangle.\end{aligned} \tag{9.8}$$

If we use (9.6) and (9.8), we shall obtain $(\bar{F}_*)^*\psi = \psi$.

Now we shall generalize the notion of the F-invariant Lagrangian function.

DEFINITION. We say that a Lagrangian function $\mathscr{L}$ is *F-invariant in a generalized sense* if there exists a complete n-form Ω on W (i.e., $\Omega = d\omega$) such that

$$(\mathscr{L}\circ F_*)(v) = \mathscr{L}(v) + \langle v \,|\, \Omega\rangle, \quad v \in \mathrm{tr}\,K^n(W). \tag{9.9}$$

In this situation we have, for $v \in \mathrm{tr}\,K^n_w(W)$, $u \in \overset{n}{\bigwedge} T_w(W)$,

$$\langle F_*(u) \,|\, \mathscr{L}'_{\mathrm{ver}}\big(F_*(v)\big)\rangle = \langle u \,|\, \mathscr{L}'_{\mathrm{ver}}(v)\rangle + \langle u \,|\, P_v\,\Omega\rangle, \tag{9.10}$$

where P_v is the projector on the subspace of 1-vertical forms on W (see (4.10)).

From formulae (9.10) and (5.32) we obtain

$$(9.11)\qquad \big((\bar{F}_*)^*\psi\big)(\bar{v}) = \psi(\bar{v}) + \pi_3^* P_{\bar{v}}\Omega, \qquad \bar{v}\in \mathrm{tr}G^n(W).$$

DEFINITION. We say that a Lagrangian function $\mathscr{L}$ is *invariant* (in the generalized sense) *with respect to a one-parameter family* $(F_\varepsilon)_{\varepsilon\in]-\delta,\delta[}$ *of transformations of* W if there exists a one-parameter family $(\Omega_\varepsilon)_{\varepsilon\in]-\delta,\delta[}$ of complete n-forms on W such that

$$(9.12)\qquad (\mathscr{L}\circ F_{\varepsilon*})(v) = \mathscr{L}(v) + \langle v\,|\,\Omega_\varepsilon\rangle \quad \varepsilon\in\,]-\delta,\,\delta[, \qquad v\in \mathrm{tr}K^n(W).$$

We assume the differentiability of mappings:

$$]-\delta,\,\delta[\,\times W \ni (\varepsilon, w)\to F_\varepsilon(w)\in W,$$

$$]-\delta,\,\delta[\,\ni \varepsilon\to\Omega_\varepsilon\in C^\infty\big(\overset{n}{\wedge}T^*(W)\big).$$

(We take the compact convergence topology in $C^\infty\big(\overset{n}{\wedge}T^*(W)\big)$.)

The main result of this section is the following

THEOREM 5 (Noether). *Let* $(F_\varepsilon)_{\varepsilon\in]-\delta,\delta[}$ *be a one-parameter family of transformations of* W *and let* $(\Omega_\varepsilon)_{\varepsilon\in]-\delta,\delta[}$ *be a one-parameter family of complete* n*-forms on* W. *Let* $\mathscr{L}$ *be a Lagrangian function which is invariant with respect to the family* (F_ε) *in the sense of* (9.12).

If $f\colon B\to W$ *is an extremal section of a variational problem* $(W, \mathscr{L}, M)$, *then there exist a vector field* X *on* $\mathrm{tr}G^n(W)$ *and an* $(n-1)$*-form* η *on* W *such that*

$$(9.13)\qquad d(X\lrcorner\,\psi - \pi_3^*\eta)\,|\bar{f}(B) = 0.$$

Proof. We known from (9.11) that

$$(9.14)\qquad \big((\bar{F}_{\varepsilon*})^*\psi\big)(\bar{v}) = \psi(\bar{v}) + \pi_3^* P_{\bar{v}}\Omega_\varepsilon.$$

If we differentiate (9.14) with respect to ε, we shall obtain

$$(9.15)\qquad \pounds_X\psi(\bar{v}) = \pi_3^*(P_{\bar{v}}\chi),$$

where $X = \dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\bar{F}_{\varepsilon*}$ is the vector field on $\mathrm{tr}G^n(W)$ which generates (F_ε), and $\chi = \dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Omega$ is a complete n-form on W (the space of complete forms is complete in the compact convergence topology); therefore there exists an $n-1$ form η on W such that $\chi = d\eta$. From (9.15) we obtain

$$(9.16)\qquad X\lrcorner\,d\psi + d(X\lrcorner\,\psi) = \pi_3^*(P_{\bar{v}}d\eta).$$

If $f\colon B\to W$ is an extremal section, then we have from (5.27)

$$(X\lrcorner\,d\psi)\,|\bar{f}(B) = 0.$$

Let $\bar{v} \in \bar{f}(B)$ and $q \in K_{\bar{v}}^n(\operatorname{tr} G^n(W))$. We have

$$\langle q \,|\, \pi_3^* P_{\bar{v}} d\eta \rangle = \langle \pi_{3*} q \,|\, P_{\bar{v}} d\eta \rangle. \tag{9.17}$$

But by the integrability of $\bar{f}(B)$, $\pi_{3*} q = a \cdot v$, $0 \neq a \in \boldsymbol{R}$.

It follows from (9.17) and lemma 12 that

$$\langle \pi_{3*} q \,|\, P_{\bar{v}} d\eta \rangle = \langle \pi_{3*} q \,|\, d\eta \rangle = \langle q \,|\, \pi_3^* d\eta \rangle. \tag{9.18}$$

From (9.16), (9.18) and (5.27) we obtain

$$d(X \lrcorner \psi - \pi_3^* \eta) \,|\, \bar{f}(B) = 0.$$

The $(n-1)$-form $a = X \lrcorner \psi - \pi_3^* \eta$ is called a *conserved current* (cf. [10]). For every extremal $\bar{C} = \bar{f}(B)$ we can define the physical quantity corresponding to the current a.

Let c_{n-1} be an $(n-1)$-dimensional submanifold of $\bar{C}$. A physical quantity Q is a functional which assigns to every extremal $\bar{C}$ the real number

$$Q(\bar{C}) = \int_{c_{n-1}} a. \tag{9.19}$$

It was proved in [6] that for every extremal $\bar{C}$ we can choose a family $\mathscr{C}$ of $(n-1)$-dimensional submanifolds of $\bar{C}$ such that for every $c^1, c^2 \in \mathscr{C}$, $\int_{c^1} a = \int_{c^2} a$. This means that $Q(\bar{C})$ does not depend on the choice of $c \subset \bar{C}$.

We shall not develop the theory of physical quantities and we refer the reader to [4], [6], [7].

Example. The energy-momentum tensor for a scalar field theory. We consider a scalar field theory in a flat space-time M with a diagonal metric tensor $(g_{\mu\nu})$, $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$. Let (ϖ^λ) denote affine coordinates in M. According to section 8.3 the Lagrangian function on $K^4(W)$ is given by

$$\mathscr{L}(v) = \lambda\left(\tfrac{1}{2} g^{\mu\nu} \gamma_\mu \gamma_\nu - G(\varphi)\right) \quad \text{(cf. (8.3.5'))},$$

where $(\varpi^\lambda, \varphi)$ are coordinates in $W = M \times \boldsymbol{R}$, $\lambda = \det[a_\mu^\nu]$,

$$\psi(\varpi^\lambda, \varphi, \gamma_\nu) = -\left(\tfrac{1}{2} g^{\mu\nu} \gamma_\mu \gamma_\nu + G(\varphi)\right) d\varpi^0 \wedge \ldots \wedge d\varpi^3 + \\ + \sum_{\nu=0}^{3} g^{\mu\nu} \gamma_\mu d\varpi^0 \wedge \ldots \wedge \underbrace{d\varphi}_{\nu} \wedge \ldots \wedge d\varpi^3.$$

In W we have a 4-parameter family of transformations which si generated by translations in M:

$$F_{a^0,\ldots,a^3}(\varpi^\lambda, \varphi) = (\varpi^\lambda + a^\lambda, \varphi),$$

$$(F_{a^0,\ldots,a^3})_*(\varpi^\lambda, \varphi, \gamma_\lambda, \lambda) = (\varpi^\lambda + a^\lambda, \varphi, \gamma_\lambda, \lambda).$$

The family $(F^{\cdot}_{\ldots\ldots\cdot})$ generates four vector fields on $\mathrm{tr}G^4(W)$:

$$X_\mu = \frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3.$$

Let $\bar{C} = \{x^\lambda \to (\varphi(x^\lambda), \gamma_\nu(x^\lambda))\}$ be a solution of field equations (8.3.9), $\gamma_\nu = \partial_\nu \varphi$, $\Box\varphi + G'(\varphi) = 0$; then

$$(X_\mu \lrcorner \psi) | C = \sum_{\nu=0}^{3} (-1)^{\nu+1} \big(\eta^\nu \eta_\mu - \delta^\nu_\mu (\tfrac{1}{2}\eta^\lambda \eta_\lambda - G(\varphi))\big) dx^0 \wedge \ldots\ldots \wedge \hat{dx^\nu} \wedge \ldots\ldots \wedge dx^3.$$

The energy momentum tensor is equal to

$$T^{\mu\nu} = \eta^\mu \eta^\nu - g^{\mu\nu}(\tfrac{1}{2}\eta^\lambda \eta_\lambda - G(\varphi)).$$

The 4-energy momentum vector of the system is

$$P^\mu = \int_\sigma T^{\mu\nu}(x) dS_\nu,$$

where σ is any space-like surface in M and dS_ν is its surface element ($\dim \sigma = 3$) (cf. [6], [7]).

10. A VARIATIONAL PROBLEM WITH CONSTRAINTS

Let $\mathscr{L}_0$ be a positive homogeneous function on $K^n(W)$. The equation

(10.1) $\quad \mathscr{L}_0(v) = 0$ defines a subset N in $K^n(W)$

and a subset $\bar{N}$ in $G^n(W)$. Let

(10.2) $$v_0 \in N \quad \text{and} \quad \mathscr{L}'_{\mathrm{ver}}(v_0) \neq 0.$$

It follows from the rank theorem that (10.1) defines locally a submanifold $\bar{\mathscr{N}}$ in $G^n(W)$ such that $\pi_3(\bar{\mathscr{N}})$ is an open set in W. In local coordinates (t^s, x^k, γ^k_s) in $\mathrm{tr}G^n(W)$ (10.2) is equivalent to the condition: there exists a (p, q) such that

(10.3) $$\frac{\partial \bar{\mathscr{L}}(t^s, x^k, \gamma^k_s)}{\partial \gamma^p_q} \neq 0.$$

We shall consider only those sections $f\colon (\pi \circ \pi_3)(\bar{\mathscr{N}}) \to W$ for which

(10.4) $$\bar{f}(\pi \circ \pi_3(\bar{\mathscr{N}})) \subset \bar{\mathscr{N}} \subset \mathrm{tr}G^n(W).$$

Definition. A variational problem with constraints is a *system* $(W, \mathscr{L}, M, \mathscr{L}_0, \bar{\mathscr{N}})$, where $\mathscr{L}_0$ and $\mathscr{L}$ are Lagrangian functions, $\bar{\mathscr{N}}$ is defined by (10.1), and M is an $(n-1)$-dimensional compact submanifold of W which is a section of π over some $(n-1)$-dimensional compact submanifold B_1 of B contained in

(10.5) $$(\pi \circ \pi_3)(\bar{\mathscr{N}}).$$

DEFINITION. A section f of π which fulfils (10.4) is called an *extremal section* of the variational problem (10.5) if, for every one-parameter family (f_ε) of sections of π over B_1 fulfilling the conditions

1° $f_\varepsilon(\partial B_1) = M, f_0 = f$,

2° $\bar{f}_\varepsilon(B_1) \subset \bar{\mathscr{N}}$,

we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\int_{B_1} \mathscr{L}(f_{\varepsilon*}\nu_{B_1})\,\omega_{B_1}\Big) = 0. \tag{10.6}$$

Now we introduce a bundle W_1 and consider a variational problem in W_1. Let $W_1 = W \oplus (\boldsymbol{R}\times B)$, i.e., fibre of W_1 over $x \in B$ is equal to $W_x \times \boldsymbol{R}$. Let $\varrho\colon W_1 \to B$, $\mathrm{pr}_1\colon W_1 \to W$, $\mathrm{pr}_2\colon W_1 \to \boldsymbol{R}\times B$ be natural projections.

These projections generate the maps

$$\mathrm{pr}_{1*}\colon K^n(W_1) \to K^n(W), \qquad \overline{\mathrm{pr}}_{1*}\colon G^n(W_1) \to G^n(W). \tag{10.7}$$

Using (10.7), we can extend functions $\mathscr{L}_0$ and $\mathscr{L}$ onto $K^n(W_1)$. We denote these extensions by $\mathscr{L}_0^e$ and $\mathscr{L}^e$. Let a be a function on W_1 defined by $a(w, \lambda) = \lambda$, for $w \in W$, $\lambda \in \boldsymbol{R}$. Let

$$\bar{a} = \varrho_3^* a = a \circ \varrho_3, \qquad \text{where } \varrho_3\colon G^n(W_1) \to W_1. \tag{10.8}$$

Let

$$\psi_0^e(\cdot) = \varrho_3^* \circ \mathscr{L}'_{0\mathrm{ver}}(\cdot), \qquad \psi^e(\cdot) = \varrho_3^* \circ \overline{\mathscr{L}}'_{\mathrm{ver}}(\cdot). \tag{10.9}$$

LEMMA 15. *If $\bar{C}$ is an integrable submanifold of $\mathrm{tr}G^n(W)$ which is a section of $\pi \circ \pi_3$ over B_1 and $\psi_0|\bar{C} = 0$, then $\bar{C} \subset \bar{\mathscr{N}}$.*

Proof. For an integrable submanifold $\bar{C}$ the condition $\psi_0|\bar{C} = 0$ is equivalent to the condition $\mathscr{L}_0(f_*\nu) = 0$ (where $\bar{C} = \bar{f}(B_1)$). This fact follows from proposition 1.

We shall use lemma 15 for the bundle $\varrho\colon W_1 \to B_1$ and the Lagrangian function $\mathscr{L}_0^e$. It is easy to see that

$$\psi_0^e = (\overline{\mathrm{pr}}_{1*})^*(\psi_0), \qquad \psi^e = (\overline{\mathrm{pr}}_{1*})^*(\psi). \tag{10.10}$$

We consider the variational problem with constraints $(W_1, B_1, \mathscr{L}^e, M^e, \mathscr{L}_0^e, \bar{\mathscr{N}}^e)$, where M^e is an $(n-1)$-dimensional submanifold of W_1 contained in $\varrho_3(\bar{\mathscr{N}}^e)$ such that $\pi \circ \varrho_3(M^e) = \partial B_1$.

THEOREM 6. *If there exists an integrable section $\bar{g}\colon B_1 \to G^n(W_1)$ such that $\mathrm{pr}_1(g(\partial B_1)) = M$ and, for every $\varrho \circ \varrho_3$-vertical vector field Y on $\bar{C}^e = \bar{g}(B_1)$, $(Y \lrcorner d(\psi^e - \bar{a}\psi_0^e))|\bar{C}^e = 0$, then the section $f = \mathrm{pr}_1 \circ g$ is an extremal section of a variational problem with constraints* (10.5).

Proof. If $Y = \partial/\partial\lambda$, we obtain $\psi_0^e|\bar{g}(B_1) = 0$. It follows from (10.10) and from lemma 15 that $\psi_0|\bar{f}(B_1) = 0$ and $\bar{f}(B_1) \subset \bar{\mathscr{N}}$.

Let (g_ε) be a one-parameter family of sections of $\varrho\colon W_1 \to B_1$ such that: $\mathrm{pr}_1 \circ g_\varepsilon(\partial B_1) = M$; $\bar{g}_\varepsilon\colon B_1 \to \bar{\mathcal{N}}^e$. Let $f_\varepsilon = \mathrm{pr}_1 \circ g_\varepsilon$ and let $\bar{f}\colon B_1 \to \bar{\mathcal{N}}$ be the canonical lift of f. From (10.10) and lemma 15 we have

$$\int_{B_1} \bar{f}_\varepsilon^* \psi = \int_{B_1} (\overline{\mathrm{pr}}_{1\varepsilon} \circ \bar{g}_\varepsilon)^* \psi = \int_{B_1} \bar{g}_\varepsilon^* (\psi^e - a\psi_0^e). \tag{10.11}$$

If we differentiate (10.11) we obtain

$$\int_{\bar{g}(B_1)} \left(\bar{X} \lrcorner d(\psi^e - a\psi_0^e)\right) + \int_{\bar{g}(\partial B_1)} \left(\bar{X} \lrcorner (\psi^e - a\psi_0^e)\right) = \frac{d}{d\varepsilon} \int_{B_1} \bar{f}_\varepsilon^* \psi \Big|_{\varepsilon=0}, \tag{10.12}$$

where

$$\bar{X} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{g}_\varepsilon.$$

The boundary term in (10.12) vanishes because $\bar{X}$ is ϱ_3-vertical on $\bar{g}(\partial B_1)$ and $\psi^e - a\psi_0^e$ is ϱ_3-horizontal. If we use the assumption of the theorem, we shall see that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{B_1} \bar{f}_\varepsilon^* \psi = 0.$$

EXAMPLE. We consider the hydrodynamics of an incompressible fluid with a constant density ϱ on an n-dimensional Riemannian manifold M. We shall use here some results published in [9]. We have

$$W = M_1 \times [t_0, t_1] \times M_2, \quad B = M_1 \times [t_0, t_1], \quad M_1 \cong M \cong M_2,$$

$$\mathrm{pr}_1\colon W \to M_1, \quad \mathrm{pr}_0\colon W \to [t_0, t_1], \quad \mathrm{pr}_2\colon W \to M_2.$$

We denote by (y^i) local coordinates in M_1 and by (x^j) local coordinates in M_2; t is a coordinate in $[t_0, t_1]$. If $\left(y_1^j, t, x^i, (\gamma_j^i)_{i,j=1}^n, \gamma_{n+1}^i\right)$ denote local coordinates in $\mathrm{tr}G^{n+1}(W)$ we have

$$\bar{\mathcal{L}}(y^j, t, x^i, \gamma_j^i, \gamma_{n+1}^i) = \frac{\varrho}{2} g_{pq}(x) \gamma_{n+1}^p \gamma_{n+1}^q - \varrho V(t, x).$$

The function $\mathcal{L}_0$ on $K^{n+1}(W)$ is given by the $(n+1)$-form η on W;

$$\mathcal{L}_0(v) = \langle v | \eta \rangle, \quad v \in K^{n+1}(W);$$

$$\eta = \left(\sqrt{\det g_{ij}(x)}\, dx^1 \wedge \ldots \wedge dx^n - \sqrt{\det g_{ij}(y)}\, dy^1 \wedge \ldots \wedge dy^n\right); \tag{10.13}$$

$$\bar{\mathcal{L}}_0(y^j, t, x^i, \gamma_j^i, \gamma_{n+1}^i) = \left(\det[\gamma_j^i] \sqrt{\det g_{ij}(x)} - \sqrt{\det g_{ij}(y)}\right);$$

$$\psi(y^j, t, x^i, \gamma_j^i, \gamma_{n+1}^i) = \gamma_{n+1}^p g_{pr}(x) \sqrt{\det g_{ij}(y)}\, dy^1 \wedge \ldots \wedge dy^n \wedge dx^r -$$

$$- \left(\frac{\varrho}{2} g_{pq}(x) \gamma_{n+1}^p \gamma_{n+1}^q + V(t, x)\right) \sqrt{\det g_{ij}(y)} \cdot dy^1 \wedge \ldots \wedge dy^n \wedge dt;$$

$$(10.14)\qquad \psi_0(y^j, t, x^i, \gamma^i_j, \gamma^i_{n+1})$$
$$= -\Big((n-1)\det[\gamma^i_j]\sqrt{\det g_{ij}(x)} + \sqrt{\det g_{ij}(y)}\Big)\cdot dy^1\wedge\ldots\wedge dy^n\wedge dt +$$
$$+\sum_{k=1}^{n}{}^{-1}\gamma^k_s \det[\gamma^i_j]\sqrt{\det g_{ij}(x)}\, dy^1\wedge\ldots\wedge\underbrace{dx^s}_{k}\wedge\ldots\wedge dy^n\wedge dt.$$

We construct the bundle W_1. $W_1 = M_1\times[t_0, t_1]\times M_2\times \boldsymbol{R}$ with local coordinates (y^j, t, x^i, λ). The forms ψ_0^e, ψ^e are given also by formulae (10.13) and (10.14). The equations of motion are

$$(10.15)\qquad \big(Y\lrcorner\, d(\psi-\lambda\psi_0)\big)|\bar{C} = 0,$$

where $\bar{C}$ is an integrable submanifold of W_1 given by

$$x^i = x^i(y^j, t),\qquad \lambda = \lambda(y^j, t),\qquad \gamma^i_j = \frac{\partial x^i}{\partial y^j},\qquad i, j = 1, \ldots, n,$$
$$\gamma^i_{n+1} = \frac{\partial x^i}{\partial t}.$$

If $Y = \partial/\partial\lambda$, we obtain $\psi_0|C = 0$, i.e.,

$$(10.16)\qquad \det[\gamma^i_j]\sqrt{\det g_{ij}(x)} = \sqrt{\det g_{ij}(y)}.$$

If $Y = A^k\partial/\partial x^k$, then we obtain

$$(10.17)\qquad \frac{\partial^2 x^s}{\partial t^2} + \Gamma^s_{rp}(x)\frac{\partial x^r}{\partial t}\cdot\frac{\partial x^p}{\partial t}$$
$$= g^{sp}(x)\left(\frac{\partial V(t, x)}{\partial x^p} + \frac{1}{\varrho}\frac{\partial\lambda\big(t, y(t, x)\big)}{\partial x^p}\right).$$

Therefore $\lambda = -P$, where $P(t, x)$ is a pressure at the point x and time t. Equations (10.16) and (10.17) are the equations of motion in the Lagrange form, cf. [9].

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