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Concerning the order approximation of periodic continuous functions by trigonometric interpolation polynomials II

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Abstract. In earlier work [2] we proved that there exists a unique trigonometric interpolation polynomial $R_n(x)$ satisfying (1.1) and (1.2). This interpolation process is called the (0, 2, 3) case. It was also shown that $R_n(x)$ converges uniformly to $f(x)$ on the real line provided $f(x) \in \operatorname{lip} \alpha$ and there are some restraints on b_{kn} at c_{kn} as shown in Theorem 1. In this work, Theorem 2, and Theorem 3 examine the question of upper and lower estimates of $\|f - R_n(f)\|$ in the uniform norm. As a corollary of these theorems, it follows that for a uniform approximation of $R_n(x)$ to $f(x)$, the $\operatorname{lip} \alpha$ class cannot be replaced by $f(x) \in c_{2\pi}$.

Let

$$x_{kn} = \frac{2\pi k}{n}, \quad k = 0, 1, \ldots, n-1. \tag{1.1}$$

Let $R_n(x)$ be the unique trigonometric polynomial determined by the interpolatory conditions

$$R_n(x_{kn}) = f(x_{kn}), \quad R_n''(x_{kn}) = b_{kn}, \quad R_n'''(x_{kn}) = c_{kn}, \quad k = 0, 1, \ldots, n-1. \tag{1.2}$$

When n is even $(= 2m)$ we require the trigonometric polynomial $R_n(x)$ to have the form

$$d_0 + \sum_{i=1}^{3m-1} (d_i \cos ix + e_i \sin ix) + d_{3m} \cos 3mx, \tag{1.3}$$

but when n is odd $(= 2m+1)$ we require it to have the form

$$d_0 + \sum_{i=1}^{3m+1} (d_i \cos ix + e_i \sin ix). \tag{1.4}$$

In our earlier work [2] we have considered the problem of existence, uniqueness, and explicit representation, and the problem of uniform convergence of $R_n(x)$ to $f(x)$ on the real line. The main theorem of [2] is as follows.

THEOREM 1. *Let $f(x)$ be a 2π periodic continuous function with $f(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 0$; then the sequence $\{R_n(x)\}$ defined by* (1.2) *converges uniformly to $f(x)$ on the real line provided*

$$b_{kn} = O\left(\frac{n^2}{\log n}\right), \quad c_{kn} = O\left(\frac{n^3}{\log n}\right) \quad \textit{for } k = 0, 1, \ldots, n-1.$$

Concerning the above theorem, it is natural to ask the following question: to what extent is the above theorem the best possible? Can the sequence $\{R_n(x)\}$ given by (1.2) converge uniformly to $f(x)$ for only 2π periodic continuous functions on the real line? We will also examine the question of lower and upper estimates of $\|f-R_n(f)\|$ in a uniform norm for a certain class of continuous periodic functions. The upper estimate of $\|f-R_n(f)\|$ is given by

THEOREM 2. *Let $\omega_2(t,f)$ be the second modulus of continuity of $f(x)$. Then we have*

$$(1.5) \qquad |R_n(x) - f(x)| = O\left(\log n\, \omega_2\left(\frac{1}{n}, f\right)\right)$$

provided

$$(1.6) \qquad |b_{kn}| = n^2\omega_2\left(\frac{1}{n}, f\right), \quad |c_{kn}| = n^3\omega_2\left(\frac{1}{n}, f\right).$$

COROLLARY. *Let $f(x)$ satisfy the condition $\lim_{n\to\infty} \log n\, \omega_2\left(\frac{1}{n}, f\right) = 0$. Then we have $|R_n(x) - f(x)| = o(1)$.*

Thus Theorem 1 is a special case of Theorem 2.

Let us denote by $c(\varphi)$ the clas of all 2π periodic continuous functions for which

$$(1.7) \qquad \omega_2(t,f) = O\big(\varphi(t)\big).$$

Let $\varphi(t)$ have the following properties:

$$(1.8) \qquad \begin{aligned} &\text{(i)} && \varphi(t) > 0 \text{ for } t > 0,\ \varphi(0) = 0,\ \varphi(T) \geqslant \varphi(t), \text{ if } T \geqslant t,\\ &\text{(ii)} && \varphi(t) \text{ is continuous for } t > 0,\\ &\text{(iii)} && \frac{t^2}{\varphi(t)} \text{ is monotonic increasing for } t \geqslant 0,\\ &\text{(iv)} && \lim_{t\to 0+} \frac{t^2}{\varphi(t)} = 0. \end{aligned}$$

THEOREM 3. *There exists a 2π periodic continuous function f belonging to $c(\varphi)$ for which*

$$(1.9) \qquad |R_n(\pi) - f(\pi)| > c \log n\, \varphi\left(\frac{1}{n}\right) \quad \textit{for } n = n_1, n_2, \ldots,$$

where $0 < n_1 < n_2 < \ldots$, n *being always an odd positive integer. Further, it is assumed that* b_{kn} *and* c_{kn} *as given by* 1.2 *are all zero.*

The proof of Theorem 2 depends on

THEOREM 4 (Stečkin [3]). *Let* k *be a positive integer; then for a given* $f \in c_{2\pi}$ *we can find a trigonometric polynomial* $t_n(x)$ *of order* n *at most such that*

$$\|f - t_n\| = O\left(\omega_k\left(\frac{1}{n}, f\right)\right) \tag{1.10}$$

and

$$\|t_n^{(k)}\| = O\left(n^k \omega_k\left(\frac{1}{n}, f\right)\right). \tag{1.11}$$

Here $\omega_k(\delta, f)$ *is the modulus of smoothness of order* k *of* $f(x)$.

2. Preliminaries. From our earlier work [2] we have the following results.

For n odd $(= 2m+1)$, $R_n(x)$ satisfying (1.2) and of the form (1.4) is given by

$$R_n(x) = \sum_{k=0}^{n-1} f(x_{kn}) u(x - x_{kn}) + \sum_{k=0}^{n-1} b_{kn} v(x - x_{kn}) + \sum_{k=0}^{n-1} c_{kn} w(x - x_{kn}), \tag{2.1}$$

where

$$w(x) = -\frac{1}{n^3}\left[\sum_{j=1}^{m} \frac{4j}{n^2 - 3j^2} \sin jx + \sum_{j=m+1}^{3m+1} \frac{(3n-2j)}{n^2 - 3(n-j)^2} \sin jx\right], \tag{2.2}$$

$$v(x) = \frac{1}{n^3}\left[1 + 2\sum_{j=1}^{m} \frac{n^2 + 3j^2}{n^2 - 3j^2} \cos jx - \frac{1}{4} \sum_{j=m+1}^{3m+1} \frac{n^2 + 3(3n-2j)^2}{n^2 - 3(n-j)^2} \cos jx\right] \tag{2.3}$$

and

$$u(x) = \frac{1}{n}\left[1 + \frac{2}{n^2} \sum_{j=1}^{m} \frac{(n^2 - j^2)}{n^2 - 3j^2} \cos jx - \frac{1}{n^2} \sum_{j=m+1}^{3m+1} \frac{(n-j)^2 (2n-j)^2}{n^2 - 3(n-j)^2} \cos jx\right]. \tag{2.4}$$

We need also the following estimates, as obtained in [2].

The following estimates are valid:

$$\sum_{k=0}^{n-1} |w(x-x_{kn})| \leqslant c_1 n^{-3}\log n, \tag{2.5}$$

$$\sum_{k=0}^{n-1} |v(x-x_{kn})| \leqslant c_2 n^{-2}\log n, \tag{2.6}$$

$$\sum_{k=0}^{n-1} |u(x-x_{kn})| \leqslant c_3 \log n. \tag{2.7}$$

Here c_1, c_2, c_3 are all positive constants independent of n and x.

From the uniqueness theorem it follows that for an arbitrary trigonometric polynomial of order n we have

$$t_n(x) = \sum_{k=0}^{n-1} t_n(x_{kn})u(x-x_{kn}) + \sum_{k=0}^{n-1} t_n''(x_{kn})v(x-x_{kn}) + \sum_{k=0}^{n-1} t_n'''(x_{kn})w(x-x_{kn}), \tag{2.8}$$

$$1 \equiv \sum_{k=0}^{n-1} u(x-x_{kn}). \tag{2.9}$$

Now, we prove the following lemma.

LEMMA 2.1. *There exists a positive constant c_4 independent of n and x such that*

$$\sum_{k=0}^{n-1} |u(\pi-x_{kn})| \geqslant c_4 \log n, \qquad n \geqslant n_0, \tag{2.10}$$

n being an odd positive integer.

We set

$$\beta_j = \frac{(n-j)^2(2n-j)^2}{n^2-3(n-j)^2} \tag{2.11}$$

and observe that

$$\sum_{j=m+1}^{3m+1} \beta_j \cos j(\pi-x_{kn}) = \sum_{j=m+1}^{2m+1} + \sum_{j=n+1}^{3m+1} = -\sum_{j=1}^{m} (\beta_{n-j}+\beta_{n+j})\cos j(\pi-x_{kn}).$$

From (2.4) and the above it follows that

$$u(\pi-x_{kn}) = \frac{1}{n}\left[1+\frac{2}{n^2}\sum_{j=1}^{m}\frac{n^4+2j^4-n^2j^2}{n^2-3j^2}\cos j(\pi-x_{kn})\right]. \tag{2.12}$$

Let us write

$$(2.13)\qquad \nu_j = \frac{n^4+2j^4-n^2j^2}{n^2-3j^2},$$

and express $u(\pi-x_{kn})$ in terms of the Fejer kernel at $x=\pi$. We set

$$(2.14)\qquad t_{j,k}(\pi)\equiv t_{j,k} = 1+\frac{2}{j}\sum_{i=1}^{j-1}(j-i)\cos i(\pi-x_{kn})$$

and observe the known properties of the Fejer kernel

$$(2.15)\qquad \sum_{k=0}^{n-1} t_{j,k} = n,\qquad t_{j,k} = \frac{1}{j}\left(\frac{\sin\frac{j(\pi-x_{kn})}{2}}{\sin\frac{\pi-x_{kn}}{2}}\right)^2$$

and

$$(2.16)\qquad (j+1)t_{j+1,k}-2jt_{j,k}+(j-1)t_{j-1,k} = 2\cos j(\pi-x_{kn}).$$

On using (2.16) we obtain

$$(2.17)\qquad u(\pi-x_{kn}) = \frac{1}{n^3}\Big[\sum_{p=1}^{m}(\nu_{p-1}-2\nu_p+\nu_{p+1})pt_{p,k}+ + \nu_m(m+1)t_{m+1,k}-\nu_{m+1}mt_{m,k}\Big].$$

From (2.14) it follows that

$$(2.18)\qquad (m+1)t_{m+1,k}-mt_{m,k} = 1+2\sum_{i=1}^{m}\cos i(\pi-x_{kn}) = \frac{\sin(m+\frac{1}{2})(\pi-x_{kn})}{\sin\frac{\pi-x_{kn}}{2}}.$$

From (2.13) it follows that

$$(2.19)\qquad |\nu_m|\geqslant n^2,\qquad |\nu_m-\nu_{m+1}|\leqslant c_5 n,$$

and

$$(2.20)\qquad |\nu_{p-1}-2\nu_p+\nu_{p+1}|\leqslant c_6,\qquad p=1,2,\ldots,n.$$

On using (2.17)-(2.18) it follows that

$$u(\pi-x_{kn}) = \frac{1}{n^3}\Big[\sum_{p=1}^{m}(\nu_{p-1}-2\nu_p+\nu_{p+1})pt_{p,k}+ + \nu_m\Big(1+2\sum_{i=1}^{m}\cos i(\pi-x_{kn})\Big)+mt_{m,k}(\nu_m-\nu_{m+1})\Big].$$

Therefore on using (2.15), (2.19) and (2.20) we have

$$\sum_{k=0}^{n-1}|u(\pi-x_{kn})|\geqslant\frac{|\nu_m|}{n^3}\sum_{k=0}^{n-1}\left|\frac{\sin\frac{n}{2}(\pi-x_{kn})}{\sin\frac{\pi-x_{kn}}{2}}\right|-\frac{1}{n^3}\sum_{p=1}^{m}|\nu_{p-1}-2\nu_p+\nu_{p+1}|pn-$$

$$-\frac{mn}{n^3}|(\nu_m-\nu_{m+1})|\geqslant\frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\sin\frac{n}{2}(\pi-x_{kn})}{\sin\frac{\pi-x_{kn}}{2}}\right|-\frac{1}{n^2}c_6\sum_{p=1}^{m}p-c_5.$$

It is well known that

$$\sum_{k=0}^{n-1}\left|\frac{\sin\frac{n}{2}(\pi-x_{kn})}{\sin\frac{\pi-x_{kn}}{2}}\right|\geqslant c_7 n\log n,$$

and we finally obtain

$$\sum_{k=0}^{n-1}|u(\pi-x_{kn})|\geqslant c_7\log n-c_6-c_5\geqslant c_4\log n,\qquad n\geqslant n_0.$$

This proves the lemma.

3. Proof of Theorem 2. On using Stečkin's theorem (for $k=2$ in Theorem 4) for given $f(x)\in c_{2\pi}$, we can find a trigonometric polynomial $t_n(x)$ of order n at most such that

$$|f(x)-t_n(x)|=0\left(\omega_2\left(\frac{1}{n},f\right)\right) \tag{3.1}$$

and

$$\|t_n''\|=O\left(n^2\omega_2\left(\frac{1}{n},f\right)\right). \tag{3.2}$$

On using the Bernstein inequality for trigonometric polynomial we obtain

$$\|t_n'''\|=O\left(n^3\omega_2\left(\frac{1}{n},f\right)\right). \tag{3.3}$$

From (2.1) and (2.8) we obtain

$$\begin{aligned} R_n(x)-f(x) &= R_n(x)-t_n(x)+t_n(x)-f(x) \\ &= \sum_{k=0}^{n-1}\big(f(x_{kn})-t_n(x_{kn})\big)u(x-x_{kn})+\sum_{k=0}^{n-1} b_{kn}v(x-x_{kn})+ \\ &\quad+\sum_{k=0}^{n-1} c_{kn}w(x-x_{kn})-\sum_{k=0}^{n-1} t_n''(x_{kn})v(x-x_{kn})- \\ &\quad-\sum_{k=0}^{k-1} t_n'''(x_{kn})w(x-x_{kn})+\big(t_n(x)-f(x)\big) \\ &= I_1+I_2+I_3+I_4+I_5+I_6. \end{aligned}$$

From (3.1) and (2.7) we obtain

$$|I_1| = O\left(\omega_2\left(\frac{1}{n}\right)\log n\right).$$

From (1.6) and (2.6) we have

$$|I_2| = n^2\omega_2\left(\frac{1}{n}\right)\sum_{k=0}^{n-1}|v(x-x_{kn})| = O\left(\omega_2\left(\frac{1}{n}\right)\log n\right).$$

Similarly from (1.6) and (2.5) we obtain

$$|I_3| = n^3\omega_2\left(\frac{1}{n}, f\right)\sum_{k=0}^{n-1}|w(x-x_{kn})| = O\left(\omega_2\left(\frac{1}{n}\right)\log n\right).$$

From (3.2) and (2.6) we have

$$|I_4| = n^2\omega_2\left(\frac{1}{n}, f\right)\sum_{k=0}^{n-1}|v(x-x_{kn})| = O\left(\omega_2\left(\frac{1}{n}, f\right)\log n\right).$$

Similarly from (3.3) and (2.5) we obtain

$$|I_5| = n^3\omega_2\left(\frac{1}{n}, f\right)\sum_{k=0}^{n-1}|w(x-x_{kn})| = O\left(\omega_2\left(\frac{1}{n}, f\right)\log n\right).$$

Lastly on using (3.1) we have

$$|I_6| = O\left(\omega_2\left(\frac{1}{n}, f\right)\right).$$

Combining all these results we have

$$|R_n(x)-f(x)| = O\left(\log n\,\omega_2\left(\frac{1}{n}, f\right)\right), \quad n > 1.$$

This proves Theorem 2.

The proof of Theorem 3 is a direct application of a recent result of Kis and Vertesi [1] and Lemma 2.1.

Let x_{kn}, $n = 1, 2, \ldots$, be an infinite point system such that $0 \leqslant x_{kn} < 2\pi$, we define

$$L_n(f, x) = \sum_{k=0}^{n-1} f(x_{kn}) P_{kn}(x), \qquad \lambda_n(x) = \sum_{k=0}^{n-1} \big(P_{nk}(x)\big),$$

where $P_{nk}(x)$ are 2π periodic continuous functions.

THEOREM 5 (Kis and Vertesi). *If* $-\infty < x_0 < \infty$ *and* $\lim_{n\to\infty} \lambda_n(x_0) \neq 1$, *then there exist a* $f(x) \in c(\varphi)$ *and integers* $0 < n_1 < n_2 < \ldots$ *such that*

$$|f(x_0) - L_{n_k}(f, x_0)| > \lambda_{n_k}(x_0)\varphi(d_{n_k})$$

for $k = 1, 2, \ldots$ *Here* $d_n = \min(x_{k+1,n} - x_{kn})$.

Proof of Theorem 3. We choose $P_{nk}(x) = u(x - x_{kn})$, $x_0 = \pi$, $d_n = 2\pi/n$, $n = 1, 3, 5, \ldots$ and observe from Lemma 2.1 that

$$\lambda_n(\pi) = \sum_{k=0}^{n-1} |u(\pi - x_{kn})| > c \log n.$$

Thus, on applying Theorem 5 we have the conclusion of Theorem 3.

References

[1] O. Kis and P. O. H. Vertesi, *On certain linear operators I*, Acta Math. Acad. Sci. Hung. 22 (1971), p. 65–71.

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[3] S. B. Stečkin, *On best approximation of continuous functions*, Izv. Akad. Nauk 15 (1951), p. 219–242.

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