ANNALES
POLONICI MATHEMATICI
XXIV (1971)

On the Phragmén-Lindelöf principle for a polyangle

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1. Let C^m denote the space of all m-tuples $z = (z_1, z_2, \ldots, z_m)$ of complex numbers. Let $G \subset C^m$ be given by

$$G = \prod_{j=1}^{m} G_j,$$

where each G_j constitutes an open angle of measure θ_j in the z_j-plane having its vertex at C_j. Such a set G is called an *open polyangle* of coordinates $(\theta_1, \theta_2, \ldots, \theta_m)$ having its vertex at $C = (C_1, C_2, \ldots, C_m)$. If the angles in each of the z_j-planes are closed, then the corresponding point set is said to be a *closed polyangle*. If each component set G_j is a ray emanating from a certain point in the z_j-plane, then the product set $G = \prod_{j=1}^{m} G_j$ is called a *polyray*.

In this paper we show the Phragmén-Lindelöf principle can be extended to the case of a function of several complex variables in a polyangle.

The maximum principle (see [3], p. 6) states that:

(i) if $f(z) = f(z_1, z_2, \ldots, z_m)$ is a holomorphic function on an open set U in C^m, and

(ii) if for every point y of ∂U, the boundary of U (which has to include the point at infinity of C^m, when U is not relatively compact),

$$\limsup_{z \to y \; z \in U} |f(z)| \leqslant M,$$

then $|f(z)| \leqslant M$ in U and if $|f(z_0)| = M$ for a point z_0 in U, then $f(z) \equiv f(z_0)$ on the connected component of U containing z_0.

In particular, when the open set U is a product set, then the boundary of U can be replaced by the distinguished boundary.

2. Taking the open set U to be a polyangle, we are going to show how the second condition can be relaxed.

THEOREM 1. *Let $f(z)$ be a holomorphic function in an open polyangle G of coordinates $(a_1\pi, a_2\pi, \ldots, a_m\pi)$ such that $0 < a_i \leqslant 2$ and $\Sigma a_i = a$ and* *let Γ be the boundary of G. Suppose that*

(i) *for every finite point $\zeta \in \Gamma$,* $\limsup_{z \to \zeta} |f(z)| \leqslant M < \infty$,

(ii) $$\liminf_{r_1, r_2, \ldots, r_m \to \infty} \frac{\ln\ln M(r_1, r_2, \ldots, r_m)}{\ln(r_1 r_2 \ldots r_m)} < 1/a,$$

where $M(r_1, r_2, \ldots, r_m)$ denotes the maximum of $|f(z)|$ on the product set

$$\prod_{i=1}^{m} \{z_i \colon |z_i| = r_i,\ z_i \in G_i\}.$$

Then $|f(z)| \leqslant M$ for all $z \in G$, and, moreover, if $|f(z_0)| = M$ for some point z_0 of G, then $f(z) \equiv f(z_0)$.

Proof. Without loss of generality, we assume the vertex to be the origin $(0, 0, \ldots, 0)$ and the coordinate-angles of the polyangle to be the sectors $|\arg z_j| < \frac{1}{2} a_j \pi$, $j = 1, 2, \ldots, m$.

From (ii) it follows that

(1) $$|f(z)| < \exp\{|z_1 z_2 \ldots z_m|^{\varrho_1}\},$$
$$\varrho_1 < 1/a, \text{ for } |z_i| = r_{in}, z_i \in G_i, 1 \leqslant i \leqslant m,$$

where $r_{in} \to \infty$ as $n \to \infty$ [1].

Consider the function

(2) $$F(z) = f(z) \exp\{-\varepsilon (z_1 z_2 \ldots z_m)^{\varrho_2}\},$$

where $\varepsilon > 0$ and $\varrho_1 < \varrho_2 < 1/a$, choosing those branches of $z_i^{\varrho_2}$ which have positive values for positive real z_i (for all $i = 1, 2, \ldots, m$).

Let G_{in} be the sector

$$G_{in} \equiv \{z_i / |\arg z_i| < \tfrac{1}{2} a_i \pi,\ |z_i| < r_{in}\}$$

in the z_i-plane and let

$$G_n = \prod_{i=1}^{m} G_{in}.$$

Let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ be a point of the distinguished boundary of G_n.

First suppose that for some value of i, ζ_i belongs to one of the line segments of the boundary of G_{in}, and

$$\zeta_j \in \overline{G}_{jn}, \quad j = 1, 2, \ldots, i-1, i+1, \ldots, m.$$

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[1] More rigorously, condition (ii) in the statement of the theorem could be replaced by this condition.

In this case it follows from (i) that

$$\limsup_{z\to\zeta\ z\in G_n} |F(z)| \leqslant M \tag{3}$$

since

$$\varrho_2 \sum_{i=1}^{m} \theta_i < \pi/2 .$$

Next suppose that ζ is such that every ζ_i belongs to the open circular arc $|z_i| = r_{in}$. For such a boundary point it follows from (1) that $F(z) \to 0$ as $|z_i| \to \infty$ through the sequences of values $\{r_{in}\}$, $|z_i| = r_{in}$, $i = 1, 2, \ldots, m$.

Hence also in this case

$$\limsup_{z\to\zeta\ z\in G_n} |F(z)| \leqslant M . \tag{4}$$

Therefore, by (3), (4) and by [3], p. 6, we get

$$|F(z)| \leqslant M, \quad \text{for all } z \in G_n .$$

Letting $\varepsilon \to 0$, we get $|f(z)| \leqslant M$ for all $z \in G_n$ and accordingly for all z belonging to the polyangle.

It again follows from [3] (p. 6) that if $|f(z_0)| = M$ for a point $z_0 \in G$, then $f(z) \equiv f(z_0)$.

Remark 1. We can very well improve the theorem by replacing the boundary of the polyangle (in the theorem) by its distinguished boundary, in which case the proof retains the same approach.

Remark 2. Condition (ii) can be replaced by

$$\liminf_{r\to\infty} \frac{\ln\ln M(r)}{\ln r} < 1/a,$$

where

$$M(r) \max_{\sum_{i=1}^{m} |z_i| = r} |f(z)| .$$

THEOREM 2. *Let $f(z) \to a$, as $z_1, \ldots, z_m \to \infty$ along every polyray $\Delta \subset \Gamma$, the distinguished boundary of the open polyangle G, and let $f(z)$ be holomorphic and bounded inside the polyangle G. Then $f(z) \to a$, uniformly in the polyangle.*

The proof is almost the same as in the case of one complex variable and is therefore omitted.

THEOREM 3. *Theorem* 1 *remains true if condition* (ii) *of the theorem is replaced by the folloving less stringent condition*:

$$\liminf_{r_1, r_2, \ldots, r_m \to\infty} \frac{\ln M(r_1, r_2, \ldots, r_m)}{(r_1 r_2 \ldots r_m)^{1/a}} \leqslant 0 . \tag{ii'}$$

Proof. Write $\varphi(z) = f(z)\exp\{-\varepsilon z_1^{1/a} z_2^{1/a} \dots z_m^{1/a}\}$, the branches of $z_i^{1/a}$ chosen to be positive valued for positive real z_i.

Then $\varphi(x) \to 0$ as $x_i \to \infty$ $(i = 1, 2, \dots, m)$ and consequently $\varphi(x)$ has an upper bound, say M'.

Let each component angle G_i of the open polyangle G be bisected by the positive real axis X_i of the respective component plane into two open angles G_i' and G_i'' so that

$$G_i = G_i' \cup G_i'' \cup X_i.$$

We consider the two polyangles

$$G' = \prod_{i=1}^{m} G_i' \quad \text{and} \quad G'' = \prod_{i=1}^{m} G_i''$$

and apply Theorem 1 to the function $\varphi(z)$ for each of them separately; the sum of the coordinates of each polyangle is $\frac{1}{2}\pi a$ and thus we need only verify that inequalities (i) and (1) hold for $\varphi(z)$ in place of $f(z)$ and for some $\varrho_1 < 2/a$. Then we see that in the polyangle G,

$$|\varphi(z)| \leqslant \max(M, M').$$

Since the value M' is attained at a point inside G, it follows that $|\varphi(z)| \leqslant M$ for all $z \in G$, as asserted.

Remark 3. Condition (ii′) replaced by

$$\liminf_{\sum_{i=1}^{m} |z_i| = r \to \infty} \frac{\ln M(r)}{r^{1/a}} \leqslant 0,$$

also serves the purpose.

THEOREM 4. *Let* $G = \prod_{j=1}^{m} G_j$, *where* G_1 *is an angle in the* z_1*-plane of measure* $a\pi$ $(0 < a < 2)$ *and* G_i $(i = 2, 3, \dots, m)$ *is an open connected set in the* z_i*-plane containing the point at infinity of the* z_i*-plane. Let* Γ *be the boundary of* G.

Let $f(z)$ *be a holomorphic function in* G *and let* $f(z)$ *satisfy the following conditions:*

(i) *for every finite point* $\zeta \in \Gamma$, $\limsup_{z \to \zeta} |f(z)| \leqslant M < \infty$,

(ii) $$\liminf_{r_m \to \infty} \dots \liminf_{r_2 \to \infty} \left\{ \liminf_{r_1 \to \infty} \frac{\ln\ln M(r_1, r_2, \dots, r_m)}{\ln r_1} \right\} < \frac{1}{a}.$$

Then $|f(z)| \leqslant M$ *for all* $z \in G$ *and, moreover, if* $|f(z_0)| = M$ *for some point* $z_0 \in G$, *then* $f(z) \equiv f(z_0)$.

Proof. From (ii) it follows that for any sufficiently large values of $r_2, r_3, \ldots, r_m$, and for a sequence $\{r_{1n}\}$ of values of $r_1, r_{1n} \to \infty$, as $n \to \infty$

$$|f(z)| < \exp\{|z_1|^{\varrho_1}\}$$

holds for some $\varrho_1 < 1/a$.

Taking $F(z)$ the same as before and proceeding essentially in the same way, we obtain the desired result.

Note. The above result is independent of the order of limits outside the brackets.

2.1. By taking the component sets G_j to be strips, a *polystrip* can be defined and the Phragmén-Lindelöf principle for a strip can be extended to the case of several complex variables.

I express my gratitude to Prof. S. K. Bose, Head of the department of Pure Mathematics, Calcutta University, for his constant guidance in the preparation of this paper.

References

[1] S. K. Bose and D. Sharma, *Integral function of two complex variables*, Compositio Mathematica, Vol. 15, Fasc. 2 (1963), p. 210-226.

[2] M. M. Džrbšyan, *On the intergral representation and expansion in generalised Taylor series of entire functions of several complex variables*, Math. Sbornik, Vol. 41 (83), No. 3 (1957), p. 257-276.

[3] M. Hervé, *Several complex variables. Local theory*, Oxford University Press, 1963.

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Reçu par la Rédaction le 11. 7. 1969