ANNALES
POLONICI MATHEMATICI
XXVI (1972)

The n-point-invariants of the projective line and cross-ratios of n-tuples

by WALTER BENZ (Waterloo, Ontario)

Consider the projective group $\Gamma(S)$ of the projective line $S' = S \cup \{\infty\}$ over the sfield S. Denote by $[n; S]$, where n is a positive integer, the set of all ordered n-tuples $(A_1, A_2, \ldots, A_n)$, where $A_1, A_2, \ldots, A_n$ are distinct points of S'. If $I \neq \emptyset$ is a set and ω a mapping of $[n; S]$ in I,

$$\omega\colon [n; S] \to I,$$

we will call the pair (I, ω) a *n-point-invariant of* $\Gamma(S)$ iff for all $\gamma \in \Gamma(S)$ and for all $(A_1, A_2, \ldots, A_n) \in [n; S]$ the equation

$$(A_1^\gamma, A_2^\gamma, \ldots, A_n^\gamma)^\omega = (A_1, A_2, \ldots, A_n)^\omega$$

holds, where — generally spoken — the image of the object A under the mapping λ is denoted by A^λ.

If (I, ω), (I', ω') are n-point-invariants we will call them equivalent in case there is a bijection

$$\alpha\colon [n; S]^\omega \to [n; S]^{\omega'}$$

such that

$$[(A_1, A_2, \ldots, A_n)^\omega]^\alpha = (A_1, A_2, \ldots, A_n)^{\omega'} \quad \text{for all } (A_1, A_2, \ldots, A_n) \in [n; S].$$

This relation is an equivalence relation on the class of n-point-invariants of $\Gamma(S)$.

Denote by Σ the set of all $\Gamma(S)$-orbits on $[n; S]$ and by σ the mapping, which associates to every element of $[n; S]$ its orbit. Then, of course, all n-point-invariants are given by $(I, \sigma\cdot\mu)$, where $I \neq \emptyset$ is an arbitrary set and $\mu\colon \Sigma \to I$ an arbitrary mapping.

Up to equivalence there is just one 1-point-invariant because $\Gamma(S)$ operates transitively on $[1; S]$. Moreover, up to equivalence there is just one 2-point-invariant and just one 3-point-invariant. If S denotes the field C of complex numbers, so S' the completed complex plane, we get at once well-known geometric examples of 4-point-invariants, which are not equivalent. For instance (I_ν, ω_ν), $\nu = 1, 2, 3, 4$ are pairwise non-equivalent, where

$$I_1 = I_2 = \{0, 1\}, \quad I_3 = \text{torus group} \bmod \pi, I_4 = C$$

and

$(A_1, A_2, A_3, A_4)^{\omega_1} = 1$ (resp. $= 0$) in case $(A_1, A_2, A_3, A_4) \in [4; S]$ is cocircular, i.e. on a common circle (resp. non-cocircular);

$(A_1, A_2, A_3, A_4)^{\omega_2} = 1$ (resp. $= 0$) in case A_1, A_2, A_3, A_4 is harmonic (non-harmonic);

$(A_1, A_2, A_3, A_4)^{\omega_3} =$ angle $\bmod \pi$, which leads around A_2 in the positive sense (defined by an indikatrix on the sphere) from the circle through A_1, A_2, A_3 to the circle through A_1, A_2, A_4 for $(A_1, A_2, A_3, A_4) \in [4; S]$;

$$(A_1, A_2, A_3, A_4)^{\omega_4} = \begin{bmatrix} A_1 & A_2 \\ A_4 & A_3 \end{bmatrix} \equiv DV(A_1, A_2, A_3, A_4).$$

All 4-point-invariants have been characterized by Aczél, Gołąb, Kuczma, Siwek in the real case, [1]; in [2] the same has been done for all 4-point-invariants in the sfield (skew-field) case. S. Topa solves in [4] a more general functional equation for the field case than that involved in the question of characterizing all 4-point-invariants.

In the present note we would like to characterize all n-point-invariants, $n \geqslant 4$, in the case of an arbitrary sfield. Let m be equal to $n-3$. We are interested in all ordered m-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ of elements $\alpha_\nu \in S \setminus \{0, 1\}$. We define $(\alpha_1, \ldots, \alpha_m) \sim (\beta_1, \ldots, \beta_m)$ iff there exists $a \in S^* \equiv S \setminus \{0\}$ with $\beta_\nu = a\alpha_\nu a^{-1}$ for all $\nu = 1, 2, \ldots, m$. This is an equivalence relation. Denote by $\langle(\alpha_1, \ldots, \alpha_m)\rangle$ the equivalence class of $(\alpha_1, \ldots, \alpha_m)$ and by E_m the set

$$\{\langle(\alpha_1, \ldots, \alpha_m)\rangle \mid \alpha_1, \ldots, \alpha_m \in S \setminus \{0, 1\}\}.$$

If $(P_1, P_2, P_3, P_4) \in [4; S]$ we define

$$\begin{pmatrix} P_1 & P_2 \\ P_4 & P_3 \end{pmatrix} = (P_2 - P_3)^{-1}(P_1 - P_3)(P_1 - P_4)^{-1}(P_2 - P_4),$$

where in case ∞ occurs in $\{P_1, P_2, P_3, P_4\}$ we just cancel both expressions $(P_\nu - P_\mu)$ containing ∞, so for instance

$$\begin{pmatrix} \infty & P_2 \\ P_4 & P_3 \end{pmatrix} = (P_2 - P_3)^{-1}(P_2 - P_4).$$

Because of $(P_1, P_2, P_3, P_4) \in [4; S]$ we have always

$$\begin{pmatrix} P_1 & P_2 \\ P_4 & P_3 \end{pmatrix} \in S \setminus \{0, 1\}.$$

By
$$\begin{bmatrix} & A_1 & A_2 \\ A_4, \ldots, & A_n & A_3 \end{bmatrix}$$
we understand the element
$$\left\langle\left(\begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}, \begin{pmatrix} A_1 & A_2 \\ A_5 & A_3 \end{pmatrix}, \ldots, \begin{pmatrix} A_1 & A_2 \\ A_n & A_3 \end{pmatrix}\right)\right\rangle$$
of E_{n-3} for $(A_1, A_2, \ldots, A_n) \in [n; S]$, $n \geqslant 4$. (For $n = 4$ we get the well-known cross rations in the sfield case.)

Then the following theorem is true:

THEOREM. *Let $n \geqslant 4$ be an integer. Consider an arbitrary non-empty set I, and an arbitrary mapping*
$$\Omega\colon E_{n-3} \to I.$$
Define
$$(A_1, A_2, \ldots, A_n)^{\omega} = \begin{bmatrix} & A_1 & A_2 \\ A_4, \ldots, & A_n & A_3 \end{bmatrix}^{\Omega}$$
for $(A_1, A_2, \ldots, A_n) \in [n; S]$. Then (I, ω) is a n-point-invariant. Moreower, there are no other n-point-invariants.

Remark. The content of this theorem is in other words the statement, that
$$(A_1, \ldots, A_n),\ (B_1, \ldots, B_n) \in [n; S]$$
belong to the same $\Gamma(S)$-orbit iff their "generalized cross ratios"
$$\begin{bmatrix} & A_1 & A_2 \\ A_4, \ldots, & A_n & A_3 \end{bmatrix}, \quad \begin{bmatrix} & B_1 & B_2 \\ B_4, \ldots, & B_n & B_3 \end{bmatrix}$$
are equal.

In this connection we would like to emphasize that
$$\begin{bmatrix} A_1 & A_2 \\ A_\nu & A_3 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_\nu & B_3 \end{bmatrix}, \quad \nu = 4, \ldots, n,$$
is not sufficient in general for having $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$ in the same orbit. Nevertheless, it is sufficient of course, in case S is a field. For the sfield case we have the following counter example:

Let S be the set of quaternions and consider $(\infty, 0, 1, i, 2i)$, $(\infty, 0, 1, -i, 2i) \in [5; S]$. Here
$$\begin{bmatrix} \infty & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} \infty & 0 \\ -i & 1 \end{bmatrix}, \quad \begin{bmatrix} \infty & 0 \\ 2i & 1 \end{bmatrix} = \begin{bmatrix} \infty & 0 \\ 2i & 1 \end{bmatrix}$$
is true because of
$$\langle -i \rangle = \langle j(-i)j^{-1} \rangle = \langle i \rangle \ (^1).$$

(1) By $1, i, j, k$ we denote the Hamiltonian basis of the quatermions over the reals.

In case $(\infty, 0, 1, i, 2i)$, $(\infty, 0, 1, -i, 2i)$ would be in the same $\Gamma(S)$-orbit there would exist an inner automorphism

$$z \to aza^{-1}, \quad a \in S^*$$

such that $-i = aia^{-1}$, $2i = a(2i)a^{-1}$; but this is obviously not true. Thus, for describing orbits it is not sufficient to define a generalized cross ratio by

$$\left(\left\langle\begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}\right\rangle, \left\langle\begin{pmatrix} A_1 & A_2 \\ A_5 & A_3 \end{pmatrix}\right\rangle, \ldots, \left\langle\begin{pmatrix} A_1 & A_2 \\ A_n & A_3 \end{pmatrix}\right\rangle\right);$$

but everything works as we have to prove, by using

$$\left\langle\left(\begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}, \begin{pmatrix} A_1 & A_2 \\ A_5 & A_3 \end{pmatrix}, \ldots, \begin{pmatrix} A_1 & A_2 \\ A_n & A_3 \end{pmatrix}\right)\right\rangle,$$

so by using the earlier defined generalized cross ratios

$$\begin{bmatrix} A_1 & A_2 \\ A_4, \ldots, A_n & A_3 \end{bmatrix}.$$

In order to prove the theorem we start with a lemma, which characterizes generalized cross ratios in another way (see [3] for the case $n = 4$):

Lemma 1. *Denote by* $\Gamma\begin{pmatrix} A & B & C \\ P & Q & R \end{pmatrix}$ *the set of all* $\gamma \in \Gamma(S)$ *such that* $A^\gamma = P$, $B^\gamma = Q$, $C^\gamma = R$.

Then for $(A, B, C, D_1, D_2, \ldots, D_{m=n-3}) \in [n; S]$ *the equation*

$$\begin{bmatrix} A & B \\ D_1, \ldots, D_m & C \end{bmatrix} = \left\{(D_1^\gamma, D_2^\gamma, \ldots, D_m^\gamma) | \gamma \in \Gamma\begin{pmatrix} A & B & C \\ \infty & 0 & 1 \end{pmatrix}\right\}$$

holds.

Proof. Let ε be an element of $\Gamma\begin{pmatrix} A & B & C \\ \infty & 0 & 1 \end{pmatrix}$ and let $a_\nu \equiv (D_\nu)^\varepsilon$, $\nu = 1, 2, \ldots, m$. Because of the bijectivity of ε we have $a_\nu \in S \setminus \{0, 1\}$. Moreover,

$$\left\{(D_1^\gamma, \ldots, D_m^\gamma) | \gamma \in \Gamma\begin{pmatrix} A & B & C \\ \infty & 0 & 1 \end{pmatrix}\right\} = \left\{(a_1^\delta, \ldots, a_m^\delta) | \delta \in \Gamma\begin{pmatrix} \infty & 0 & 1 \\ \infty & 0 & 1 \end{pmatrix}\right\}$$

holds. Because of the fact that $\Gamma\begin{pmatrix} \infty & 0 & 1 \\ \infty & 0 & 1 \end{pmatrix}$ contains exactly inner automorphisms of S, we get

$$\left\{(D_1^\gamma, \ldots, D_m^\gamma) | \gamma \in \qquad\qquad = \langle(a_1, \ldots, a_m)\rangle \in E_m.\right.$$

Denote for the moment $\left\{(D_1^\gamma, \ldots, D_m^\gamma) | \gamma \in \Gamma\begin{pmatrix} A & B & C \\ \infty & 0 & 1 \end{pmatrix}\right\}$ by

$$\begin{bmatrix} A & B \\ D_1, \ldots, D_m & C \end{bmatrix}^*.$$

Thus we put

$$\begin{bmatrix} A & B \\ D_1, \ldots, D_m & C \end{bmatrix}^* = \langle (a_1, \ldots, a_m) \rangle \in E_m,$$

where $a_\nu = (D_\nu)^\varepsilon$, $\nu = 1, 2, \ldots, m$ with an arbitrary $\varepsilon \in \Gamma\begin{pmatrix} A & B & C \\ \infty & 0 & 1 \end{pmatrix}$.

Case 1: $A = \infty$.

If we take (be aware of $(A, B, C, D_1, \ldots, D_m) \in [n; S]$)

$$z^\varepsilon = \frac{1}{C-B}(z-B) \quad \text{for } z \in S$$

and $\infty^\varepsilon = \infty$, so

$$\begin{bmatrix} A & B \\ D_1, \ldots, D_m & C \end{bmatrix}^* = \left\langle \begin{pmatrix} A & B \\ D_1 & C \end{pmatrix}, \begin{pmatrix} A & B \\ D_2 & C \end{pmatrix}, \ldots, \begin{pmatrix} A & B \\ D_m & C \end{pmatrix} \right\rangle = \begin{bmatrix} A & B \\ D_1, \ldots, D_m & C \end{bmatrix}.$$

Case 2: $B = \infty$.

Take

$$z^\varepsilon = \begin{cases} (C-A)\dfrac{1}{z-A} & \\ 0 & \text{for} \\ \infty & \end{cases} \begin{matrix} z \neq \infty, A, \\ z = \infty, \\ z = A. \end{matrix}$$

Then

$$\langle (a_1, \ldots, a_m) \rangle = \left\langle \left((C-A)\frac{1}{D_1 - A}, \ldots, (C-A)\frac{1}{D_m - A} \right) \right\rangle$$

$$= \left\langle \begin{pmatrix} A & B \\ D_1 & C \end{pmatrix}, \ldots, \begin{pmatrix} A & B \\ D_m & C \end{pmatrix} \right\rangle = \begin{bmatrix} A & B \\ D_1, \ldots, D_m & C \end{bmatrix}.$$

Case 3: $C = \infty$.

Take

$$z^\varepsilon = \begin{cases} \dfrac{1}{z-A}(A-B)+1 & \\ 1 & \text{for} \\ \infty & \end{cases} \begin{matrix} z \neq \infty, A, \\ z = \infty, \\ z = A. \end{matrix}$$

Then

$$\frac{1}{D_\nu - A}(A-B)+1 = \frac{1}{D_\nu - A}\big((A-B)+(D_\nu - A)\big) = \frac{1}{A - D_\nu}(B - D_\nu)$$

and therefore,

$$\langle (a_1, \ldots, a_m) \rangle = \begin{bmatrix} & A & B \\ D_1, \ldots, D_m & & C \end{bmatrix}.$$

Case 4: $\infty \notin \{A, B, C\}$.
Define

$$a \equiv (B-A)(B-C)^{-1}(C-A), \qquad c \equiv a(A-B)^{-1}$$

and

$$z^s = \begin{cases} a\dfrac{1}{z-A} + c & z \neq \infty, A, \\ c & \text{for } z = \infty, \\ \infty & z = A. \end{cases}$$

This is the mapping $\psi(a, 1, c, A)$ in the notation of [3]. For $D_\nu = \infty$ we have

$$D_\nu^s = c = (A-B)\begin{pmatrix} A & B \\ D_\nu & C \end{pmatrix}(A-B)^{-1}.$$

For $D_\nu \neq \infty$ we get (be aware of $D_\nu \notin \{A, B, C\}$)

$$D_\nu^s = a\frac{1}{D_\nu - A} + c = (A-B)\begin{pmatrix} A & B \\ D_\nu & C \end{pmatrix}(A-B)^{-1}.$$

Thus,

$$\langle (a_1, \ldots, a_m) \rangle = \left\langle \left(t\begin{pmatrix} A & B \\ D_1 & C \end{pmatrix} t^{-1}, \ldots, t\begin{pmatrix} A & B \\ D_m & C \end{pmatrix} t^{-1} \right) \right\rangle$$

$$= \left\langle \left(\begin{pmatrix} A & B \\ D_1 & C \end{pmatrix}, \ldots, \begin{pmatrix} A & B \\ D_m & C \end{pmatrix} \right) \right\rangle = \begin{bmatrix} & A & B \\ D_1, \ldots, D_m & & C \end{bmatrix}$$

holds with $t \equiv A - B$. Now altogether we have

$$\begin{bmatrix} & A & B \\ D_1, \ldots, D_m & & C \end{bmatrix}^* = \begin{bmatrix} & A & B \\ D_1, \ldots, D_m & & C \end{bmatrix}.$$

Lemma 2. *Consider $(A_1, \ldots, A_n) \in [n; S]$ and $\gamma \in \Gamma(S)$. Then*

$$\begin{bmatrix} & A_1^\gamma & A_2^\gamma \\ A_4^\gamma, \ldots, A_n^\gamma & & A_3^\gamma \end{bmatrix} = \begin{bmatrix} & A_1 & A_2 \\ A_4, \ldots, A_n & & A_3 \end{bmatrix}.$$

Proof. Lemma 1 implies

$$\begin{bmatrix} A_1^\gamma & A_2^\gamma \\ A_4^\gamma, \dots, A_n^\gamma & A_3^\gamma \end{bmatrix} = \left\{ (A_4^{\gamma\tau}, \dots, A_n^{\gamma\tau}) \mid \tau \in \Gamma\begin{pmatrix} A_1^\gamma & A_2^\gamma & A_3^\gamma \\ \infty & 0 & 1 \end{pmatrix} \right\}$$

$$= \left\{ (A_4^\delta, \dots, A_n^\delta) \mid \delta \in \Gamma\begin{pmatrix} A_1 & A_2 & A_3 \\ \infty & 0 & 1 \end{pmatrix} \right\} = \begin{bmatrix} A_1 & A_2 \\ A_4, \dots, A_n & A_3 \end{bmatrix}.$$

As a consequence of Lemma 2 we get that the pairs (I, ω) of the theorem are n-point-invariants.

It remains to prove that every n-point-invariant can be described as was done in the theorem.

So let us consider an arbitrary n-point-invariant (I, ω). We have to prove that

$$\begin{bmatrix} A_1 & A_2 \\ A_4, \dots, A_n & A_3 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_4, \dots, B_m & B_3 \end{bmatrix}$$

implies $(A_1, \dots, A_n)^\omega = (B_1, \dots, B_n)^\omega$ for elements $(A_1, \dots, A_n)$, $(B_1, \dots, B_n)$ of $[n; S]$. With

$$\xi \in \Gamma\begin{pmatrix} A_1 & A_2 & A_3 \\ \infty & 0 & 1 \end{pmatrix}, \qquad \eta \in \Gamma\begin{pmatrix} B_1 & B_2 & B_3 \\ \infty & 0 & 1 \end{pmatrix}$$

and Lemma 2 we get

$$\begin{bmatrix} \infty & 0 \\ A_4^\xi, \dots, A_n^\xi & 1 \end{bmatrix} = \begin{bmatrix} \infty & 0 \\ B_4^\eta, \dots, B_n^\eta & 1 \end{bmatrix}$$

and so

$$\langle (A_4^\xi, \dots, A_n^\xi) \rangle = \langle (B_4^\eta, \dots, B_n^\eta) \rangle.$$

This implies the existence of an element $t \in S^*$ such that $A_\nu^\xi = t B_\nu^\eta t^{-1}$ for $\nu = 4, \dots, n$.

Denote the inner automorphism of S, induced by t, by ι. Then we have

$$A_\nu^\gamma = B_\nu, \qquad \nu = 1, 2, \dots, n,$$

with $\gamma = \xi\iota^{-1}\eta^{-1} \in \Gamma(S)$. Thus

$$(A_1, \dots, A_n)^\omega = (A_1^\gamma, \dots, A_n^\gamma)^\omega = (B_1, \dots, B_n)^\omega$$

is true.

As an application we determine in case S is the sfield of quaternions

$$\begin{bmatrix} \infty & 0 \\ i, 2i & 1 \end{bmatrix} = \langle (i, 2i) \rangle \quad \text{and} \quad \begin{bmatrix} \infty & 0 \\ -i, 2i & 1 \end{bmatrix} = \langle (-i, 2i) \rangle.$$

Because there is no inner automorphism ϱ such that $(-i)^\varrho = i$ and $(2i)^\varrho = 2i$ we get $\langle(i, 2i)\rangle \neq \langle(-i, 2i)\rangle$, what implies that $(\infty, 0, 1, i, 2i)$, $(\infty, 0, 1, -i, 2i)$ are in different orbits in correspondence with our previous example.

References

[1] J. Aczél, S. Gołąb, M. Kuczma und E. Siwek, *Das Doppelverhältnis als Lösung einer Funktionalgleichung*, Ann. Polon. Math. 9 (1960), p. 183-187.

[2] W. Benz, *Die 4-Punkt-Invarianten in der projektiven Geraden über einem Schiefkörper*, ibidem 21 (1968), p. 97-101.

[3] W. Benz, *Zur Geometrie der Körpererweiterungen*, Can. J. Math. XXI (1969), p. 1097-1122.

[4] S. Topa, *On a generalization of the functional equation for the harmonic ratio of four points on a projective line over an arbitrary commutative field*, Ann. Polon. Math. 23 (1970), p. 65-72.

Reçu par la Rédaction le 10. 1. 1971