ANNALES
POLONICI MATHEMATICI
XLVI (1985)

On the maximum of $a_4 - 3a_2 a_3 + \mu a_2$ and some related functionals for bounded real univalent functions

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Abstract. By using Grunsky-type estimation the authors maximize the functional $a_4 + (p-2)a_2 a_3 + \mu a_2$ for bounded real functions $S_R(b)$ in some algebraic extremal cases. By aid of the same estimation technique the special case $p = -1$ is treated in essentially all algebraic extremal cases.

1. Introduction. Consider functions f which are analytic and univalent in the unit disc U: $|z| < 1$. The functions are further supposed to be bounded and normalized to form the class

$$S(b) = \{f \mid f(z) = b(z + a_2 z^2 + \ldots),\ |f(z)| < 1,\ 0 < b \leqslant 1\}.$$

In the sense of uniform approximation this class of bounded functions includes the class S of unbounded functions. Therefore, we may briefly denote $S = S(0)$. The corresponding real classes, for which $a_\nu \in R$, are denoted by $S_R(b)$ and $S_R = S_R(0)$.

There are two main methods for estimating the functionals: First, the classical variational method by aid of which the freedom of the extremal function is restricted and, second, the direct estimation method which is based on Grunsky-type inequalities. The two procedures are often complementary, as was seen with regard to the functional $a_3 + \lambda a_2^2$ studied in the complex class $S(b)$ [5], [6].

In the present paper the functional chosen appears to be more receptive to Grunsky-type estimations. Therefore, we choose these estimations as our main tool and limit ourselves to reaffirm some estimation by aid of variations.

2. Estimating the combination $a_4 + (p-2)a_2 a_3 + \mu a_2$. The two inequalities to be used will be called the *Power* and the *Jokinen inequalities*, the theory and the notations of which are presented in [7] and [8]. These inequalities are applied in [9] to a functional closely related to the present one.

Therefore, it is advisable sometimes to refer directly to [9] instead of [7] and [8].

The unoptimized Power inequality in $S_R(b)$ is [9]

$$a_4-2a_2a_3 \leqslant \tfrac{2}{3}(1-b^3)-\tfrac{1}{2}ba_2^2-\tfrac{13}{12}a_2^3-2\lambda(a_3-\tfrac{3}{4}a_2^2+ba_2)+ \\ +\lambda^2[2(1-b)-a_2], \quad \lambda\in R.$$

The choice

$$\lambda = \frac{a_3-\frac{3}{4}a_2^2+ba_2}{2(1-b)-a_2}$$

yields the optimized Power inequality

$$a_4-2a_2a_3 \leqslant \tfrac{2}{3}(1-b^3)-\tfrac{1}{2}ba_2^2-\tfrac{13}{12}a_2^3-\frac{(a_3-\frac{3}{4}a_2^2+ba_2)^2}{2(1-b)-a_2};$$

$$a_4+(p-2)a_2a_3 \leqslant \tfrac{2}{3}(1-b^3)-\tfrac{1}{2}ba_2^2-\tfrac{13}{12}a_2^3+G;$$

$$G = pa_2a_3-\frac{(a_3-\frac{3}{4}a_2^2+ba_2)^2}{2(1-b)-a_2}.$$

Here G is a polynomial of a_3. Rewrite it in the form that presupposes a_3 in a perfect square term. The use of the abbreviations

$$\Delta = 2(1-b)-a_2, \qquad h = \tfrac{3}{4}a_2^2-ba_2$$

yields

$$\begin{aligned} G &= pa_2a_3-\frac{1}{\Delta}(a_3-h)^2 \\ &= -\frac{1}{\Delta}\{a_3-[(\tfrac{3}{4}-\tfrac{1}{2}p)a_2^2+(p-(p+1)b)a_2]\}^2+\tfrac{1}{2}p\{\tfrac{1}{2}(3-p)a_2^3+ \\ &\qquad +(p(1-b)-2b)a_2^2\}; \end{aligned}$$

$$\begin{aligned} (1)\quad & a_4+(p-2)a_2a_3+\mu a_2 \\ & \leqslant \tfrac{2}{3}(1-b^3)+\mu a_2+\tfrac{1}{2}[p^2-(p+1)^2b]a_2^2-\tfrac{1}{4}(p^2-3p+\tfrac{13}{3})a_2^3- \\ & \qquad -\frac{1}{\Delta}\{a_3-[(\tfrac{3}{4}-\tfrac{1}{2}p)a_2^2+(p(1-b)-b)a_2]\}^2 \\ & = \bar{F} \leqslant \tfrac{2}{3}(1-b^3)+\mu a_2+\tfrac{1}{2}[p^2-(p+1)^2b]a_2^2-\tfrac{1}{4}(p^2-3p+\tfrac{13}{3})a_2^3 = F. \end{aligned}$$

In the latter estimation the equality holds for

1° $a_3 = (\frac{3}{4}-\frac{1}{2}p)a_2^2+(p(1-b)-b)a_2$.

The optimizing λ assumes for 1° the value

$$\lambda = \tfrac{1}{2}pa_2.$$

The sharpness of the estimation in (1) is guaranteed as far as the maximizing point (a_2, a_3) on 1° lies in the algebraic part I of the coefficient body (a_2, a_3) (cf. [8], p. 149).

The other device to be used is the Jokinen inequality in $S_R(b)$. Its unoptimized form reads (cf. [8] and [9], (10)):

(2) $$a_4-2a_2a_3+a_2^3-b^2a_2+2\lambda(a_3-a_2^2+1-b^2) \leqslant \tfrac{2}{3}(1+\lambda)^3, \qquad -1 \leqslant \lambda \leqslant 0.$$

Its optimized form reads ([9], (15)):

$$a_4 \leqslant a_2^3+(3b^2-2)a_2+2(a_2+1)x_0^2-\tfrac{4}{3}x_0^3,$$
$$0 \leqslant x_0 = \lambda+1 = \sqrt{a_3-a_2^2+1-b^2} \leqslant 1.$$

This yields for our functional the estimation

$$\begin{aligned} a_4+(p-2)a_2a_3+\mu a_2 &\leqslant (3b^2-2+\mu)a_2+a_2^3+(p-2)a_2a_3+2(a_2+1)x_0^2-\tfrac{4}{3}x_0^3 \\ &= [(p+1)b^2+\mu-p]a_2+(p-1)a_2^3+G_0, \\ G_0 &= (pa_2+2)x_0^2-\tfrac{4}{3}x_0^3. \end{aligned}$$

Take G_0 as a polynomial in x_0. Because $x_0 = \frac{1}{2}pa_2+1$ is the point yielding the local extremum and because $x_0 \geqslant 0$ we see that the following alternatives occur.

For $pa_2+2 > 0$, G_0 is maximized in a_3 (or x_0) for

$$x_0 = \lambda+1 = \tfrac{1}{2}pa_2+1; \qquad \lambda = \tfrac{1}{2}pa_2,$$

i.e. for the curve

$$\sqrt{a_3-a_2^2+1-b^2} = \tfrac{1}{2}pa_2+1;$$

2° $a_3 = (\frac{1}{4}p_2+1)a_2^2+pa_2+b^2$.

The use of the sharp maximum (which is not required here) is permissible if $-2 \leqslant pa_2 \leqslant 0$.

For $pa_2+2 < 0$, G_0 is similarly maximized for

$$x_0 = \lambda+1 = 0,$$

i.e. for the curve

3° $a_3 = a_2^2-(1-b^2)$,

which is the lower boundary arc of the coefficient body (a_2, a_3). The estimation on 2° is sharp as far as the parabola 2° lies in the algebraic part II of the coefficient body ([8], p. 149).

In the previous successful applications the curves 1°, 2° and 3° have formed a smooth arc in the permissible part of the coefficient body, thus allowing the sharp estimation on that arc (cf. e.g. [9]). This would hold true also for properly chosen p-values. As a matter of fact, the parabolae 1° and 2° have a joint tangent at the point, where

$$a_2 = -\frac{2b}{p+1}, \qquad p \neq -1.$$

From this we see that the case $p = -1$, which is our main concern here, will be the only exception where the curves 1° and 2°, assuming the forms

1° $a_3 = \frac{5}{4}a_2^2 - a_2$,

2° $a_3 = \frac{5}{4}a_2^2 - a_2 + b^2$,

do not intersect. Thus, in the case $p = -1$ it is not possible to advance as before, along the complete arc $1^\circ \cup 2^\circ \cup 3^\circ$. However, for a set of p-values we find sharp estimations already from the form (1) of the Power-inequality.

3. The use of the Power inequality for $-\frac{1}{2} \leqslant p \leqslant \frac{2}{3}$, $0 \leqslant b \leqslant 1$. Consider F to be a polynom in a_2. From

$$F' = -\tfrac{1}{4}(3p^2 - 9p + 13)a_2^2 + [p^2 - (p+1)^2 b]a_2 + \mu$$

we see, because $3p^2 - 9p + 13 > 0$, that the local maximum and minimum of F occur correspondingly at

$$\alpha, \beta = 2\frac{p^2 - (p+1)^2 b}{3p^2 - 9p + 13} \pm \frac{2}{3p^2 - 9p + 13}\sqrt{[p^2 - (p+1)^2 b]^2 + \mu(3p^2 - 9p + 13)}.$$

Compare first α and $2(1-b)$ by requiring that

$$\alpha \geqslant 2(1-b); \tag{3}$$

$$\sqrt{[p^2 - (p+1)^2 b]^2 + \mu(3p^2 - 9p + 13)} \geqslant 2p^2 - 9p + 13 - (p-4)(2p-3)b. \tag{4}$$

This holds automatically, if the right-hand side is negative:

$$2p^2 - 9p + 13 + (p-4)(2p-3)b < 0. \tag{5}$$

For those values that interest us, $-\frac{1}{2} \leqslant p \leqslant \frac{2}{3}$, $0 \leqslant b \leqslant 1$, this condition never holds. Thus, in that case, (4) is equivalent to the condition obtained from (4) by squaring:

$$\mu \geqslant (1-b)[p^2 - 9p + 13 - (p^2 - 13p + 11)b]. \tag{6}$$

Hence, for these values (b, μ), (3) holds.

Require now that at the endpoints $\pm\varrho = \pm 2(1-b)$ the following order is valid:

$$F(-\varrho) \leqslant F(\varrho). \tag{7}$$

Condition (3) for the local maximizing point α then guarantees that the polynomial F for $-\varrho \leqslant a_2 \leqslant \varrho$ is necessarily maximized at the endpoint $a_2 = \varrho = 2(1-b)$;

$$F(\varrho) - F(-\varrho) = 2\varrho[\mu - \tfrac{1}{4}(p^2 - 3p + \tfrac{13}{3})\varrho^2] \geqslant 0$$

for $[\ ] \geqslant 0$, i.e. for

$$\mu \geqslant (p^2 - 3p + \tfrac{13}{3})(1-b)^2. \tag{8}$$

Conditions (3) and (7) are true simultaneously provided that the equality-curve in (6) runs above the equality-curve in (8) on the entire interval $0 \leqslant b \leqslant 1$. The direction of the two parabolae is the same for the interval $-\frac{1}{2} \leqslant p \leqslant \frac{2}{3}$. They meet each other at $b = 1$ and at b_0 for which we require

$$b_0 = \frac{9p-13}{15p-10} \geqslant 1; \quad \frac{2p+1}{3p-2} \leqslant 0;$$

$$-\tfrac{1}{2} \leqslant p \leqslant \tfrac{2}{3}. \tag{9}$$

Thus, in this interval (9) the parabola in (6) is above that in (8) for $0 \leqslant b \leqslant 1$.

The maximizing value $a_2 = 2(1-b)$ belongs to the left radial-slit mapping 1:1 (we are referring here to the notation $m{:}n$ for a slit-mapping where the image domain consists of a unit disc minus a system of slits with m starting points and $n \geqslant m \geqslant 1$ endpoints [7]).

If now (8) and hence (7) remains to hold, but we are below the equality curve of (6) we know that, on $|a_2| \leqslant 2(1-b)$, F is necessarily maximized at $a_2 = \alpha > 0$ ($\mu > 0$). In this case the extremal domain is of the type 3:3 or 1:3. In order to check the existence of this extremal mapping we have to consider the location of the curve 1° in the coefficient body.

We refer here to [8], pp. 149–152, where the boundary curves of the algebraic part of the coefficient body (a_2, a_3, a_4) are given (cf. especially Figure 38, p. 149 of [8]). The part I is determined by the boundary curves

2: $$a_3 = a_2^2+b^2-1+\tfrac{1}{4}[2(1-b)-a_2]^2;$$

1′: $$a_2 = -\tfrac{2}{3}-2b+8\sigma-\tfrac{16}{3}\sigma^{3/2}, \quad a_3 = a_2^2+\tfrac{7}{9}+b^2-16\sigma^2+\tfrac{64}{9}(\sigma^{3/2}+\sigma^3),$$
$$\sigma \in [b, 1];$$

2′: $$a_2 = -\tfrac{2}{3}(1-b)+\tfrac{4}{3}(1-3\lambda)(1-b^{-1/2}),$$
$$a_3 = a_2^2+\tfrac{7}{9}(1-b^2)-\tfrac{8}{9}(1-3\lambda)(1-b^{1/2})+\tfrac{4}{9}(1-3\lambda)^2(1-b^{-1}),$$
$$\tfrac{1}{3}-\tfrac{4}{3}b^{3/2} \leqslant \lambda \leqslant \tfrac{1}{3}+\tfrac{8}{3}b^{3/2}.$$

Moreover, the parabola

3: $$a_3 = \tfrac{5}{4}a_2^2-(1+3b)a_2+6b(1-b)$$

divides I in the lower part of 3:3 and in the upper part of 1:3.

Compare first the location of 1° to the lower boundary 2:

$$\begin{aligned}(\tfrac{3}{4}-\tfrac{1}{2}p)a_2+[p(1-b)-b]a_2-\{a_2^2+b^2-1+\tfrac{1}{4}[2(1-b)-a_2]^2\}\\ = \tfrac{1}{2}[2(1-b)-a_2][(p+1)a_2+2b] > 0 \quad \text{for } p\in[-\tfrac{1}{2}, \tfrac{2}{3}],\ a_2 \geqslant 0.\end{aligned} \tag{10}$$

Thus, 1° is always above 2. The location of 1° with respect to the upper boundary arc of I, $1' \cup 2'$, becomes evident by comparing the tangents of 1′ and 1° at the point $a_2 = 2(1-b)$. We have for the derivatives:

$$2' \ni (da_3/da_2)_{a_2=2(1-b)} = -4b < -4b+(b-1)p+3 = (da_3/da_2)_{a_2=2(1-b)} \in 1^\circ.$$

This strongly indicates that 1° lies below $1' \cup 2'$ for $a_2 > 0$, which can be verified from the corresponding equations. Hence, $1° \subset \mathrm{I}$ for $a_2 > 0$, $p \in [-\frac{1}{2}, \frac{2}{3}]$.

Compare finally 1° to 3 by forming

$$(11) \quad \tfrac{5}{4}a_2^2-(1+3b)a_2+6b(1-b)-\{(\tfrac{3}{4}-\tfrac{1}{2}p)a_2^2+[p(1-b)-b]a_2\} = [2(1-b)-a_2][3b-\tfrac{1}{2}(p+1)a_2].$$

This shows that 1° is below 3 as far as

$$0 \leqslant a_2 < \frac{6b}{p+1}$$

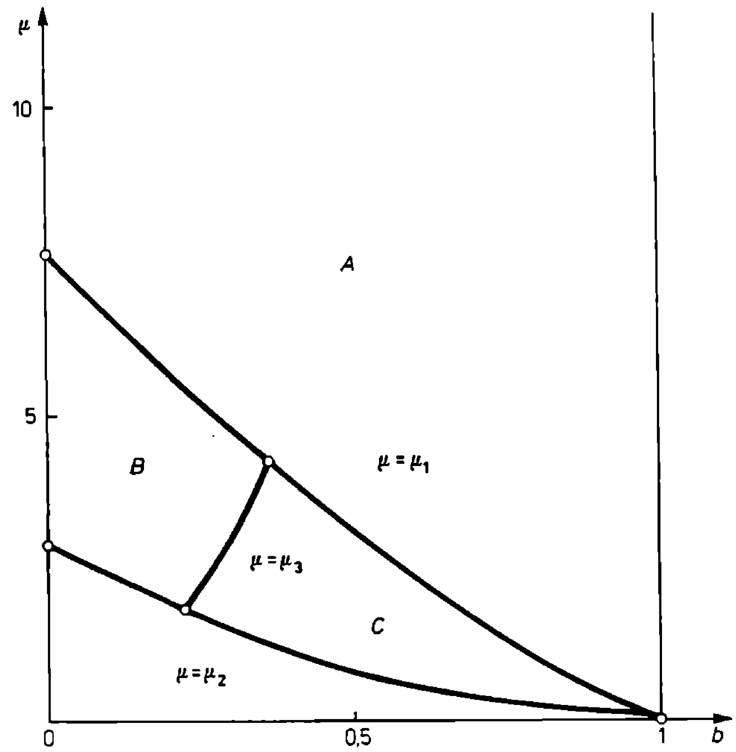
and above it from the point

$$a_2 = \frac{6b}{p+1}$$

onwards. This condition together with $F' = 0$ yields for the boundary curve separating the types 1:3 and 3:3:

$$(12) \qquad \mu = 9\frac{3p^2-9p+13}{(p+1)^2}b^2+6(p+1)b^2-\frac{6p^2 b}{p+1}.$$

Fig. 1 shows an example of the regions found. As to the notations, we refer to the end of Section 4.



$p = 2/3$

Fig. 1

THEOREM 1. *In* $S_R(b)$ *the functional* $a_4+(p-2)a_2a_3+\mu a_2$ *for* $-\frac{1}{2} \leqslant p \leqslant \frac{2}{3}$ *is maximized by the left radial-slit mapping if*

$$\mu \geqslant [p^2-9p+13-(p^2-13p+11)b](1-b) = \mu_1, \quad 0 \leqslant b \leqslant 1.$$

If

$$\mu_2 = (p^2-3p+\tfrac{13}{3})(1-b)^2 \leqslant \mu < \mu_1, \quad 0 \leqslant b \leqslant 1,$$

then the extremal domain is of the type 1:3 *for*

$$\mu \leqslant 9\frac{3p^2-9p+13}{(p+1)^2}b^2+6(p+1)b^2-\frac{6p^2b}{p+1} = \mu_3$$

and of the type 3:3 *for*

$$\mu > \mu_3.$$

4. The use of the Power inequality for $p = -1$. In the main case $p = -1$ the order $\mu_1 \geqslant \mu_2$ holds up to the point

$$b_0 = \tfrac{22}{25} = 0.88.$$

From (11) we see that now 1° is below 3 for $0 \leqslant a_2 < 2(1-b)$. Hence, above the curve $\mu = \mu_1$ the extremal domain is of left radial-slit type and between $\mu = \mu_1$ and $\mu = \mu_2$ of the type 3:3.

THEOREM 2. *In* $S_R(b)$ *the functional* $a_4-3a_2a_3+\mu a_2$ *is maximized by the left radial-slit mapping if*

$$\mu \geqslant \begin{cases} 23-48b+25b^2, & 0 \leqslant b \leqslant 0.88, \\ \frac{25}{3}(1-b)^2, & 0.88 \leqslant b \leqslant 1. \end{cases} \tag{13}$$

The type 3:3 *is the maximal one for*

$$\tfrac{25}{3}(1-b)^2 \leqslant \mu < 23-48b+25b^2, \quad 0 \leqslant b \leqslant 0.88. \tag{14}$$

Clearly, on the interval (0.88, 1] the result is not sharp. We may try using the Power-estimate (1) more effectively as follows.

The upper bound of $a_4-3a_2a_3+\mu a_2$ is taken from

$$F = \tfrac{2}{3}(1-b^3)-\tfrac{25}{12}a_2^3+\tfrac{1}{2}a_2^2+\mu a_2 \quad \text{for } r \leqslant a_2 \leqslant 2(1-b), \tag{15}$$

where r is the intersection point of

1° $a_3 = \frac{5}{4}a_2^2-a_2$

with the upper boundary arc of the coefficient body (a_2, a_3). This consists of the transcendental arc (cf. [7])

$$\begin{aligned} a_2 &= 2(\sigma\ln\sigma-\sigma+b)\in[-2(1-b), 2b\ln b], \\ a_3 &= a_2^2+2\sigma a_2+2(\sigma-b)^2+1-b^2, \quad b \leqslant \sigma \leqslant 1, \end{aligned} \tag{16}$$

and the algebraic arc

$$a_3 = \left(1+\frac{1}{\ln b}\right)a_2^2+1-b^2, \qquad 2b\ln b \leqslant a_2 \leqslant 0. \tag{17}$$

If 1° meets the latter arc we denote $r = a_2^0$. If the intersection point lies on the former arc, then $r = a_2^*$.

From the expression of $\tilde{F}$,

$$\tilde{F} = F-\frac{1}{\Delta}(a_3-\tfrac{5}{4}a_2^2+a_2)^2, \tag{18}$$

we read that for $-2(1-b) \leqslant a_2 \leqslant r$ the unsharp upper bound of $\tilde{F}$ is reached on the upper boundary arc of the coefficient body. By using F for $r \leqslant a_2 \leqslant 2(1-b)$ and $\tilde{F}$ on the upper boundary arc for $-2(1-b) \leqslant a_2 \leqslant r$ we find the maximum for the whole interval $|a_2| \leqslant 2(1-b)$. If the maximizing point is

$$a_2 = \alpha = \tfrac{2}{25}(1+\sqrt{1+25\mu}) > 0 \quad \text{ or } \quad a_2 = 2(1-b)$$

the corresponding point $(a_2, a_3) \in 1° \subset \mathrm{I}$ and the maximum found is sharp. In the other cases F and $\tilde{F}$ can not yield a sharp upper bound.

Numerical comparisons based on the above analysis allow a slight improvement of Theorem 2. Thus it appears that the left radial-slit mapping remains to be the maximum case still for

$$\mu \geqslant \begin{cases} 23-48b+25b^2, & 0 \leqslant b \leqslant 0.92, \\ 0, & 0.92 \leqslant b \leqslant 1. \end{cases} \tag{19}$$

The maximal character of the left radial-slit mapping can be squeezed somewhat below the b-axis. However, although it is most probable that the parabola $y = 23-48b+25b^2$ forms the expected boundary also for all the values $-\frac{1}{25} \leqslant \mu < 0$ our method fails in proving this. Clearly, the improvement would require sharp estimations also in the elliptic cases, not yet available.

Similarly, the maximizing type 3:3 can be extended to hold below the curve $\mu = \frac{25}{3}(1-b)^2$. The unsharp boundary is indicated in Fig. 2 by a dotted line. The true boundary of the 3:3-region seems to meet the parabola $\mu = 23 -48b+25b^2$ before the endpoint $(1, 0)$.

5. The use of the Jokinen inequality for $p = -1$. Start from the unoptimized Jokinen inequality (2). The permissible choice $\lambda = -1$ yields

$$a_4-2a_2a_3+a_2^3-b^2a_2-2(a_3-a_2^2+1-b^2) \leqslant 0;$$

$$a_4-3a_2a_3-\mu a_2 \leqslant (2-a_2)a_3+2(1-b^2)+(b^2+\mu)a_2-2a_2^2-a_2^3 = \tilde{\tilde{F}}. \tag{20}$$

As can be directly checked, the equality holds for the right radial-slit mapping, for which

$$a_2 = -2(1-b), \quad a_3 = 3-8b+5b^2, \quad a_4 = -(4-20b+30b^2-14b^3).$$

Because in $\tilde{\tilde{F}}$ the factor $2-a_2 > 0$, we obtain a valid upper bound by maximizing a_3 for any fixed a_2, i.e. $\tilde{\tilde{F}}$ is to be taken on the upper boundary arc of the coefficient body (a_2, a_3) for $-2(1-b) \leqslant a_2 \leqslant r$. As before r has the meaning a_2^0 or a_2^*. For $r \leqslant a_2 \leqslant 2(1-b)$ the upper bound F can, of course, be used as before.

Try to find the region where the right radial-slit mapping gives the maximum. It is necessary for (20) yielding the result that $\tilde{\tilde{F}}$ is locally maximized at $a_2 = -2(1-b)$. The left upper boundary arc has the parametric presentation (16). The left corner point is reached for $\sigma = 1$.

Introduce

$$1-\sigma = d > 0.$$

In d we have, according to (16),

$$a_2 = -2(1-b)+d^2+o(d^2), \quad a_3 = 3-8b+5b^2+4bd^2+o(d^2).$$

This, when substituted in $\tilde{\tilde{F}}$, yields

$$\tilde{\tilde{F}} = 14-46b+48b^2-16b^3-2(1-b)\mu+(\mu-7+40b-24b^2)d^2+o(d^2).$$

From this we see that (20) can yield the right radial-slit mapping as a maximal case only for

$$\mu \leqslant 7-40b+24b^2. \tag{21}$$

On the other hand the right radial-slit mapping cannot be maximal in a larger region. This is seen by considering the inequality of Jokinen in the optimized form, yielding

$$a_4 - 3a_2a_3 + \mu a_2 \leqslant (\mu+1)a_2 - 2a_2^3 + (2-a_2)x_0^2 - \tfrac{4}{3}x_0^3.$$

The equality is reached here in the algebraic part II of the coefficient body (cf. [8], p. 149) where the lower boundary arc

$$a_3 = a_2^2 - 1 + b^2$$

belongs. On it we have a valid equation:

$$a_4 - 3a_2a_3 + \mu a_2 = (\mu+1)a_2 - 2a_2^3.$$

Consider the values close to the point $a_2 = -2(1-b)$ by denoting

$$a_2 = -2(1-b)+d^2;$$

$$\begin{aligned} a_4 &- 3a_2a_3 + \mu a_2 \\ &= 14-46b+48b^2-16b^3-2(1-b)\mu+(\mu-7+40b-24b^2)d^2+o(d^2). \end{aligned}$$

Thus, if $\mu > 7+40b-24b^2$, the value of the functional for $d^2 > 0$ exceeds its value at $a_2 = -2(1-b)$. This shows, that the limit $\mu = 7-40b+24b^2$ in (21) cannot be improved.

By using the estimation (20) we can confirm numerically that if (21) holds, then the right radial-slit mapping does indeed give the maximum except in the triangular region close to the point (0, 7).

In order to facilitate the comparisons consider $\tilde{\tilde{F}}$ on the upper boundary arc of the coefficient body. On the transcendental part (16) we have

$$\text{(22)} \quad \frac{d\tilde{\tilde{F}}}{da_2} = \mu-1+8\sigma+4b\sigma-2\sigma^2-6\sigma a_2-6a_2^2, \qquad a_2 = 2(\sigma\ln\sigma-\sigma+b),$$

and on the parabolic arc (17)

$$\text{(23)} \qquad \frac{d\tilde{\tilde{F}}}{da_2} = \mu-1+2b^2+\frac{4}{\ln b}a_2-3\left(2+\frac{1}{\ln b}\right)a_2^2.$$

If $\mu \leqslant 7-40b+24b^2$, then $d\tilde{\tilde{F}}/da_2 \leqslant 0$ for the interval $-2(1-b) \leqslant a_2 \leqslant r$, provided that $s \leqslant b \leqslant 1$, where s is somewhat smaller than 0.16. For $0 \leqslant b < s$ the derivative is non-positive up to a certain limit $\mu_0(b)$ determinable numerically. Moreover, if $\mu \leqslant -\frac{1}{25}$, then $dF/da_2 < 0$. These observations allow us to restrict the comparisons to the values $\tilde{\tilde{F}}(-2(1-b))$, $\tilde{\tilde{F}}(r)$, $F(r)$, $F(\alpha)$ when considering the cases $\mu \leqslant 7-40b+24b^2$. From these we infer that the right radial-slit mapping remains to be the extremal one as far as $F(\alpha) < F(-2(1-b))$. However, if μ is large enough this condition fails to hold and the curve

$$F(-2(1-b)) = F(\alpha)$$

forms a limit case between the right radial-slit mapping and 3:3.

THEOREM 3. *In $S_R(b)$ the functional $a_4-3a_2a_3+\mu a_2$ is maximized by the right radial-slit mapping if*

$$\text{(24)} \qquad \mu \leqslant \begin{cases} 7-40b+24b^2, & 0.076\cdot 604\cdot 918 \leqslant b \leqslant 1, \\ \mu(b), & 0 \leqslant b \leqslant 0.076\cdot 604\cdot 918, \end{cases}$$

where $\mu(b)$ is the root of the equation

$$\text{(25)} \qquad F(-2(1-b)) = F(\alpha),$$

with

$$F(-2(1-b)) = 14-46b+48b^2-16b^3-2(1-b)\mu,$$

$$F(\alpha) = \tfrac{2}{3}(1-b^3)+\frac{4}{75}\frac{1+25\mu}{25}(1+\sqrt{1+25\mu})+\frac{2\mu}{75}.$$

The boundary curve (25) is sharp; above it the extremal type 3:3 holds

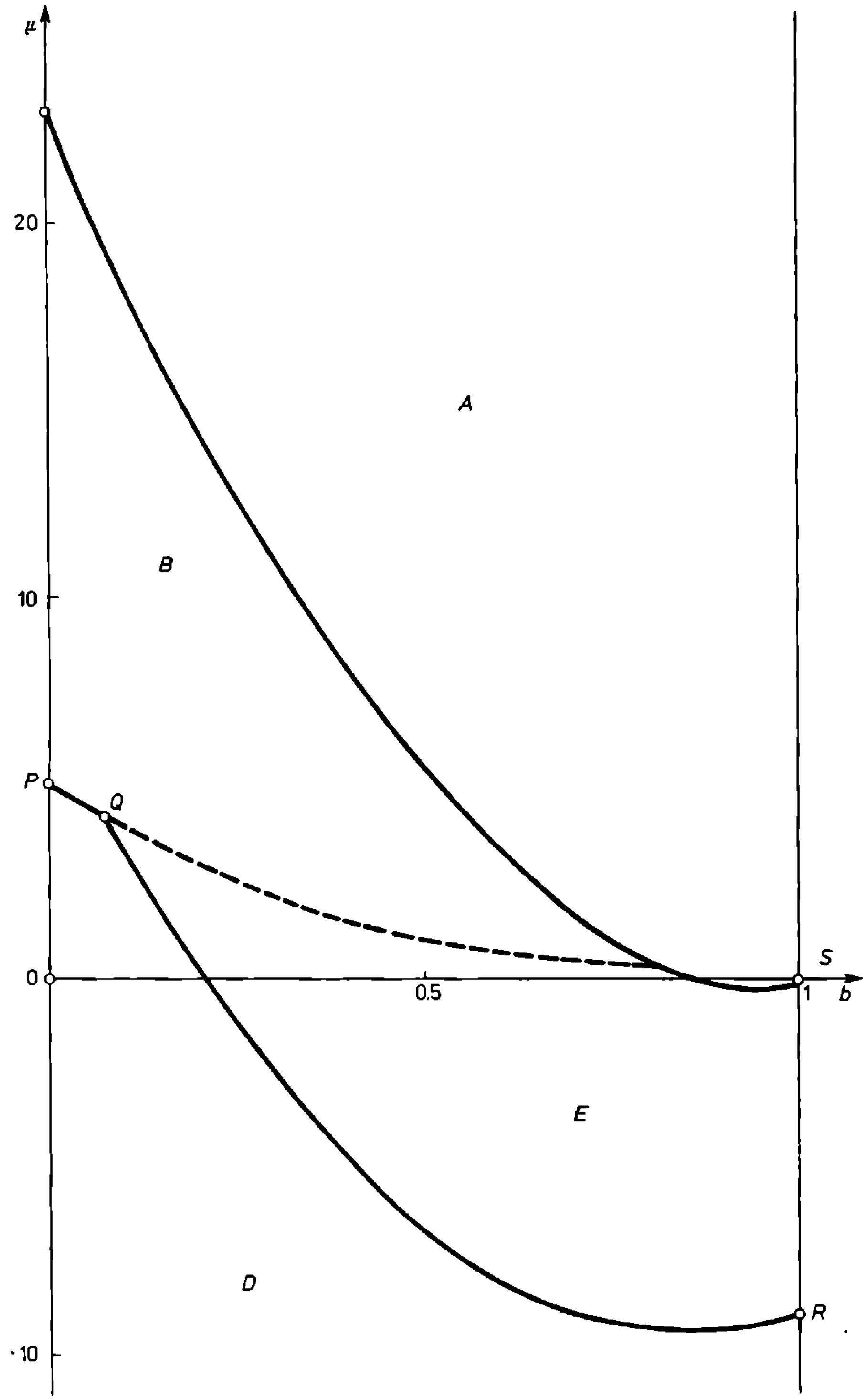


Fig. 2

and on it there are thus two extremal domains, 3:3 and the right radial slit-mapping.

Fig. 2 shows the boundary arcs separating the regions of different extremal domains. Let us denote these regions as well as the corresponding extremal domais by the letters:

A: left radial-slit, 1:1,
B: 3:3,
C: 1:3 (Fig. 1),
D: right radial-slit, 1:1,
E: elliptic cases.
The endpoints of separate arcs are:

$$P = (0, 4.971\cdot 084\cdot 667), \qquad Q = (0.076\cdot 604\cdot 918, 4.076\cdot 642\cdot 803),$$

$$R = (1, 9), \qquad S = (1, 0).$$

As was stated above, the arc $B \cap D$ is only an approximative one. Similarly, the lower part of $A \cap B$ and $A \cap E$ remains open for b close to 1.

6. The use of the variational method for $p = -1$. In [2] the variational condition of Dziubiński type [1] was derived for the functional in question (cf. [2], (5.5)–(5.7), p. 229). Rewrite this for the b-formalism used previously in the present paper. Thus, for the extremal function $f \in S_R(b)$ maximizing the functional $a_4 - 3a_2 a_3 + \mu a_2$ we have the necessary condition

$$\left(\frac{f'(z)}{f(z)}\right)^2 M[f(z)] = \frac{1}{z^2} N(z), \qquad 0 < |z| < 1, \tag{26}$$

with

$$M(w) = D_3(w^3 + w^{-3}) + D_2(w^2 + w^{-2}) + D_1(w + w^{-1}) - 2P,$$

$$N(z) = E_3(z^3 + z^{-3}) + E_2(z^2 + z^{-2}) + E_1(z + z^{-1}) + 2E_0 - 2P,$$

$$D_3 = 2b^4, \qquad D_2 = 0, \qquad D_1 = 2b^2(\mu - a_3 - 5a_2^2),$$

$$E_3 = 2b, \qquad E_2 = -2ba_2, \qquad E_1 = 2b(\mu - 6a_2^2), \qquad E_0 = b(3a_4 - 9a_3 a_2 + \mu a_2),$$

$$P = 2b^4 \min_{0 \leqslant x < 2\pi} u(x), \qquad u(x) = \cos 3x + u \cos x,$$

$$u = b^{-2}(\mu - a_3 - 5a_2^2).$$

The functions M and N, which are non-negative on the unit circumference possess the symmetry property: If t is a zero of M or N so is also t^{-1} and $\bar{t}$. For P we obtain [2]

$$P = \begin{cases} 2b^4(u+1) & \text{if } u < -9, \\ -2b^4\left(\dfrac{3-u}{u}\right)^{3/2} & \text{if } -9 \leqslant u \leqslant 0, \\ -2b^4(u+1) & \text{if } 0 < u. \end{cases}$$

We want to choose μ so that a desired factorization leading to the left radial-slit mapping occurs in M and N. Following the lines of [2] we observe that the best possible choice for μ is

$$\mu > (1-b)(27-25b). \tag{27}$$

This yields

$$M(w) = 2b^4 w^{-3}(w+1)^2 [w^2 - (1 - i\sqrt{u})w + 1][w^2 - (1 + i\sqrt{u})w + 1],$$
$$N(z) = 2bz^{-3}(z+1)^2 [z^2 - (1 + \tfrac{1}{2}a_2 - i\sqrt{v})z + 1][z^2 - (1 + \tfrac{1}{2}a_2 + i\sqrt{v})z + 1],$$
$$u > 0, \quad v = \mu - \tfrac{25}{4}a_2^2 + a_2 > 0.$$

According to [4] and [3] this implies that the extremal function can only be the left radial-slit mapping.

In order to achieve the correct factorization for the right radial-slit mapping we observe similarly that the choice

$$\mu < \begin{cases} -\frac{29}{3} & \text{for } 0 < b \leqslant \frac{5}{6}, \\ 7 - 40b + 24b^2 & \text{for } \frac{5}{6} \leqslant b < 1, \end{cases} \tag{28}$$

yields for (26)

$$\left(z\frac{w'}{w}\right)^2 b^3 w^{-3}(w-1)^2 [w^2 + (1 + \sqrt{-u})w + 1][w^2 + (1 - \sqrt{-u})w + 1]$$
$$= z^{-3}(z-1)^2 [z^2 + (1 - \tfrac{1}{2}a_2 + \sqrt{v_1})z + 1][z^2 + (1 - \tfrac{1}{2}a_2 - \sqrt{v_1})z + 1];$$
$$u = b^{-2}(\mu - a_3 - 5a_2^2) < -9, \quad v_1 = \tfrac{25}{4}a_2^2 + a_2 - \mu > 4.$$

Thus the right radial-slit mapping is the extremal one [4], [3].

Observe that although not the best possible the limits (27) and (28) confirm in part the results obtained from Grunsky-type estimation. In (28) the latter limit is the one in (24).

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Reçu par la Rédaction le 3.02.1984