ON OPERATORS ON $L_0$

BY

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In [4] the author showed that operators on the spaces $L_p$ \((0, 1)\) for \(0 \leq p < 1\) cannot be too small. Precisely, if $T: L_p \to X$ is a non-zero operator where $X$ is an $F$-space, then there is a closed subspace $V$ of $L_p$ isomorphic to $l_2$ such that $T|V$ is an embedding (isomorphism into $X$). This note* represents an attempt to push these results a little further in the case $p = 0$.

It was originally the author's belief that if $T: L_0 \to X$ is a non-zero operator, then there is a Borel subset $A$ of $(0, 1)$ such that the restriction of $T$ to $L_0(A)$ (functions supported on $A$) is an isomorphism. This is, however, not the case although it is true when $X = L_0$. An example in [5] shows that it is possible to find a proper closed subspace $N$ of $L_0$ such that the quotient mapping does not have this property.

We therefore introduce the notion of a weakly singular operator on $L_0$, which is defined to be an operator which fails to be an isomorphism on any subspace $L_0(A)$, where $A$ has positive measure. In Theorem 1 we characterize weakly singular operators and we use this characterization to show that if $X$ is an ordered $F$-space whose positive cone is closed and $T: L_0 \to X$ is positive and weakly singular, then $T = 0$. This result is related to recent work of Aliprantis and Burkinshaw [1] who show that the topology of $L_0$ is minimal with respect to locally solid topologies on $L_0$ considered as a vector lattice. Stated in terms of operators the result of Aliprantis and Burkinshaw shows that a lattice isomorphism $T: L_0 \to X$, where $X$ is a locally solid topological vector lattice, is a topological isomorphism.

We conclude by mentioning that the existence of non-zero weakly singular operators implies that certain subspaces of $L_0$ and $L_\infty$ cannot be ultrabarrelled.

Our notation is as follows. $\mathcal{B}$ will denote the collection of Borel subsets of $(0, 1)$ and $\mathcal{C}$ the sub-algebra of $\mathcal{B}$ generated by the dyadic intervals

$$[(k-1)\cdot 2^{-n}, k\cdot 2^{-n}) \cap (0, 1).$$

$\lambda$ will denote the usual Lebesgue measure on $\mathcal{B}$.

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The space $L_0 = L_0(0, 1)$ consists of all real Borel functions on $(0, 1)$, where, as usual, functions differing only on a set of measure zero will be identified. $L_0$ is an $F$-space (complete metrizable topological vector space) when equipped with the topology of convergence in measure; this topology is given by the $F$-norm

$$\|f\| = \int_0^1 \frac{|f(t)|}{1 + |f(t)|} dt.$$  

$A$ will denote the subspace of $L_0$ of all simple functions and $\Gamma$ the subspace of all countably simple functions. $L_0$ is a lattice and, for $f \in L_0$, we write, as usual, $f^+ = \sup (f, 0)$ and $f^- = \sup (-f, 0)$. If $A \in \mathcal{A}$, then $L_0(A)$ is the subset of $L_0$ of all $f$ supported in $A$ and, for $g \in L_0$, $R_A g$ is defined by

$$R_A g(t) = \begin{cases} 
  g(t), & t \in A, \\
  0, & t \notin A.
\end{cases}$$

Thus $R_A$ is a projection of $L_0$ onto its subspace $L_0(A)$.

If $\mathcal{I}$ denotes the ideal of sets in $\mathcal{A}$ of measure zero, then $\mathcal{A}/\mathcal{I}$ is a complete Boolean algebra. We shall refer to the lattice infimum of a collection of sets $(A_i)_{i \in I}$ in $\mathcal{A}$, and by this we mean the set $A$ in $\mathcal{A}$ unique up to sets of measure zero such that

$$\lambda(A \setminus A_i) = 0, \quad i \in I,$$

and if $\lambda(B \setminus A_i) = 0$ (i.e., I), then $\lambda(B \setminus A_i) = 0$.

The term operator will mean a continuous linear map and all spaces under consideration will be $F$-spaces.

**Definition.** An operator $T : L_0 \to X$ (where $X$ is an $F$-space) is weakly singular if, for every Borel subset $A$ of $(0, 1)$ with $\lambda(A) > 0$, $T|L_0(A)$ fails to be an isomorphism.

**Remarks.** (1) If $T : L_0 \to X$ is weakly singular and $S : X \to Y$ is any linear operator, then $ST : L_0 \to Y$ is also weakly singular. However, other "ideal" properties of weak singularity are not clear. In particular, if $S : L_0 \to X$ and $T : L_0 \to X$ are weakly singular, is $S + T$ weakly singular? (P 1273)

(2) If $T : L_0 \to L_0$ is weakly singular, then $T = 0$ (see [5]).

(3) There is a closed subspace $M$ of $L_0$ such that the quotient map $q : L_0 \to L_0/M$ is a non-zero weakly singular operator [5].

**Lemma 1.** Suppose $T : L_0 \to X$ is an operator, $\varepsilon > 0$, and

$$B_\varepsilon = \{ f \in L_0 : \lambda(|f| > 1) \leq \varepsilon \}.$$  

Suppose $0$ is not in the closure of $T(L_0 \setminus B_\varepsilon)$. Then there is a Borel subset $A$ of $(0, 1)$ such that $T|L_0(A)$ is an isomorphism and $\lambda(A) \geq 1 - \varepsilon$.

**Proof.** We may suppose $X$ is $F$-normed in such a way that if $f \in L_0$ and $\|Tf\| \leq 1$, then $f \in B_\varepsilon$. 


Let \( \mathcal{F} \) be the set of sequences \( \Phi = (\phi_n) \) in \( L_0 \) such that \( \sum \|T\phi_n\| < \infty \). For each \( \Phi \in \mathcal{F} \), let \( A(\Phi) = \{ t : \phi_n(t) \to 0 \} \). Let \( A \) be the lattice infimum of \( (A(\Phi) : \Phi \in \mathcal{F}) \) so that \( \lambda(A \setminus A(\Phi)) = 0 \) for each \( \Phi \in \mathcal{F} \). Then, if \( f_n \in L_0(A) \) and \( \sum \|Tf_n\| < \infty \), we have \( f_n \to 0 \) a.e. Thus it follows easily that \( f_n \in L_0(A) \) and \( Tf_n \to 0 \). Then \( f_n \to 0 \) in measure.

The proof will therefore be complete if we can show that \( \lambda(A) \geq 1 - \varepsilon \).

Now

\[
\lambda(A) = \inf \{ \lambda\left( \bigcap_{i=1}^k A(\Phi^i) \right) : \Phi^i \in \mathcal{F}, \ i = 1, 2, \ldots, k \}.
\]

Fix \( k \) and \( (\Phi^i : i = 1, \ldots, k) \) in \( \mathcal{F} \). For \( (s_1, \ldots, s_k) \in (0, 1)^k \) we have \( (s_1 \phi^{1}_{\infty}_n + \ldots + s_k \phi^{k}_{\infty}_n)_{n=1}^{\infty} \in \mathcal{F} \). Let \( \Delta \) be the set of \( (t, s_1, \ldots, s_k) \in (0, 1)^{k+1} \) such that

\[
\lim_{n \to \infty} \sum_{i=1}^k s_i \phi^{i}_{\infty}(t) = 0.
\]

For fixed \( t \), let

\[
\Delta_t = \{(s_1, \ldots, s_k) \in (0, 1)^k : (t, s_1, \ldots, s_k) \in \Delta \}.
\]

Then \( \lambda_k(\Delta_t) = 0 \) unless \( t \in \bigcap_{i=1}^k A(\Phi^i) \), and in this case \( \lambda_k(\Delta_t) = 1 \). (Here \( \lambda_k \) is product Lebesgue measure on \( (0, 1)^k \).) Hence

\[
\lambda_{k+1}(\Delta) = \lambda\left( \bigcap_{i=1}^k A(\Phi^i) \right).
\]

It follows that there exists \( (s_1, \ldots, s_k) \in (0, 1)^k \) such that if

\[
\Psi = (s_1 \phi^{1}_{\infty}_n + \ldots + s_k \phi^{k}_{\infty}_n),
\]

then

\[
\lambda(A(\Psi)) \leq \lambda\left( \bigcap_{i=1}^k A(\Phi^i) \right).
\]

Thus

\[
\lambda(A) = \inf_{\Phi \in \mathcal{F}} \lambda(A(\Phi)).
\]

We complete the proof by showing \( \lambda(A(\Phi)) \geq 1 - \varepsilon \).

To do this let \( \eta \) be a normally distributed random variable with mean zero and variance one, defined on some probability space \( (\Omega, \Sigma, P) \); let \( (\eta_n)_{n=1}^{\infty} \) be a sequence of independent copies of \( \eta \) defined also in \( (\Omega, \Sigma, P) \). Then, for \( \omega \in \Omega \),

\[
\|\eta_n(\omega) T\phi_m\| \leq (|\eta_n(\omega)| + 1) \|T\phi_m\|,
\]

and

\[
\lambda(A) \geq 1 - \varepsilon.
\]
and hence
\[ \int \sum_{n=1}^{\infty} \| \eta_n(\omega) T \varphi_n \| dP(\omega) \leq \left( E(\|\eta\|) + 1 \right) \sum_{n=1}^{\infty} \| T \varphi_n \|. \]

In particular,
\[ \sum_{n=1}^{\infty} \| \eta_n(\omega) T \varphi_n \| < \infty \quad P\text{-a.e.} \]

Thus, if
\[ \Omega_m = \{ \omega : \sum_{n=1}^{\infty} \| m^{-1} \eta_n(\omega) T \varphi_n \| \leq 1 \}, \]

then \( P(\Omega_m) \to 1 \) as \( m \to \infty \).

For fixed \( m, N \), define \( g_{m,N} : (0, 1) \times \Omega \to \mathbb{R} \) by
\[ g_{m,N}(t, \omega) = \sum_{n=1}^{N} \frac{1}{m} \eta_n(\omega) \varphi_n(t) \]

and let \( Q_{m,N} = \{ (t, \omega) : |g_{m,N}(t, \omega)| > 1 \} \). Then
\[ (\lambda \times P)(Q_{m,N}) = \int_{0}^{1} \int_{\Omega} \chi_{Q_{m,N}}(t, \omega) dP(\omega) dt = \int_{0}^{1} P(\|\eta\| > m \left( \sum_{n=1}^{N} |\varphi_n(t)|^2 \right)^{-1/2}) dt \]

while, on the other hand,
\[ (\lambda \times P)(Q_{m,N}) = \int_{\Omega} \int_{0}^{1} \chi_{Q_{m,N}}(t, \omega) dt dP(\omega) \leq P(\Omega - \Omega_m) + \int_{\Omega_m} \int_{0}^{1} \chi_{Q_{m,N}}(t, \omega) dt dP(\omega) \leq P(\Omega - \Omega_m) + \varepsilon P(\Omega_m). \]

Letting \( N \to \infty \), by the dominated convergence theorem, we obtain
\[ \int_{0}^{1} P(\|\eta\| > m \left( \sum_{n=1}^{\infty} |\varphi_n(t)|^2 \right)^{-1/2}) dt \leq P(\Omega - \Omega_m) + \varepsilon P(\Omega_m), \]

and hence
\[ \lambda\left( t : \sum_{n=1}^{\infty} |\varphi_n(t)|^2 = \infty \right) \leq P(\Omega - \Omega_m) + \varepsilon P(\Omega_m). \]

Let \( m \to \infty \). We have \( \sum_{n=1}^{\infty} |\varphi_n(t)|^2 < \infty \) except on a set of measure at most \( \varepsilon \). Thus \( \lambda(A(\Phi)) \geq 1 - \varepsilon \) and the result follows.

We derive two immediate consequences from Lemma 1.

**Lemma 2.** Suppose \( T : L_0 \to X \) is weakly singular. Then for every \( m \in N \) and \( \varepsilon > 0 \) there exists \( f \in L_0 \) such that \( |f(t)| \geq m \) a.e. and \( ||Tf|| \leq \varepsilon \).
Proof. If the lemma were false, then for some \( \varepsilon (0 < \varepsilon < 1) \) and \( m \in \mathbb{N} \) the inequality \( |f| \geq m \) a.e. implied \( \| Tf \| > \varepsilon \). Pick \( \delta > 0 \) so that \( \| f \| \leq \delta \) implies \( \| Tf \| < \frac{1}{2} \varepsilon \). Then if \( f \notin B_{1-\delta} \), there exists \( g \in L_0 \) with \( \| g \| \leq \delta \) and \( |mf + g| \geq m \) a.e. Hence

\[
\| T(mf + g) \| > \varepsilon, \quad \| T(mf) \| > \varepsilon/2, \quad \| Tf \| > \varepsilon/2m.
\]

Thus \( T(L_0 \setminus B_{1-\delta}) \) does not have 0 as a closure point, and so there is a set \( A \) of measure \( \delta \) such that \( T|L_0(A) \) is an isomorphism. This contradiction proves the lemma.

Lemma 3. Suppose \( (X_n)_{n=1}^\infty \) is a sequence of \( F \)-spaces and \( \omega(X_n) \) their countable product. Suppose \( T: L_0 \to \omega(X_n) \) is a linear operator given by \( Tf = (T_n f)_{n=1}^\infty \), where \( T_n: L_0 \to X_n \). Suppose for every \( n \) the map

\[
S_n: L_0 \to X_1 \oplus \ldots \oplus X_n
\]

given by \( S_n = T_1 \oplus \ldots \oplus T_n \) is weakly singular. Then \( T \) is also weakly singular.

Proof. If \( T \) is not weakly singular, there exists a Borel set \( A \) of positive measure such that \( T|L_0(A) \) is an isomorphism. Thus, it suffices to consider the case where \( T \) is an isomorphism. Then there is a neighborhood of zero in \( \omega(X_n) \) of the form

\[
V = \{(x_n)_{n=1}^\infty : \|x_i\| \leq \varepsilon_i, \; i = 1, \ldots, N\}
\]

such that \( V \cap T(L_0 \setminus B_{1/2}) = \emptyset \). Hence, if \( V' \subset X_1 \oplus \ldots \oplus X_N \) is the set

\[
V' = \{(x_n)_{n=1}^N : \|x_i\| \leq \varepsilon_i, \; i = 1, \ldots, N\},
\]

then \( V' \cap S_N(L_0 \setminus B_{1/2}) = \emptyset \). By Lemma 1, \( S_N \) fails to be weakly singular and this proves Lemma 3.

Theorem 1. Suppose \( T: L_0 \to X \) is a linear operator. Then the following conditions are equivalent:

(i) \( T \) is weakly singular.

(ii) There is a sequence \( u_n \) with \( u_n \in A, \|u_n\| \geq n \), and \( TR_A u_n \to 0 \) for every \( A \in \mathcal{A} \).

(iii) There is a sequence \( v_n \) in \( L_0 \) with \( |v_n| \to \infty \) uniformly and \( TR_A v_n \to 0 \) for every \( A \in \mathcal{A} \).

(iv) There is a sequence \( v_n \in L_0 \) with \( v_n \geq 0 \) and \( v_n \to \infty \) uniformly so that \( TR_A v_n \to 0 \) for every \( A \in \mathcal{A} \).

Proof. (i) \( \Rightarrow \) (ii). Let \( (D(n) : n \in \mathbb{N}) \) be an enumeration of the sets of \( \mathcal{A} \) and consider the map \( \tilde{T}: L_0 \to \omega(X) \) (countable product of copies of \( X \)) given by

\[
\tilde{T}f = (TR_{D(n)} f)_{n=1}^\infty.
\]
If \( \tilde{T} \) is not weakly singular, then there exist \( N \in \mathbb{N} \) and \( A \in \mathcal{B} \) such that 

\[
S_N f = (T_{R_D(1)} f, \ldots, T_{R_D(N)} f)
\]

(\( S_N : L_0 \to \bigoplus X_1 \oplus \ldots \oplus X_N \)). Let \( A_0 \) be an atom of the algebra of sets generated by \((A, D(1), \ldots, D(N))\), which is contained in \( A \). Consequently, \( T_{R_A} \) is an embedding and \( T \) is not weakly singular. We conclude that \( \tilde{T} \) is weakly singular.

Now, by Lemma 2 we may find a sequence \( f \in L_0 \) with \( |f_n| \geq n \) and \( \tilde{T} f_n \to 0 \). Choose \( |u_n| \geq n \) so that \( u_n \in A, u_n - f_n \to 0, \) and \( |u_n| \geq n \). Then \( \tilde{T} u_n \to 0 \).

If \( A \in \mathcal{B} \) and \( \varepsilon > 0 \), then we may choose \( \delta > 0 \) so that if \( ||f|| \leq \delta \), then \( ||Tf|| \leq \varepsilon \). Choose \( D \in \mathcal{D} \) so that \( \lambda (A \Delta D) \leq \delta \). Then \( ||R_A u_n - R_D u_n|| \leq \delta \), and so \( ||T_{R_A} u_n - T_{R_D} u_n|| \leq \varepsilon \). Thus

\[
\limsup_{n \to \infty} ||T_{R_A} u_n|| \leq \varepsilon
\]

since \( \tilde{T} u_n \to 0 \) implies \( T_{R_D} u_n \to 0 \). As \( \varepsilon > 0 \) is arbitrary, the implication (i) \( \Rightarrow \) (ii) is established.

(ii) \( \Rightarrow \) (iv). First, observe that by the preceding argument it suffices to show that \( T_{R_D} v_n \to 0 \) for \( D \in \mathcal{D} \), i.e. that \( \tilde{T} u_n \to 0 \). Next, observe that if \( s \) is a simple function in \( L_0 \) (i.e. a finite linear combination of characteristic functions), then \( \tilde{T}(s u_n) \to 0 \).

For each \( \varepsilon > 0 \) let \( K(\varepsilon) = \{ f : \lambda(supp f^-) \leq \varepsilon \} \). Suppose \( f \in K(\varepsilon) \). Then for \( n \geq 2 \)

\[
\lambda(f^+ (1 + u_n)) + \lambda(f^-(1 - u_n)) \leq \varepsilon,
\]

and hence, for some \( \theta_n = \pm 1 \), if \( g_n = f^+ - f^-(1 + \theta_n u_n) \), then \( g_n \in K(\frac{\varepsilon}{2}) \). Now

\[
\tilde{T} g_n = \tilde{T} f - \tilde{T}(\theta_n u_n f^-) \to \tilde{T} f \quad \text{as} \quad n \to \infty.
\]

Hence \( \tilde{T} f \in \overline{TK}(\frac{\varepsilon}{2}) \) so that \( \overline{TK}(\varepsilon) \subset \overline{TK}(\frac{\varepsilon}{2}) \). It now follows that \( \tilde{T} A = \overline{TK}(1) \subset \overline{TK}(\delta) \) for every \( \delta > 0 \).

Thus there exist \( f_n \in K(1/n) \) such that \( \tilde{T}(f_n + n) \to 0 \). Since \( f_n \in K(1/n) \), we have \( f_n - f_n^+ \to 0 \), and so \( \tilde{T}(f_n^+ + n) \to 0 \). Writing \( v_n = f_n^+ + n \) we complete the proof.

(iv) \( \Rightarrow \) (iii) is immediate.

(iii) \( \Rightarrow \) (i). For \( A \in \mathcal{B} \) with \( \lambda(A) > 0 \), the sequence \( (R_A u_n)_{n=1}^\infty \) does not converge to zero in \( L_0(A) \) and \( T_{R_A} u_n \to 0 \).

For the next theorem we shall use the term "ordered \( F \)-space" to denote an \( F \)-space \( X \) equipped with a partial ordering \( \leq \) satisfying

1. \( x + u \leq y + u \) whenever \( x \leq y \) and \( u \in X \);
2. \( \lambda x \leq \lambda y \) whenever \( x \leq y \) and \( \lambda \geq 0 \);
3. \( x \leq y \) and \( y \leq x \) if and only if \( y = x \);
4. the positive cone \( \{ x : x \geq 0 \} \) is closed.
Theorem 2. Suppose $X$ is an ordered $F$-space and $T: L_0 \to X$ is positive (i.e. $x \geq 0$ implies $Tx \geq 0$) and weakly singular. Then $T = 0$.

Proof. Suppose $u \in L_x$. Choose, by Theorem 1 (iii), a sequence $v_n \in L_0$ with $v_n \to \infty$ uniformly, $v_n \geq 0$, and $Tv_n \to 0$. Then, eventually, $u \leq v_n$, and so $Tu \leq Tv_n$. Since the positive cone is closed, $Tu \leq 0$ and, analogously, $T(-u) \leq 0$. Thus, as $L_x$ is dense in $L_0$, we have $T = 0$.

Corollary. Suppose $X$ is an $F$-space and $T: L_0 \to X$ is a weakly singular operator. Let $P$ denote the positive cone of $L_0$. Then $T(P)$ is dense in $T(X)$.

Proof. Let $M = \overline{T(X)}$ and $Q = \overline{T(P)}$. Then $Q$ is a wedge (see [3]). Let $N = Q \cap (-Q)$; $N$ is a closed subspace of $M$.

Consider the quotient space $M/N$ and let $\pi: M \to M/N$ be the quotient map. Then $\pi(Q)$ is a cone in $M/N$ (i.e. $\pi(Q) \cap -\pi(Q) = \{0\}$) and may easily be checked to be closed. Consider the space $M/N$ ordered by the cone $\pi(Q)$ and consider $\pi \circ T: L_0 \to M/N$. Then $\pi \circ T$ is weakly singular, and hence $\pi \circ T = 0$. Consequently, $M = N$ and the corollary follows.

Finally, we remark that the existence of non-zero weakly singular operators implies the failure of certain spaces to be ultrabarrelled (see [2] and [6]). If $T: L_0 \to X$ is non-zero and weakly singular, we may find $v_n \geq 0$, $v_n \in A$, $v_n \to \infty$ uniformly so that $TR_A v_n \to 0$ for all $A \in B$. In fact, it is easy to see that $T(v_n s) \to 0$ for all countably simple functions $s$. Let $\Gamma$ be the space of countably simple functions in $L_0$. If $\Gamma$ were an ultrabarrelled subspace of $L_0$, then we could conclude from the Banach-Steinhaus theorem that the operators $f \to T(v_n f)$ are equicontinuous. Hence, if $f \in L_0$, then

$$T(v_n v_n^{-1} f) \to 0,$$

i.e. $Tf = 0$. Thus $T = 0$ and we have a contradiction.

A similar argument shows that $\Gamma \cap L_x$ cannot be ultrabarrelled in $L_x$. Of course, $\Gamma$ is barrelled in $L_x$; see [6] for further information.

References


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