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Absolute Nörlund summability and orthogonal series

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1. Let $\sum a_n$ be a given series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of real constants, and let us write

$$P_n = p_0 + p_1 + \dots + p_n .$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \qquad (P_n \neq 0) \tag{1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the sequence $\{p_n\}$. The transforms are called the *Nörlund means* of the sequence $\{s_n\}$ or of the series $\sum a_n$.

The series $\sum a_n$ is said to be (N, p_n)-*summable* to the sum s if $\lim t_n$ exists and equals s. Moreover, it is said to be *absolutely* (N, p_n)-*summable*, or shortly $|N, p_n|$-*summable*, if the sequence $\{t_n\}$ is of bounded variation, i.e. if the series $\sum |t_n - t_{n-1}|$ is convergent.

Obviously, $|N, p_n|$-summability implies (N, p_n)-summability. However, not conversely. There exist certain series (N, p_n)-summable but not $|N, p_n|$-summable ([1]).

If $\{p_n\}$ is non-negative, then

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \tag{2}$$

is a necessary and sufficient condition for the regularity of the method of summation (N, p_n).

([1]) This proves the following example by L. Mc. Fadden (*Absolute Nörlund summability*, Duke Math. Journ. 9(1942), pp. 168-207). Let $p_n = 1$, $n = 0, 1, \dots$, and let $a_n = (-1)^n$, $n = 0, 1, \dots$ Then

$$t_k = \begin{cases} 1/(k+1) & \text{for } k \text{ even}, \\ 0 & \text{for } k \text{ odd}. \end{cases}$$

Clearly t_k converges to zero, but

$$|t_k - t_{k-1}| = \begin{cases} 1/(k+1) & \text{for } k \text{ even}, \\ 1/k & \text{for } k \text{ odd}. \end{cases}$$

Hence, $\sum_{k=1}^{\infty} |t_k - t_{k-1}|$ diverges.

The object of this note is to examine the $|N, p_n|$-summability of orthogonal series of the form

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)\,, \tag{3}$$

where $\{\varphi_n(x)\}$ denotes an arbitrary orthonormal system defined in the interval $\langle 0, 1\rangle$, and $\{a_n\} \in l^2$, i.e.

$$\sum_{n=0}^{\infty} a_n^2 < +\infty\,. \tag{4}$$

In the sequel we shall restrict ourselves mainly to the special classes $\overline{M}^a$ of Nörlund means:

A sequence $\{p_n\}$ will be said to belong to the class $\overline{M}^a$, with $-1 < a < 0$, if

(i) $p_0 > 0 \quad \text{and} \quad p_n < 0 \quad \text{for} \quad n = 1, 2, \ldots,$

(ii) $p_1 < p_2 < \ldots < p_n < p_{n+1} < \ldots,$

(iii) $\lim\limits_{n\to\infty} \dfrac{n(p_n - p_{n-1})}{p_n} = a - 1\,.$

A sequence $\{p_n\}$ will be said to belong to the class $\overline{M}^a$, with $a > 0$, if

(j) $0 < p_{n+1} < p_n \quad \text{or} \quad 0 < p_n < p_{n+1} \quad (n = 0, 1, 2, \ldots)\,,$

(jj) $p_0 + p_1 + \ldots + p_n = P_n \nearrow +\infty\,,$

(jjj) $\lim\limits_{n\to\infty} \dfrac{n(p_n - p_{n-1})}{p_n} = a - 1\,.$

In particular, if instead of condition (jjj) we shall assume the condition $\lim\limits_{n\to\infty} np_n/P_n = a$, retaining conditions (j) and (jj), then $\{p_n\}$ will be said to belong to the class M^a.

The theorems presented below generalize the corresponding results due to K. Tandori [7] and L. Leindler (see [2], Satz I, p. 244 and Satz II, p. 253) concerning the $|C, a|$-summability (with $a > \frac{1}{2}$ and $-1 < a \leqslant \frac{1}{2}$, $a \neq 0$) of orthogonal series. They establish at the same time an analogue to a result of F. Móricz [5] concerning the absolute Riesz-summability of orthogonal series. The idea and the proofs themselves of the theorems presented here are similar to F. Móricz's results.

LEMMA 1. *Let $\{r_n(x)\}$ be the Rademacher-system. Then for every sequence $\{a_n\}$ of real coefficients we have the inequality*

$$C(E) \int_E \Big| \sum_{k=N}^{n} a_k r_k(x) \Big|\, dx \geqslant \Big\{ \sum_{k=N}^{n} a_k^2 \Big\}^{1/2} \qquad (n = N, N+1, \ldots)\,,$$

where E ($\subset \langle 0,1\rangle$) denotes an arbitrary set of positive measure and N is a positive integer dependent only on E.

This lemma is known (see [6], Hilfsatz IV, pp. 31-32).

LEMMA 2. *Let $\{p_n\} \in \bar{M}^a$, with $a > -1$ and $a \neq 0$. Then $p_n/P_n \nearrow$ as $-1 < a < 0$, and $p_n/P_n \searrow$ as $a > 0$, for sufficiently large n.*

Proof. Let $\{p_n\} \in \bar{M}^a$, with $-1 < a < 0$. First we shall give some properties of the sequence $\{p_n\}$ belonging to this class.

(a) If $P_n = p_0 + p_1 + \ldots + p_n$, then $\{P_n\}$ is a positive, decreasing and null-convergent sequence.

In fact, since $p_n < 0$ for $n = 1, 2, \ldots$, we have

$$P_{n-1} > P_{n-1} + p_n = P_n \quad (n = 1, 2, \ldots).$$

Considering that $P_n \searrow$ and applying the Stolz lemma, we find that

$$\lim_{n\to\infty} \frac{np_n}{P_n} = a < 0 .$$

Hence $P_n > 0$ for $n > N$, where N denotes a positive integer sufficiently large. Suppose then that $P_n \leqslant 0$ for $1 \leqslant n \leqslant N$. Then we should have $P_n > P_N > P_{N+1} > 0$, which would imply $P_n > 0$, contrary to hypothesis. Arguing as in the proof of a lemma (see [4], Lemma 3, p. 249), we easily state that $\{P_n\}$ is a null-convergent sequence.

(b) $p_n/P_n \nearrow$ for sufficiently large n.

This is evident because

$$\frac{p_n}{P_n} - \frac{p_{n-1}}{P_{n-1}} = \frac{p_n}{nP_{n-1}} \left[\frac{n(p_n - p_{n-1})}{p_n} - \frac{np_n}{P_n}\right].$$

(c) Let $W_n = p_{n-1}P_n - p_nP_{n-1}$ and $p_{n+1}/p_n \nearrow$. Then $W_n/p_n \searrow$.

In fact,

$$\frac{W_{n+1}}{p_{n+1}} = \frac{p_n}{p_{n+1}} P_n - P_{n-1} < \frac{p_{n-1}}{p_n} P_n - P_{n-1} = \frac{W_n}{p_n}.$$

(d) $\lim\limits_{n\to\infty} \dfrac{W_n}{p_n^2} = \dfrac{1}{a}$ because

$$\frac{W_n}{p_n^2} = 1 - \frac{P_n}{np_n} \cdot \frac{n(p_n - p_{n-1})}{p_n}.$$

Remark. Property (d) is also valid if $\{p_n\} \in \bar{M}^a$, with $a > 0$. Now let $\{p_n\} \in \bar{M}^a$, with $a > 0$. If $0 < p_n \searrow$, then

$$p_{n-1}/P_{n-1} - p_n/P_n = \frac{1}{P_nP_{n-1}} (p_{n-1}P_n - p_nP_{n-1}) > \frac{1}{P_n^2} (p_nP_n - p_nP_n) = 0$$

for every n. If $0 < p_n \nearrow$, then our statement is evident because

$$p_{n-1}/P_{n-1} - p_n/P_n = \frac{P_n}{np_{n-1}} \left[\frac{np_n}{P_n} - \frac{n(p_n - p_{n-1})}{p_n}\right],$$

and because the expression in the square brackets tending to $1/a$ is positive for sufficiently large n.

Remark. If we assume only that $0 < p_n \nearrow$ and $p_{n+1}/p_n \searrow$ or that $\{p_n\}$ is concave, then $p_n/P_n \searrow$ for every n. In fact, considering the expression W_n, which is positive for $n = 1$, let us suppose that $W_n > 0$ for a given $n \neq 1$. If $\{p_n\}$ is concave, then

$$\begin{aligned} W_{n+1} &= p_n P_{n+1} - p_{n+1} P_n = p_n p_{n+1} - P_n(p_{n+1} - p_n) \\ &> p_n^2 - (p_n - p_{n-1}) P_n = p_{n-1} P_n - p_n P_{n-1} = W_n > 0 . \end{aligned}$$

If $p_{n+1}/p_n \searrow$, then

$$\begin{aligned} W_{n+1} &= p_n P_{n+1} - p_{n+1} P_n = p_n \left(P_n - \frac{p_{n+1}}{p_n} P_{n-1} \right) \\ &> p_n \left(P_n - \frac{p_n}{p_{n-1}} P_{n-1} \right) = \frac{p_n}{p_{n-1}} W_n > \frac{p_{n+1}}{p_n} W_n > 0 . \end{aligned}$$

Applying complete induction, we infer the validity of our statement in both cases of monotony of the sequence $\{p_n\}$. At the same time we have proved that $\{W_n\}$ and $\{W_n/p_n\}$, respectively, are increasing sequences in the case under examination.

LEMMA 3. *If* $\{p_n\} \in M^a$, $a > 1$, *then*

$$\frac{P_{n-[n/4a]}}{P_n} > \frac{1}{4}$$

for sufficiently large n.

Proof. Let r be a positive integer such that $2^{r-1} < a \leqslant 2^r$. Since $a > 1$, then $0 < p_n \nearrow$. Hence

$$\begin{aligned} \frac{P_{n-[n/4a]}}{P_n} &> \frac{P_{n-[n/2^{r+1}]}}{P_n} = 1 - \frac{1}{P_n} (p_{n-[n/2^{r+1}]+1} + \dots + p_n) \\ &> 1 - \left[\frac{n}{2^{r+1}} \right] \frac{p_n}{P_n} > 1 - \frac{1}{2a} \cdot \frac{np_n}{P_n} , \end{aligned}$$

whence Lemma 3 follows because the last expression is greater than $1/4$ for sufficiently large n.

LEMMA 4. *If* $\{p_n\} \in \overline{M}^a$, *with* $a > -1$ *and* $a \neq 0$, *then*

$$C_1(a) k p_n p_{n-k} < |p_{n-k} P_n - p_n P_{n-k}| < C_2(a) k p_n p_{n-k} \text{ (2)} \tag{5}$$
$$(n = N, N+1, \dots; \ k = 1, 2, \dots, n),$$

where $C_1(a)$ *and* $C_2(a)$ *denote positive constants dependent, in general, on* a, *and* N *denotes a natural number, which will be defined in the proof.*

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(2) The sign of the absolute value can be omitted as $a > 0$.

Proof. First we shall prove the second inequality of estimation (5). On writing

$$W_{n,k} = p_{n-k}P_n - p_nP_{n-k} \qquad (n = 1, 2, \ldots;\ k = 1, 2, \ldots, n),$$

we have

$$W_{n,k} = p_np_{n-k}\left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right) = p_np_{n-k}\left[\left(\frac{P_n}{p_n} - \frac{P_{n-1}}{p_{n-1}}\right) + \ldots + \left(\frac{P_{n-k+1}}{p_{n-k+1}} - \frac{P_{n-k}}{p_{n-k}}\right)\right]$$

$$= p_np_{n-k}\left[\frac{W_n}{p_np_{n-1}} + \ldots + \frac{W_{n-k+1}}{p_{n-k+1}p_{n-k}}\right].$$

In virtue of property (d), we have

$$\lim_{n\to\infty} \frac{|W_n|}{p_n^2} = \frac{1}{|a|}.$$

Hence it follows that

$$|W_{n,k}| < C_2(a)kp_np_{n-k} \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots).$$

Passing to the proof of the first inequality of (5), we shall first examine the case $-1 < a < 0$.

Let $W_n < 0$ and $|W_n|/p_n^2 > 1/2|a|$ for $n > N_1$, where N_1 denotes a natural number sufficiently large. If $n-k+1 > N_1$, then

$$|W_{n,k}| = p_np_{n-k}\left(\frac{|W_n|}{p_np_{n-1}} + \ldots + \frac{|W_{n-k+1}|}{p_{n-k+1}p_{n-k}}\right)$$

$$> p_np_{n-k}\left(\frac{|W_n|}{p_{n-1}^2} + \ldots + \frac{|W_{n-k+1}|}{p_{n-k}^2}\right)$$

and

$$|W_{n,k}| > \frac{1}{2|a|}kp_np_{n-k} \qquad (k = 1, 2, \ldots, n;\ n = N_1, N_1+1, \ldots).$$

If $n-k+1 \leqslant N_1$, then $|p_{n-k}| > |p_{N_1}|$. Hence

$$|W_{n,k}| > p_np_{n-k}\left(\frac{P_n}{|p_n|} - \frac{P_{n-k}}{|p_{n-k}|}\right) > kp_np_{n-k}\left(\frac{P_n}{n|p_n|} - \frac{p_0}{n|p_{N_1}|}\right).$$

Considering that the expression in the last brackets tending to $1/|a|$ as $n\to\infty$ is greater than $1/2|a|$ for $n > N_2$, we find that

$$|W_{n,k}| > \frac{1}{2|a|}kp_np_{n-k} \qquad \text{for} \qquad n > N = \max(N_1, N_2),$$

which ends the proof in the case examined.

Now let $\{p_n\} \in \bar{M}^a$, with $a > 0$. If $0 < p_n \searrow$, then

$$W_{n,k} \geqslant p_{n-k}P_n - p_{n-k}P_{n-k} = p_{n-k}(P_n - P_{n-k})$$

$$= p_{n-k}(p_{n-k+1} + p_{n-k+2} + \ldots + p_n) > kp_np_{n-k},$$

whence it follows that the first inequality of (5) holds for $n = 1, 2, \ldots$, $k = 1, 2, \ldots, n$.

If $0 < p_n \nearrow$, then we can write

$$W_{n,k} > p_n p_{n-k}\left(\frac{W_n}{p_n^2} + \ldots + \frac{W_{n-k+1}}{p_{n-k+1}^2}\right)$$

$$> \frac{1}{2a} k p_n p_{n-k} \quad \text{for} \quad n-k+1 > N_1,$$

where N_1 denotes a natural number sufficiently large and $k = 1, 2, \ldots, (n-N_1)$. If $n-k+1 \leqslant N_1$, then denoting by N_2 a natural number such that

$$\frac{P_n}{np_n} - \frac{N_1}{n} > \frac{1}{2a} \quad \text{for} \quad n > N_2,$$

we can write

$$W_{n,k} > p_n p_{n-k}\left[\frac{P_n}{p_n} - (n-k+1)\right] \geqslant k p_n p_{n-k}\left(\frac{P_n}{np_n} - \frac{N_1}{n}\right) > \frac{1}{2a} k p_n p_{n-k}$$

for $n > N_2$ and $k = n-N_1+1, n-N_1+2, \ldots, n$.

Taking $N = \max(N_1, N_2)$, we find that

$$W_{n,k} > C_1(a) k p_n p_{n-k} \qquad (k = 1, 2, \ldots, n;\ n = N, N+1, \ldots),$$

with $C_1(a) = 1/2a$.

Thus we have proved the first inequality of (5) in both cases of the sequence $\{p_n\}$, which, together with the first part of the proof, completes the proof of Lemma 4.

Remark. If $0 < p_n \nearrow$ and if $\{p_n\}$ is concave or if $p_{n+1}/p_n \searrow$, then the first inequality of (5) holds also for every natural number n. In fact, we can write

$$W_{n,k} = p_n p_{n-k}\left[\left(\frac{W_n}{p_n p_{n-1}} + \ldots + \frac{W_{i+1}}{p_{i+1} p_i}\right) + \left(\frac{W_i}{p_i p_{i-1}} + \ldots + \frac{W_{n-k+1}}{p_{n-k+1} p_{n-k}}\right)\right],$$

where $(i+1)$ denotes the least natural number greater than N_1 defined above. According to the remark put at the end of Lemma 2, the sequences $\{W_n\}$ and $\{W_n/p_n\}$ are increasing if $\{p_n\}$ is concave or if $p_{n+1}/p_n \searrow$, respectively. Therefore the second expression appearing in the last square brackets is greater than

$$W_1\left(\frac{1}{p_i p_{i-1}} + \ldots + \frac{1}{p_{n-k+1} p_{n-k}}\right)$$

or than

$$\frac{W_1}{p_1}\left(\frac{1}{p_{i-1}} + \ldots + \frac{1}{p_{n-k}}\right),$$

respectively. Hence we get

$$W_{n,k} > p_n p_{n-k}\left[\frac{n-i}{2a} + \frac{p_0^2}{p_{N_1}^2}(i-n+k)\right]$$

and ultimately

$$W_{n,k} > C_1(a) k p_n p_{n-k} \qquad (n = 1, 2, \ldots;\ k = 1, 2, \ldots, n),$$

where

$$C_1(a) = \min\left(\frac{1}{2a}, \frac{p_0^2}{p_{N_1}^2}\right).$$

Now, putting

$$A_m = \Big\{\sum_{k=2^m+1}^{2^{m+1}} a_k^2\Big\}^{1/2} \qquad (m = 0, 1, 2, \ldots),$$

we can formulate the following theorem.

THEOREM 1. *Let $\{p_n\} \in \bar{M}^a$, with $a > \frac{1}{2}$. In order that series* (3) *be $|N, p_n|$-summable in the interval $\langle 0, 1\rangle$ almost everywhere for every orthonormal system $\{\varphi_n(x)\}$, the condition*

$$\sum_{m=0}^{\infty} A_m < +\infty \tag{6}$$

is necessary and sufficient.

Proof. Sufficiency. Let $\{p_n\} \in \bar{M}^a$, with $a > \frac{1}{2}$. We can write (omitting the argument x for the sake of brevity)

$$t_n - t_{n-1} = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n} (p_{n-k}P_n - P_{n-k}p_n) a_k \varphi_k .$$

Applying the Schwarz inequality, we obtain with the aid of Lemma 4 the following estimate:

$$\sum_{n=2}^{\infty} \int_0^1 |t_n - t_{n-1}|\,dx = \sum_{m=0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \int_0^1 |t_n - t_{n-1}|\,dx$$

$$\leqslant \sum_{m=0}^{\infty} \Big\{2^m \sum_{n=2^m+1}^{2^{m+1}} \int_0^1 (t_n - t_{n-1})^2 dx\Big\}^{1/2}$$

$$= O(a) \sum_{m=0}^{\infty} \Big\{2^m \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{n^2 P_n^2} \sum_{k=1}^{n} k^2 p_{n-k}^2 a_k^2\Big\}^{1/2}.$$

If $\{p_n\} \in M^a$, $a > \frac{1}{2}$, then according to a lemma of the author (see [3], Lemma 3, pp. 232-233) the sequence $\{P_n^2/n\}$ is increasing for sufficiently large n and tends to infinity. Applying the Stolz lemma, we get the relation

$$\lim_{n\to\infty}\frac{n}{P_n^2}\sum_{k=0}^{n}p_k^2=\frac{a^2}{2a-1}\,. \tag{7}$$

In view of this relation and by condition (5) the proof proceeds after the following estimate:

$$\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\int_0^1|t_n-t_{n-1}|\,dx$$

$$=O(a)\sum_{m=0}^{\infty}\left\{\frac{1}{2^mP_{2^m}^2}\sum_{n=2^m+1}^{2^{m+1}}\sum_{l=0}^{m}\sum_{k=2^l+1}^{\min(2^{l+1},n)}k^2p_{n-k}^2a_k^2\right\}^{1/2}$$

$$=O(a)\sum_{m=0}^{\infty}\left\{\frac{1}{2^mP_{2^m}^2}\sum_{l=0}^{m}\sum_{k=2^l+1}^{2^{l+1}}k^2a_k^2\sum_{n=\max(2^m+1,k)}^{2^{m+1}}p_{n-k}^2\right\}^{1/2}$$

$$=O(a)\sum_{m=0}^{\infty}\left\{\frac{1}{2^{2m}}\cdot\frac{2^m}{P_{2^m}^2}\sum_{k=1}^{2^m}p_k^2\sum_{l=0}^{m}\sum_{k=2^l+1}^{2^{l+1}}k^2a_k^2\right\}^{1/2}$$

$$=O(a)\sum_{m=0}^{\infty}\left\{\frac{1}{2^{2m}}\sum_{l=0}^{m}2^{2l}\sum_{k=2^l+1}^{2^{l+1}}a_k^2\right\}^{1/2}$$

$$=O(a)\sum_{m=0}^{\infty}\frac{1}{2^m}\sum_{l=0}^{m}2^lA_l=O(a)\sum_{l=0}^{\infty}2^lA_l\sum_{m=l}^{\infty}\frac{1}{2^m}=O(a)\sum_{l=0}^{\infty}A_l<+\infty\,.$$

Necessity. It suffices to prove the following statement:

If the Rademacher series

$$\sum_{n=0}^{\infty}a_nr_n(x)$$

is $|N,p_n|$-summable, with $\{p_n\}\in\overline{M}^a$, $a>\frac{1}{2}$, on a set of positive measure, then condition (6) *holds.*

In fact, in virtue of the Egoroff theorem there exist a measurable set E, with $|E|>0$, and a positive constant M such that

$$\sum_{n=2}^{\infty}|t_n(x)-t_{n-1}(x)|<M$$

for every $x\in E$ and that

$$\sum_{n=2}^{\infty}\int_E|t_n(x)-t_{n-1}(x)|\,dx\leqslant M|E|\,. \tag{8}$$

We can write (omitting the argument x)

$$\sum_{k=N}^{n} \frac{W_{n,k}}{P_n P_{n-1}} a_k r_k = (t_n - t_{n-1}) - \sum_{k=1}^{N-1} \frac{W_{n,k}}{P_n P_{n-1}} a_k r_k ,$$

where N denotes a positive integer suitably chosen in view of Lemmas 1-4. Hence

$$\sum_{n=N}^{\infty} \left| \sum_{k=N}^{n} \frac{W_{n,k}}{P_n P_{n-1}} a_k r_k \right| \geqslant \sum_{n=N}^{\infty} |t_n - t_{n-1}| - \sum_{n=N}^{\infty} \left| \sum_{k=1}^{N-1} \frac{W_{n,k}}{P_n P_{n-1}} a_k r_k \right| . \tag{9}$$

The last series is of course convergent. Therefore, without loss of generality, we may suppose that $a_k = 0$ for $k = 1, 2, \ldots, N-1$. Now, applying Lemmas 1 and 4, we obtain after (8) and (9) the following estimate:

$$M|E|\,C(E) > \sum_{n=N}^{\infty} C(E) \int_E \left| \sum_{k=N}^{n} \frac{W_{n,k}}{P_n P_{n-1}} a_k r_k(x) \right| dx$$

$$> \sum_{n=N}^{\infty} \left\{ \sum_{k=N}^{n} \frac{W_{n,k}^2}{P_n^2 P_{n-1}^2} a_k^2 \right\}^{1/2} > C_2^2(a) \sum_{n=N}^{\infty} \frac{p_n}{P_n^2} \left\{ \sum_{k=N}^{n} k^2 p_{n-k}^2 a_k^2 \right\}^{1/2} .$$

In order to estimate the last expression we shall distinguish two cases: (1) $0 < p_n \searrow$ and (2) $0 < p_n \nearrow$.

Let $0 < p_n \searrow$. Denoting by m_0 the least positive integer x satisfying the inequality $2^x + 1 \geqslant N$, and considering that $p_{n-k} \geqslant p_n$ $(k = 0, 1, 2, \ldots)$, we find that

$$\sum_{n=N}^{\infty} \frac{p_n}{P_n^2} \left\{ \sum_{k=N}^{n} k^2 p_{n-k}^2 a_k^2 \right\}^{1/2} > \sum_{n=N}^{\infty} \frac{p_n^2}{P_n^2} \left\{ \sum_{k=N}^{n} k^2 a_k^2 \right\}^{1/2} > C_3 \sum_{n=N}^{\infty} \frac{1}{n^2} \left\{ \sum_{k=N}^{n} k^2 a_k^2 \right\}^{1/2}$$

$$> C_3 \sum_{m=m_0+1}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{n^2} \left\{ \sum_{k=N}^{n} k^2 a_k^2 \right\}^{1/2}$$

$$> C_3 \sum_{m=m_0}^{\infty} \left\{ \sum_{k=2^m+1}^{2^{m+1}} k^2 a_k^2 \right\}^{1/2} \sum_{n=2^{m+1}+1}^{2^{m+2}} \frac{1}{n^2}$$

$$> C_3 \sum_{m=m_0}^{\infty} \frac{1}{2^{m+3}} \left\{ \sum_{k=2^m+1}^{2^{m+1}} k^2 a_k^2 \right\}^{1/2} > \frac{1}{8} C_3 \sum_{m=0}^{\infty} A_m ,$$

where C_3 is a suitably chosen constant. This and the last but one estimate imply condition (6).

Now let $0 < p_n \nearrow$, and let r be a positive integer such that $2^{r-1} < a \leqslant 2^r$. Further, let m_0 denote the least positive integer x fulfilling the inequality $2^{x-r-2} \geqslant N$.

With the aid of Lemmas 2 and 3 we can write

$$\sum_{n=N}^{\infty}\frac{p_n}{P_n^2}\Big\{\sum_{k=N}^{n}k^2p_{n-k}^2a_k^2\Big\}^{1/2} > \sum_{n=2^{m_0}+1}^{\infty}\frac{p_n}{P_n^2}\Big\{\sum_{k=N}^{[n/2^{r+1}]}\frac{k^2p_{n-k}^2}{P_{n-k}^2}P_{n-k}^2a_k^2\Big\}^{1/2}$$

$$> \sum_{n=2^{m_0}+1}^{\infty}\frac{p_n^2}{P_n^2}\cdot\frac{P_{n-[n/2^{r+1}]}}{P_n}\Big\{\sum_{k=N}^{[n/2^{r+1}]}k^2a_k^2\Big\}^{1/2}$$

$$> \frac{1}{4}\sum_{n=2^{m_0}+1}^{\infty}\frac{1}{n^2}\Big\{\sum_{k=N}^{[n/2^{r+1}]}k^2a_k^2\Big\}^{1/2}$$

$$> \frac{1}{4}\sum_{m=m_0}^{\infty}\Big\{\sum_{n=2^{m-r-2}}^{2^{m-r-1}}k^2a_k^2\Big\}\sum_{n=2^m+1}^{2^{m+1}}\frac{1}{n^2}$$

$$> \frac{1}{32a}\sum_{m=m_0}^{\infty}A_{m-r-2} = \frac{1}{32a}\sum_{m=0}^{\infty}A_m\,.$$

Collecting the above results, we infer the necessity of condition (6). This ends the proof of Theorem 1.

2. In this section we shall occupy ourselves with the case of $\{p_n\}\in\bar{M}^a$, $-1<a<\frac{1}{2}$, $a\neq 0$, giving certain conditions for absolute (N,p_n)-summability. The case $a=\frac{1}{2}$ proves more difficult and requires additional assumptions about the sequence $\{p_n\}$.

First we remark that

$$\lim_{n\to\infty} n\left(\frac{p_n^2}{p_{n+1}^2}-1\right)=2(1-a)>1 \qquad\text{as}\qquad -1<a<\tfrac{1}{2},\ a\neq 0\,,$$

whence we infer by the Raabe criterion the convergence of the series $\sum_{n=0}^{\infty}p_n^2$. Since $0<p_n^2\searrow$, we have by a well-known theorem $\lim_{n\to\infty} np_n^2=0$. Therefore $\lim_{n\to\infty} n/P_n^2=\infty$, whence we get the relation

$$\frac{n}{P_n^2}\sum_{k=0}^{n}p_k^2\sim\frac{n}{P_n^2}\qquad(-1<a<\tfrac{1}{2},\ a\neq 0)\,. \tag{10}$$

At once we state that in this case the sequence $\{n/P_n^2\}$ is increasing for n sufficiently large and tends to infinity.

THEOREM 2. *Let $\{p_n\}\in\bar{M}^a$, $-1<a<\frac{1}{2}$, $a\neq 0$. In order that series (3) be $|N,p_n|$-summable in the interval $\langle 0,1\rangle$ almost everywhere for every orthonormal system $\{\varphi_n(x)\}$, the condition*

$$\sum_{m=1}^{\infty}\frac{2^{m/2}}{P_{2^m}}A_m<\infty\qquad\Big(A_m=\Big\{\sum_{n=2^m+1}^{2^{m+1}}a_k^2\Big\}^{1/2}\Big)$$

is sufficient; simultaneously, it is necessary if $\{a_n\}$ is a monotone sequence (³) *of coefficients and if the summability is required for all orthonormal systems* $\{\varphi_n(x)\}$.

Proof. Sufficiency. Let $-1 < a < \frac{1}{2}$. In view of the estimate deduced in the proof of Theorem 1, and according to relation (10), we can write

$$\sum_{n=2}^{\infty} \int_0^1 |t_n - t_{n-1}|\,dx = O(1) \sum_{m=2}^{\infty} \Big\{ \frac{1}{2^m P_{2^m}^2} \sum_{n=2^m+1}^{2^{m+1}} \sum_{l=0}^{m} \sum_{k=2^l+1}^{\min(2^{l+1},n)} p_{n-k}^2 k^2 a_k^2 \Big\}^{1/2}$$

$$= O(1) \Big(\sum_{m=2}^{\infty} \Big\{ \frac{1}{2^m P_{2^m}^2} \sum_{l=0}^{m} \sum_{k=2^l+1}^{2^{l+1}} k^2 a_k^2 \sum_{n=\max(2^m+1,k)}^{2^{m+1}} p_{n-k}^2 \Big\}^{1/2} \Big)$$

$$= O(1) \Big(\sum_{m=2}^{\infty} \Big\{ \frac{1}{2^m P_{2^m}^2} \Big[2^{2m} A_m^2 + 2^{2(m-1)} A_{m-1}^2 + \sum_{l=2}^{m-2} 2^{2l} A_l^2 \sum_{n=2^m+1}^{2^{m+1}} p_{n-2^{m-1}}^2 \Big] \Big\}^{1/2} \Big)$$

$$= O(1) \Big(\sum_{m=2}^{\infty} \frac{2^{m/2}}{P_{2^m}} A_m + \sum_{l=2}^{\infty} 2^l A_l \sum_{m=l}^{\infty} \frac{1}{2^m} \Big) = O(1) \sum_{m=2}^{\infty} \frac{2^{m/2}}{P_{2^m}} A_m < \infty .$$

Necessity. Under the assumptions of Theorem 2 relation (10) holds and the sequence $\{n/P_n^2\}$ is increasing for n sufficiently large. Therefore

$$M|E|\,C(E) > C_1(a) \sum_{n=N}^{\infty} \frac{|p_n|}{P_n^2} \Big\{ \sum_{k=N}^{n} k^2 p_{n-k}^2 a_k^2 \Big\}^{1/2}$$

$$> C_1(a) \sum_{m=m_0+1}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{|p_n|}{P_n^2} \Big\{ \sum_{k=N}^{n} k^2 p_{n-k}^2 a_k^2 \Big\}^{1/2}$$

$$> C_2(a) \sum_{m=m_0}^{\infty} \Big\{ \sum_{k=2^m+1}^{2^{m+1}} k^2 p_{2^{m+1}-k}^2 a_k^2 \Big\}^{1/2} \sum_{n=2^{m+1}+1}^{2^{m+2}} \frac{1}{nP_n}$$

$$> C_3(a) \sum_{m=m_0}^{\infty} \frac{2^m a_{2^{m+1}} 2^{m+1}}{2^{m+2}\, P_{2^{m+2}}} \Big\{ \sum_{k=0}^{2^m} p_k^2 \Big\}^{1/2}$$

$$> C_4(a) \sum_{m=m_0}^{\infty} \frac{2^{(m+1)/2)}}{P_{2^{m+1}}} \Big\{ \sum_{k=2^{m+1}+1}^{2^{m+2}} a_k^2 \Big\}^{1/2}$$

$$= C_4(a) \sum_{m=m_0+1}^{\infty} \frac{2^{m/2}}{P_{2^m}} A_m ,$$

where $C_1(a)$-$C_4(a)$ are suitably chosen constants.

(³) positive and non-increasing.

3. We consider here the case of $\{p_n\} \in \overline{M}^a$, with $a = \frac{1}{2}$. Examining this case without any additional assumptions about the sequence $\{p_n\}$, we state the relation

$$\log\frac{1}{p_n^2} \sim \log n\,. \tag{11}$$

If $\{p_n\}$, in addition to satisfying the last assumption, is such that

$$\left[1-\frac{2np_n}{P_n}\right]\log n \to 0 \quad \text{and} \quad \left[1-\frac{2n(p_{n-1}-p_n)}{p_n}\right]\log n \to 0\,, \tag{12}$$

then

$$\frac{n}{P_n^2}\sum_{k=0}^{n} p_k^2 \sim \log n\,. \tag{13}$$

In fact, applying the Stolz lemma and the first relation of (12), we state that

$$\lim_{n\to\infty}\frac{\log np_n^2}{\log\log n} = 0\,,$$

whence it follows that for sufficiently large n

$$\frac{\log np_n^2}{\log\log n} > -1$$

holds. The series $\sum_{k=0}^{\infty} p_k^2$ is then divergent. Applying the Stolz lemma to the expression

$$\frac{P_n^2\log n}{n}\Big/\sum_{k=0}^{n} p_k^2\,,$$

we state, in view of the first relation of (12), relation (13).

In virtue of relation (13), we get the following theorem:

THEOREM 3. *Let $\{p_n\} \in \overline{M}^a$, with $a = \frac{1}{2}$, and moreover let $\{p_n\}$ satisfy conditions (12). Then in order that series (3) be $|N, p_n|$-summable in the interval $\langle 0, 1\rangle$ almost everywhere for every orthonormal system $\{\varphi_n(x)\}$, the condition*

$$\sum_{m=1}^{\infty}\sqrt{m}A_m < +\infty$$

is sufficient; simultaneously, it is necessary if $\{a_n\}$ is a monotone sequence [4] *of coefficients and if the summability is required for all systems $\{\varphi_n(x)\}$.*

Proof. Sufficiency. Under the hypothesis of the theorem, we can write

$$\sum_{n=2}^{\infty}\int_0^1 |t_n-t_{n-1}|\,dx = O(a)\sum_{m=1}^{\infty}\left\{\frac{m}{2^{2m}}\sum_{l=0}^{m}\sum_{k=2^l+1}^{2^{l+1}} k^2a_k^2\right\}^{1/2}$$

$$= O(a)\sum_{m=1}^{\infty}\left\{\frac{m}{2^m}\sum_{l=0}^{m}\sum_{k=2^l+1}^{2^{l+1}} a_k^2\right\}^{1/2}$$

[4] positive and non-increasing.

$$= O(a) \sum_{m=1}^{\infty} \frac{\sqrt{m}}{2^m} \sum_{l=0}^{m} 2^l A_l = O(a) \sum_{l=1}^{\infty} 2^l A_l \sum_{m=l}^{\infty} \frac{\sqrt{m}}{2^m}$$

$$= O(a) \sum_{l=1}^{\infty} \sqrt{l} A_l < \infty .$$

Hence follows the sufficiency of Theorem 3.

Necessity. Arguing as in the proof of Theorem 1, we can write

$$M|E|C(E) > C_1(a) \sum_{n=N}^{\infty} \frac{p_n}{P_n^2} \Big\{ \sum_{k=N}^{n} k^2 p_{n-k}^2 a_k^2 \Big\}^{1/2}$$

$$> C_2(a) \sum_{m=m_0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{nP_n} \Big\{ \sum_{k=2^m+1}^{n} p_{n-k}^2 \Big\}^{1/2} a_{2^m} 2^m$$

$$> C_2(a) \sum_{m=m_0}^{\infty} \frac{2^m}{2^m P_{2^m}} \Big\{ \sum_{k=0}^{2^m} p_k^2 \Big\}^{1/2} 2^{m/2} 2^{m/2} a_{2^m} .$$

In virtue of relation (13) the last expression is less than

$$C_3(a) \sum_{m=m_0}^{\infty} \sqrt{m} A_m = C_3(a) \sum_{m=1}^{\infty} \sqrt{m} A_m ,$$

which completes the proof of Theorem 3.

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