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A Szegö type property for two doubly commuting contractions

by Marek Słociński (Kraków)

Abstract. In this paper we consider scalar measures for a pair of doubly commuting contractions. If we take the spectral measure E (on the two-dimensional torus) connected with the special unitary dilation of two doubly commuting contractions, we can ask about continuity of scalar measures $\mu_x(\sigma) = (E(\sigma)x, x)$ with respect to the measure $m \times \mu_x^2$ on the torus (m is the Lebesgue measure on the unit circle). It is proved in the paper that if $T_n^1 x \to 0$, then the measure μ_x is mutually absolutely continuous with respect to $m \times \mu_x^2$ and, for almost all z_1, $\log f(z_1, z_2)$ is summable, where

$$f(z_1, z_2) = \frac{d\mu_x}{d(m \times \mu_x^2)}; \quad \mu_x^2(\sigma) = \mu_x(\sigma \times \Gamma).$$

In what follows H is a complex Hilbert space with inner product (x, y), $x, y \in H$, and norm $\|x\| = \sqrt{(x, x)}$, $x \in H$. $L(H)$ stands for the algebra of all linear bounded operators on the space H. For $T \in L(H)$, T^* denotes the adjoint of T. If $T \in L(H)$ is a contraction, then (see [3]) T has a unitary dilation, i.e., there is a Hilbert space K including H and a unitary operator $U \in L(K)$ such that $T^n x = PU^n x$ for every $x \in H$ and $n = 0, 1, 2, \ldots$, where P is the projection of K onto H. A unitary dilation $U \in L(K)$ is called *minimal* if $K = \bigvee_{n=-\infty}^{\infty} U^n H$. It is proved (see [5]) that two minimal unitary dilations are unitarily equivalent. Ando in [1] proved that if $T_1, T_2 \in L(H)$ are commuting contractions, then there are a Hilbert space $K \supset H$ and two commuting unitary operators $U_1, U_2 \in L(K)$ such that $T_1^n T_2^m x = PU_1^n U_2^m x$ for every $x \in H$ and $n, m = 0, 1, 2, \ldots$, P is the projection of K onto H. The couple (U_1, U_2) is called a *unitary dilation* of T_1, T_2. A unitary dilation U_1, U_2 on K is called *minimal* if $K = \bigvee_{n,m=-\infty}^{\infty} U_1^n U_2^m H$. In the case of two commuting contractions, two minimal unitary dilations need not be unitarily equivalent. It is proved in [3] and [5] that if T is a completely non-unitary contraction on H and E is the spectral measure of its minimal unitary dilation, then for every $x \in H$ the scalar measure $\mu(\sigma) = (E(\sigma)x, x)$ is absolutely continuous with respect to m (m is the Lebesgue measure on the unit circle) and $\log \frac{d\mu}{dm}$ is summable with respect to m. In this paper we give a similar result for the special unitary dilation of a pair of doubly commuting contractions.

Let T_1, T_2 be a pair of doubly commuting contractions on the space H. If S_2 is the minimal unitary dilation of T_2 on the space $K_0 = \bigvee_{n=-\infty}^{\infty} S_2^n H$, then (see for example Lemma 1 of [4]) there is exactly one extension S_1 on K_0 of the operator T_1 such that S_1 commutes with S_2. Moreover, the space H reduces S_1 and $\|S_1\| = \|T_1\|$. Now, for $x \in H$ and $n, m = 0, 1, \ldots$, we have

$$(1) \qquad P_H S_1^n S_2^m x = P_H S_2^m S_1^n x = P_H S_2^m T_1^n x = T_2^m T_1^n x = T_1^n T_2^m x.$$

Since S_2 is a unitary operator, S_1 and S_2 doubly commute. If V_1 is the minimal unitary dilation of S_1 on the space $K_1 = \bigvee_{n=-\infty}^{\infty} V_1^n K_0$, then (see Lemma 1 of [4]) there is exactly one unitary extension V_2 of S_2 such that V_2 commutes with V_1. It follows, by (1), that for every $x \in H$ and $n, m = 0, 1, 2, \ldots$ we have the equality

$$(2) \quad P_H V_1^n V_2^m x = P_H P_{K_0} V_1^n V_2^m x = P_H P_{K_0} V_1^n S_2^m x = P_H S_1^n S_2^m x = T_1^n T_2^m x;$$

hence the pair V_1, V_2 is a unitary dilation of T_1, T_2.

Let $x \in H$. Then we can define

$$(3) \qquad H_x = \bigvee_{n=-\infty}^{\infty} \bigvee_{m=-\infty}^{\infty} V_1^n V_2^m x = \bigvee_{n=-\infty}^{\infty} \bigvee_{m=-\infty}^{\infty} V_1^n S_2^m x,$$

$$(4) \qquad H_x(n) = \bigvee_{p \leqslant n} \bigvee_{m=-\infty}^{\infty} V_1^p V_2^m x = \bigvee_{p \leqslant n} \bigvee_{m=-\infty}^{\infty} V_1^p S_2^m x,$$

$$(5) \qquad S_x = \bigcap_{n \leqslant 0} H_x(n),$$

$$(6) \quad R_-(L) = \bigcap_{n \geqslant 0} V_1^{-n} M_-(L), \qquad \text{where } M_-(L) = \bigvee_{n \geqslant 0} V_1^{-n} L \text{ and } L = \bigvee_{m=-\infty}^{\infty} S_2^m x.$$

We shall show that $S_x = R_-(L)$. We start from considering the spaces $H_x(n)$. We have the following equalities:

$$H_x(n) = \bigvee_{p \leqslant n} \bigvee_{m=-\infty}^{\infty} V_1^p S_2^m x = \bigvee_{p \leqslant n} V_1^p \Big(\bigvee_{m=-\infty}^{\infty} S_2^m x \Big) = \bigvee_{p \leqslant n} V_1^p L = \bigvee_{p-n \leqslant 0} V_1^n V_1^{p-n} L$$
$$= \bigvee_{q \leqslant 0} V_1^n V_1^q L = V_1^n \Big(\bigvee_{q \leqslant 0} V_1^q L \Big) = V_1^n \Big(\bigvee_{q \geqslant 0} V_1^{-q} L \Big) = V_1^n M_-(L).$$

It follows

$$S_x = \bigcap_{n\leqslant 0} H_x(n) = \bigcap_{n\leqslant 0} V_1^n M_-(L) = \bigcap_{n\geqslant 0} V_1^{-n} M_-(L) = R_-(L).$$

Suppose additionally that $T_1^n x \to 0$ for $n \to \infty$. Since S_1 is an extension of T_1, we have $S_1^n x = T_1^n x \to 0$ for $n \to \infty$. Since V_1 is the minimal unitary dilation of S_1, it follows, by Lemma 6.4 of [3], that the following equality holds:

$$\text{(7)} \qquad S_1^{*n} P_{K_0} V_1^n y = P_{K_0} y \quad \text{for } y \in V_1^{-n} M_-(L) \text{ and } n = 1, 2, \ldots$$

Let P_n be the projection of K_1 onto $V_1^{-n} M_-(L)$. It follows, by the definition of P_n (see [3]), that there is $Q = s-\lim P_n$ and Q is the projection of K onto $R_-(L)$. Let $y \in L$. It follows by (7) that

$$\|P_n y\|^2 = (P_n y, y) = (S_1^{*n} P_{K_0} V_1^n P_n y, y) = (P_{K_0} V_1^n P_n y, S_1^n y).$$

Let $y = S_2^m x$. Then $\|P_n S_2^m x\|^2 = (P_{K_0} V_1^n P_n S_2^m x, S_1^n S_2^m x)$. Since $S_1^n x \to 0$ for $n \to \infty$, we get $\|S_1^n S_2^m x\| = \|S_2^m S_1^n x\| = \|S_1^n x\| \to 0$. Since $L = \bigvee_{m=-\infty}^{\infty} S_2^m x$, we have $QL = \{0\}$. But Q commutes with V_1 because $R_-(L)$ reduces V_1. Hence we have $QV_1^{-n} L = V_1^{-n} QL = \{0\}$. Consequently $QM_-(L) = \{0\}$ and $R_-(L) = QR_-(L) = Q\big(\bigcap_{n\geqslant 0} V_1^{-n} M_-(L)\big) \subset \bigcap_{n\geqslant 0} QV_1^{-n} M_-(L) = \bigcap_{n\geqslant 0} V_1^{-n} QM_-(L) = \{0\}$. Now we consider the following process: $x(n, m) = V_1^n V_2^m x$. If E_i is the spectral measure of V_i $(i = 1, 2)$, then $E = E_1 \times E_2$ is the spectral measure of the pair V_1, V_2. It is easy to show that for $x(n, m)$ the following condition holds true

$$\text{(8)} \quad B_x(n, m) = \big(x(s+n, t+m), x(s, t)\big) \quad \text{does not depend on } s, t.$$

Indeed,

$$\big(x(s+n, t+m), x(s, t)\big) = (V_1^{s+n} V_2^{t+m} x, V_1^s V_2^t x) = (V_1^n V_2^m x, x).$$

Also, by this computation we have

$$\text{(9)} \qquad B_x(n, m) = \int_{\Gamma^2} z_1^n z_2^m \, d\mu_x(z_1, z_2),$$

where $\mu_x(\sigma) = \big(E(\sigma)x, x\big)$; σ is a Borel set on Γ^2.

If $H_x'(n) = \bigvee_{p\leqslant n} \bigvee_{m=-\infty}^{\infty} x(p, m)$ and $S_x' = \bigcap_{n\leqslant 0} H_x'(n)$, then by the definition of the process $x(n, m)$ we get that $H_x'(n) = H_x(n)$ and $S_x' = S_x$. It follows that if $T_1^n x \to 0$ for $n \to \infty$, then $S_x' = \{0\}$ and, consequently, for the process $x(n, m)$ the assumptions of the following theorem hold true (Theorem 1 of [2]):

Suppose that for the process $x(n, m)$ condition (9) holds. Then $S'_x = \{0\}$ if and only if:

(10) the measures μ_x and $m\times\mu''_x$ are mutually absolutely continuous, where m is the Lebesgue measure on the unit circle Γ and $\mu''_x(\sigma) = \mu_x(\Gamma\times\sigma)$ for every Borel subset σ of Γ,

(11) if $f(z_1, z_2) = \dfrac{d\mu_x}{d(m\times\mu''_x)}$, then for almost all z_2 (with respect to μ) we have the inequality

$$\Big|\int_\Gamma \log f(z_1, z_2)\,dm(z_1)\Big| < \infty.$$

From this theorem we get the following

THEOREM 1. *Let T_1, T_2 be a pair of doubly commuting contractions on the space H. Suppose that $E = E_1\times E_2$ is the spectral measure of V_1, V_2 (V_1, V_2 is the unitary dialation of T_1, T_2 defined as above), where E_i is the spectral measure of V_i ($i = 1, 2$). Let $x\in H$ be a vector satisfying $T_1^n x\to 0$ for $n\to\infty$. Then:*

(12) *the measures μ_x and $m\times\mu''_x$ are mutually absolutely continuous, where $\mu_x(\sigma) = (E(\sigma)x, x)$ and $\mu''_x(\delta) = (E_2(\delta)x, x)$,*

(13) *if $f(z_1, z_2) = \dfrac{d\mu_x}{d(m\times\mu''_x)}$, then for almost all z_2 (with respect to μ''_x) the following inequality holds:*

$$\Big|\int \log f(z_1, z_2)\,dm(z_1)\Big| < \infty.$$

Using this theorem we can prove

COROLLARY. *Suppose that the assumptions of Theorem 1 are fulfilled. If, additionally, $T_2^m x\to 0$ for $m\to\infty$, or T_2 is completely non-unitary, then the closed support of the measure μ_x is equal to Γ^2.*

Proof. It is sufficient to show that, for every open subset δ of Γ, $\mu_x(\delta\times\delta) > 0$. By Theorem 1 we have $\mu_x(\delta\times\delta) = \int_{\delta\times\delta} f(z_1, z_2)\,d(m\times\mu''_x)$. It follows, by Fubini's theorem, that $\mu_x(\delta\times\delta) = \int_\delta\int_\delta f(z_1, z_2)\,dm\,d\mu''_x$. Now we consider the function $g(z_2) = \int_\delta f(z_1, z_2)\,dm(z_1)$. Since for almost all $z_2(\mu''_x)$, $\log f(z_1, z_2)$ is summable (m), we have, in particular, that for almost all $z_2(\mu''_x)$ and almost all $z_1(m)$, $f(z_1, z_2) > 0$. This implies that $g(z_2) > 0$ for almost all $z_2(\mu''_x)$, because $m(\delta) > 0$. Consequently, it suffices to show that $\mu''_x(\delta) > 0$, because then we get $\mu_x(\delta\times\delta) = \int_\delta g(z_2)\,d\mu''_x > 0$.

It is easy to see that if E' is the spectral measure of the minimal unitary dilation of T_2 (for example, the spectral measure of S_2), then $\mu_x''(\sigma) = (E'(\sigma)x, x)$. Now, by Theorem 8.5 of [3] we get that μ_x'' is absolutely continuous with respect to m and if $f(z) = d\mu_x''/dm$, then $\log f(z)$ is summable. In particular, $f(z) > 0$ for almost all $z(m)$ and consequently $\mu_x''(\delta) = \int_\delta f(z)\,dm(z) > 0$ which finishes the proof.

References

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, SECTION CRACOV

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