On a generalization of the Perron integral on one-dimensional intervals

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Introduction

The integral studied in the present paper is a generalization of the one-dimensional Perron integral. We call it an $H$-integral and denote by

$\int_a^b f dt$, where $H$ specifies a system of pointed intervals used in the definition.

The integral is defined as a certain limit of the sums

$$S(f, \Delta) = \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1}),$$

where $x_0 < a = t_1 < x_1 < t_2 < \ldots < x_{k-1} < t_k = b < x_k$. The intervals $[x_{j-1}, x_j]$ forming a covering (rather than a partition) of the interval $[a, b]$, the integral has some unexpected properties. For example, it is possible that $(H) \int_a^b f dt$ exists but $(H) \int_c^d f dt$ does not for some $c \in (a, b)$. For some choices of the set $H$ we have $(H) \int_1^{-1} dx/x = 0$.

In our paper [4], such examples were presented and the transformation of the integral for a special choice of the set $H$ was discussed. In the general case, the main results were given without proofs. The aim of the present paper is to give brief proofs of the results announced in [4], Section 3.

1. Preliminaries

Let $[a, b]$ be an interval. Then any finite set

$\Delta = \{(t_j, [x_{j-1}, x_j]); j = 1, \ldots, k\}$
such that
\[ x_0 < t_1 = a < x_1 < t_2 < \ldots < t_{k-1} < x_{k-1} < t_k = b < x_k \]
is called a covering of \([a, b]\).

If \(\delta\) is a gauge on \([a, b]\), i.e., \(\delta: [a, b] \rightarrow (0, +\infty)\), and
\[ [x_j, x_{j-1}] \subseteq B(t_j, \delta(t_j)), \quad j = 1, \ldots, k, \]
then \(\Delta\) is said to be \(\delta\)-fine. (Here and in the sequel, \(B(t, r) = (t-r, t+r)\).)

Write
\[ J = J[a, b] = \{(t, [x, y]); t \in [a, b], x < t < y\}, \]
\[ \text{Sym} = \text{Sym}[a, b] = \{(t, [x, y]) \in J; t = \frac{1}{2}(x+y)\}. \]

Let \(H = H[a, b]\) be a set such that \(\text{Sym} \subseteq H \subseteq J\) and
\[ (1) \quad \text{for every } (t, [x, y]) \in H \text{ there is } \xi > 0 \text{ such that } (t, [x+h, y-h]) \in H \text{ for every } h, |h| < \xi. \]

A covering \(\Delta\) such that \(\Delta \subseteq H\) will be called an \(H\)-covering.

1.1. Remark. Let \(K > 0, \theta > 1\) be constants. Then
\[ AS_{K, \theta} = \{(t, [x, y]) \in J; 0 < t-x < y-t + K(y-t)^{\theta}, 0 < y-t < t-x + K(t-x)^{\theta}\} \]
satisfies \(\text{Sym} \subseteq AS_{K, \theta} \subseteq J\) and has property (1). (Cf. [4], Note 3.2.)

1.2. Remark. Given a set
\[ \Xi = \{(\tau_i, [\xi_{i-1}, \xi_i]); i = 1, \ldots, l\} \subseteq H \]
and a gauge \(\delta\) on \([\tau_1, \tau_l]\) such that
\[ \xi_0 < \tau_1 < \xi_1 < \tau_2 < \ldots < \tau_{l-1} < \xi_{l-1} < \tau_l < \xi_l, \quad [\xi_{i-1}, \xi_i] \subseteq B(\tau_i, \delta(\tau_i)); \]
then there exists \(\eta > 0\) such that
\[ \xi_0 + h < \tau_1 < \xi_1 - h < \tau_2 < \ldots < \tau_{l-1} < \xi_{l-1} - (1)^{l-1}h < \tau_l < \xi_l + (1)^{l}h, \]
\[ [\xi_{i-1} + (1)^{l-1}h, \xi_i + (1)^{l}h] \subseteq B(\tau_i, \delta(\tau_i)) \]
for \(i = 1, \ldots, l\) provided \(|h| < \eta\). (This follows from (1) and from the fact that \(\Xi\) is finite.)

The set
\[ \Xi_h = \{(\tau_i, [\xi_{i-1} + (1)^{l-1}h, \xi_i + (1)^{l}h]); i = 1, \ldots, l\} \]
will be called an \(h\)-modification (or briefly a modification) of the set \(\Xi\).

In particular, if \(\Xi\) is a \(\delta\)-fine \(H\)-covering of \([a, b]\), then its \(h\)-modification
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(with $h$ sufficiently small) is a $\delta$-fine $H$-covering of $[a, b]$ as well. This fact will be frequently used in the proofs throughout the paper without further notice.

2. Definition and main properties of the $H$-integral

2.1. Definition. A function $f: [a, b] \to R$ is called $H$-integrable (on $[a, b]$) if there is $q \in R$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that

$$|q - S(f, \Delta)| < \varepsilon$$

for every $\delta$-fine $H$-covering $\Delta$ of $[a, b]$, where

$$S(f, \Delta) = \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1}).$$

The number $q$ is the $H$-integral of $f$ over $[a, b]$ and we write

$$q = (H)\int_{a}^{b} f(t) dt = (H)\int_{a}^{b} f dt.$$

2.2. Remark. If $H = J$, then the $H$-integral is the Perron integral. Indeed, let $f: [a, b] \to R$, $\gamma \in R$. It is well known (cf. [2], Section 1.2; [3], Theorem 3.5; [1], Appendix A, Proposition 4.3) that the Perron integral of $f$ exists and $\gamma = (P)\int_{a}^{b} f dt$ iff for every $\varepsilon > 0$ there exists such a gauge $\delta$ on $[a, b]$ that

$$|\gamma - \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1})| \leq \varepsilon$$

holds for every sequence

(*)

$$a = x_0 \leq t_1 \leq x_1 \leq \ldots \leq x_{k-1} \leq t_k \leq x_k = b$$

satisfying

$$t_j - \delta(t_j) < x_{j-1}, \quad x_j < t_j + \delta(t_j).$$

It can be assumed without loss of generality that $a < t - \delta(t)$, $t + \delta(t) < b$ for $t \in (a, b)$, so that we have in addition $t_1 = a$, $t_k = b$. It is not difficult to prove that the same concept of integral is obtained if (*) is replaced by

$$x_0 < a = t_1 < x_1 < \ldots < x_{k-1} < t_k = b < x_k,$$

and this modified concept is at the same time the $H$-integral for $H = J$.

2.3. Proposition. A function $f$ is $H$-integrable if and only if for every $\varepsilon > 0$ there is a gauge $\delta$ such that

$$|S(f, \Delta_1) - S(f, \Delta_2)| < \varepsilon$$

for any two $\delta$-fine $H$-coverings $\Delta_1, \Delta_2$.

Proof is standard.
2.4. Remark. Let $\delta$ be a gauge on $[a, b]$, let $d \in (a, b)$. The point $d$ is said to be $\delta$-reachable from $a$ (more precisely, $\delta$-$H$-reachable from $a$) if there is a set $\theta = \{(\sigma_i, [\delta_{i-1}, \delta_i]); i = 1, \ldots, m\} \subset H$

such that

$$\delta_0 < \sigma_1 = a < \delta_1 < \sigma_2 < \ldots < \delta_{m-1} < \sigma_m < \delta_m = d,$$

$$[\delta_{i-1}, \delta_i] \subset B(\sigma_i, \delta(\sigma_i)).$$

(Notice that $\theta$ is a covering of $[a, \sigma_m]$ but not of $[a, d]$.)

Similarly, $d$ is called $\delta$-reachable from $b$ if $\theta$ satisfies (2) and

$$d = \delta_0 < \sigma_1 < \delta_1 < \sigma_2 < \ldots < \delta_{m-1} < \sigma_m = b < \delta_m.$$

The set $\theta$ will be called a $\delta$-chain from $a$ to $d$ (or, as the case may be, from $b$ to $d$). Lemma 2.4 [4] asserts that the set of points $d \in (a, b)$ which are not $\delta$-reachable from either $a$ or $b$ is at most countable.

Indeed, let $s$ be the supremum of all $c \in (a, b)$ such that in $(a, c)$ there are at most countably many points not $\delta$-reachable from $a$. We have $s \geq a + \delta(a)$ since every $x \in (a, a + \delta(a))$ is $\delta$-reachable from $a$ (it suffices to put $\delta_0 = 2a - x < \sigma_1 = a < \delta_1 = x$). Assume $s < b$. Then in the interval $(s, \min(s + \delta(s), b))$ there exist uncountably many points not $\delta$-reachable from $a$. Let $x$ be such a point. Then $y = s - (s - x) = 2s - x$ cannot be $\delta$-reachable from $a$ since otherwise we could extend the corresponding chain from $a$ to $y$ by the pair $(s, [2s - x, x]) \in \text{Sym}$, thus obtaining a $\delta$-chain from $a$ to $x$. But there are only countably many points $y < s$ not $\delta$-reachable from $a$, which is a contradiction.

2.5. Remark. It follows from [4], Lemma 2.3 or from Remark 2.4 above that for every gauge $\delta$ there exists a $\delta$-fine $H$-covering. In fact, there always exists a $\delta$-fine Sym-covering. Indeed, by Remark 2.4 we can find a point $d, b - \delta(b) < d < b$, which is $\delta$-Sym-reachable from $a$. By adding the element $(b, [d, 2b - d])$ to the corresponding chain we obtain a $\delta$-fine Sym-covering of $[a, b]$.

2.6. Theorem. Let $f: [a, b] \to R$ be $H$-integrable. Denote by $E = E_f$ the set of all $c \in (a, b)$ such that $f \mid_{[a, c]}$ is not $H$-integrable. Then $E$ is at most countable.

Proof. Set $\varepsilon_j = 2^{-j}$ and find the corresponding gauge $\delta_j$ from the definition of the $H$-integral $(H) \int_a^b f \, dt$. Denote by $W_j$ the set of all points $c \in (a, b)$ which are not $\delta_j$-reachable from either $a$ or $b$, and put $W = \bigcup_{j=1}^\infty W_j$. By Remark 2.4 the set $W$ is at most countable.

Let $c \in (a, b) \setminus W$. Then $(H) \int_a^c f \, dt$ exists.

Indeed, let $\varepsilon > 0$. Find $j$ such that $\frac{1}{2} \varepsilon > 2^{-j}$. Since $c$ is $\delta_j$-reachable from
there exists a set
\[ \theta = \{(\sigma_i, [u_{i-1}, u_i]); \ i = 1, \ldots, m\} \subset H \]
such that
\[ c = u_0 < \sigma_1 < u_1 < \sigma_2 < \ldots < \sigma_{m-1} < u_{m-1} < \sigma_m = b < u_m, \]
\[ \[u_{i-1}, u_i]\subset B(\sigma_i, \delta_j(\sigma_i)). \]
For this set, find \( \eta > 0 \) such that its every \( h \)-modification with \( 0 < h < \eta \) is a \( \delta_j \)-fine \( H \)-covering of \( [\sigma_1, b] \) (cf. Remark 1.2).

Further, choose \( h, \ 0 < h < \eta \), so that
\[ 2h \sum_{i=1}^{m} |f(\sigma_i)| < 2^{-j}, \]
and a gauge \( \delta_j \) such that
\[ \delta_j(t) < \min(\delta_j(t, c-t)) \text{ for } a \leq t < c, \]
\[ \delta_j(c) \leq \min(h, \delta_j(c)), \]
\[ \delta_j(t) = \delta_j(t) \text{ for } c < t \leq b. \]

Let \( A^1, A^2 \) be \( \delta_j \)-fine \( H \)-covering of \( [a, c] \) where
\[ A^p = \{(t^p_j, [x^p_{j-1}, x^p_j]); \ j = 1, \ldots, k^p\}, \quad p = 1, 2. \]
Write \( \varepsilon_p = x^p_k - c; \) then \( 0 < \varepsilon_p < h. \) Construct \( \varepsilon_p \)-modifications of the set \( \theta \) for \( p = 1, 2, \) and denote them by \( \theta^1, \theta^2. \) By the choice of \( h \) and \( \delta_j, \) the sets \( A^p \cup \theta^p, p = 1, 2, \) are \( \delta_j \)-fine \( H \)-covering of \( [a, b]. \) Evidently,
\[ S(f, A^p \cup \theta^p) = S(f, A^p) + S(f, \theta^p) \]
and
\[ |S(f, \theta^1) - S(f, \theta^2)| = \left| \sum_{i=1}^{m} f(\sigma_i)[(u_i + (-1)^i \varepsilon_1 - u_{i-1} - (-1)^i \varepsilon_2)] \right| \]
\[ \leq \sum_{i=1}^{m} |f(\sigma_i)| 2(\varepsilon_1 - \varepsilon_2) \leq 2h \sum_{i=1}^{m} |f(\sigma_i)| < \frac{1}{2} \varepsilon. \]

Consequently, we have
\[ |S(f, A^1) - S(f, A^2)| \leq |S(f, A^1 \cup \theta^1) - S(f, A^2 \cup \theta^2)| + |S(f, \theta^1) - S(f, \theta^2)| < \varepsilon, \]
and the desired integrability (over \( [a, c] \)) follows by Proposition 2.3.

2.7. THEOREM. Let \( a < c < b \) and let two of the integrals in the equality
\[ (H) \int_a^c f \ dt + (H) \int_c^b f \ dt = (H) \int_a^b f \ dt \]
exist. Then the third integral exists as well and the equality holds.

Proof. Consider the case where the first and the last integral exist. Let \( \varepsilon > 0 \), and find gauges \( \delta_1, \delta_2 \) on \([a, c], [a, b] \), respectively, corresponding to \( \varepsilon \) in the sense of Definition 2.1. Without loss of generality we can and will assume that

\[
\delta_2(t) \leq \delta_1(t) \quad \text{for } t \in [a, c],
\]

\[
\delta_2(t) \leq |t - c| \quad \text{for } t \in [a, b]\setminus\{c\},
\]

\[
2|f(c)||\delta_2(c)| < \varepsilon.
\]

Let \( A \) be a \( \delta_2 \)-fine \( H \)-covering of \([c, b] \), where

\[
A = \{ (t_j, [x_{j-1}, x_j]) ; j = 1, \ldots, k \}.
\]

Let \( h > 0 \) be such that the \( h \)-modification \( A_h \) of \( A \) is a \( \delta_2 \)-fine \( H \)-covering of \([c, b] \), and \( x_0 + h \) is \( \delta_2 \)-reachable from \( a \). (Existence of such an \( h \) follows from Remarks 1.2 and 2.4.) Moreover, let \( h \) be so small that

\[
2h \sum_{j=1}^{k} |f(t_j)| < \varepsilon.
\]

Let \( \theta \) be the first set from Remark 2.4 with \( d = x_0 + h \) and \( \delta = \delta_2 \). Then the set \( \theta \cup A_h \) is a \( \delta_2 \)-fine \( H \)-covering of \([a, b] \), and the set \( \theta \cup \{(c, [x_0 + h, x_1 - h])\} \) is a \( \delta_1 \)-fine \( H \)-covering of \([a, c] \). Consequently,

\[
|S(f, A) - (H)\int_c^b f dt + (H)\int_a^c f dt| \leq |S(f, A) - S(f, A_h)|
\]

\[
+ |S(f, \theta \cup \{(c, [x_0 + h, x_1 - h])\}) - (H)\int_a^c f dt|
\]

\[
+ |f(c)|(x_1 - h - x_0 - h) + |S(f, \theta \cup A_h) - (H)\int_a^b f dt|
\]

\[
\leq 2h \sum_{j=1}^{k} |f(t_j)| + \varepsilon + 2|f(c)||\delta_2(c)| + \varepsilon < 4\varepsilon
\]

which proves the existence of the integral \( (H)\int_c^b f dt \) as well as the validity of equality (3).

Proofs of the other cases are analogous.

2.8. Remark. Let \( (H)\int_c^b f dt \) exist, let \( c, d \in [a, b] \), \( c < d \). If \( c, d \in [a, b]\setminus E \), then by Theorems 2.6, 2.7 the integral \( (H)\int_c^d f dt \) exists. Conversely, if \( (H)\int_c^d f dt \) exists, then by Theorem 2.6 there is \( \tau \in (c, d) \) such that the integrals \( (H)\int_a^\tau f dt \)
and \((H) \int_0^t f \, dt\) exist, and by Theorem 2.7 we obtain \(c \not\in \mathcal{E}\) and similarly \(d \not\in \mathcal{E}\).

3. Indefinite \(H\)-integral and its properties

In this section let \(f: [a, b] \to \mathbb{R}\) be \(H\)-integrable, and define

\[
F(a) = 0, \quad F(t) = (H) \int_a^t f(s) \, ds \quad \text{for} \quad t \in [a, b] \setminus \mathcal{E},
\]

where \(a \not\in \mathcal{E}\) is the (at most countable, cf. Theorem 2.6) set of points \(c \in (a, b)\) such that \((H) \int_a^c f \, dt\) does not exist.

3.1. Theorem. The function \(F\) is continuous on \([a, b] \setminus \mathcal{E}\).

Proof. Let \(c \in [a, b] \setminus \mathcal{E}\), let \((c_n)\) be an increasing sequence, \(c_n \not\in \mathcal{E}\), \(\lim_{n \to \infty} c_n = c\). Let \(\varepsilon > 0\). Find the gauge \(\delta\) on \([a, c]\) corresponding to the definition of the integral \((H) \int_a^c f \, dt\), assuming without loss of generality that

\[
\delta(t) < c - t \quad \text{for} \quad t < c, \quad |f(c)| \delta(c) < \varepsilon.
\]

Let \(k\) be such an integer that \(c_k > c - \delta(c)\). Let \(\delta_k\) be the gauge from Definition 2.1 corresponding to \(\varepsilon\) and \((H) \int_a^c f \, dt\). Without loss of generality let us assume that \(\delta_k(t) \leq \delta(t)\) for \(t \in [a, c_k]\). Let \(A\) be a \(\delta_k\)-fine \(H\)-covering of \([a, c_k]\) whose last element is \((c_k, [x_{k-1}, x_k])\). We have \(x_k < c\) by (4). Hence \(A \cup \{(c_k, [x_k, 2c - x_k])\}\) is a \(\delta\)-fine covering of \([a, c]\). Consequently,

\[
|\int_a^c f \, dt - (H) \int_a^c f \, dt| \leq |(H) \int_a^c f \, dt - S(f, A) \cup \{(c, [x_k, 2c - x_k])\}| + |f(c)2\delta(c) + S(f, A) - (H) \int_a^c f \, dt| < 3\varepsilon,
\]

which proves the theorem.

Before formulating a converse result, we will prove the version of the Saks–Henstock lemma corresponding to the \(H\)-integral.

3.2. Theorem (Saks–Henstock lemma). Let \(f: [a, b] \to \mathbb{R}\) be \(H\)-integrable, \(\varepsilon > 0\). Let \(\delta\) be the gauge from Definition 2.1 and let

\[
\{(t_j, [u_j, v_j]); \ j = 1, \ldots, k\} \subset \mathcal{H}
\]

satisfy

\[
[u_j, v_j] \subset B(t_j, \delta(t_j));
\]

\[
a \leq u_1 < t_1 < v_1 \leq u_2 < t_2 < \ldots < v_{k-1} \leq u_k < t_k < v_k \leq b
\]
and \( u_j, v_j \notin E \) for \( j = 1, \ldots, k \). Then

\[
\sum_{j=1}^{k} \left( (H) \int_{u_j}^{v_j} \frac{dt}{f(t_j)(v_j - u_j)} \right) \leq \varepsilon.
\]

**Proof.** Let us first make two remarks. First, we may and will assume, without loss of generality, that all inequalities in (7) are strict. Indeed, we can pass from the points \( u_j, v_j \) to \( u_j' > u_j, v_j' < v_j \) so that conditions (5)–(7) are fulfilled with the new points, and the “error” made by replacing \( u_j, v_j \) by \( u_j', v_j' \) on the left-hand side of (8) is arbitrarily small. Second, notice that by Remark 2.8 the \( H \)-integrals of \( f \) over \([u_j, v_j]\) and \([v_j, u_{j+1}]\) exist for \( j = 1, \ldots, k \) (\( k-1 \), respectively).

Since the proof of Theorem 3.2 is technically rather complicated, we first prove a lemma.

3.3. **Lemma.** Let \( \delta, t_j, u_j, v_j \) be from Theorem 3.2, let \( \varrho > 0 \). Then there exists a \( \delta \)-fine \( H \)-covering \( \Omega \) of \([a, b]\),

\[
\Omega = \{ (\tau_i, [\omega_{i-1}, \omega_i]); \ i = 1, \ldots, l \},
\]

and integers \( 0 < m_1 < m_2 < \ldots < m_k \) such that

\[
\tau_{m_j} = u_j < \omega_{m_j} < \tau_{m_j+1} = t_j < \omega_{m_j+1} < v_j = \tau_{m_j+2},
\]

\[
\omega_{m_j} - u_j = v_j - \omega_{m_j+1} < \varrho.
\]

**Proof.** Denote \( s_j = \frac{1}{2}(v_j+u_j+1) \) for \( j = 1, \ldots, k-1 \), \( s_k = b \). Set

\[
\delta'(s_j - \lambda) = \delta'(s_j + \lambda) = \min(\delta(s_j - \lambda), \delta(s_j + \lambda))
\]

for \( 0 \leq \lambda \leq s_j - v_j, j = 1, \ldots, k-1, \)

\[
\delta'(t) = \delta(t) \quad \text{otherwise,} \ t \in [a, b].
\]

For \( j = 1, \ldots, k-1 \) find \( \sigma_j, 0 < \sigma_j < \delta'(s_j) \) such that

\[
v_j < s_j - \sigma_j < s_j + \sigma_j < u_{j+1} \quad \text{and} \quad s_j - \sigma_j, s_j + \sigma_j \notin E
\]

(cf. Theorem 2.6). Then there exists \( h, 0 < h < \varrho \), satisfying

\[
h < \xi(t_j, [u_j, v_j]) \quad \text{(cf. (1))}, \quad h < s_j - \sigma_j - v_j, \quad h < \delta'(u_j), \quad h < \delta'(v_j)
\]

for \( j = 1, \ldots, k, \) and such that the points \( v_j + h \) are \( \delta' \)-reachable from \( s_j - \sigma_j \) and \( u_1 - h \) is \( \delta' \)-reachable from \( a \) (cf. Remark 2.4).

Now we will construct the desired covering.

By the choice of \( h \) there exists a \( \delta' \)-chain from \( a \) to \( u_1 - h \); let it consist of points

\[
\omega_0, \tau_1 = a, \omega_1, \ldots, \tau_{m_1-1}, \omega_{m_1-1} = u_1 - h.
\]

Put \( \tau_{m_1} = u_1, \omega_{m_1} = u_1 + h, \tau_{m_1+1} = t_1, \omega_{m_1+1} = v_1 - h, \tau_{m_1+2} = v_1, \omega_{m_1+2} = v_1 + h \). Again by the choice of \( h \), the last point is \( \delta' \)-reachable from \( s_1 - \sigma_1 \).
Suppose we have found the points of $\Omega$ up to a point $\omega_{m_j+2} = v_j + h$ in such a way that (9) is fulfilled. Then there is a $\delta'$-chain from $s_j - \sigma_j$ to $\omega_{m_j+2}$, and a "symmetric" chain from $s_j + \sigma_j$ to $u_{j+1} - h$. (Here "symmetric" means that the points of the latter chain are symmetric about $s_j$ to the corresponding points of the former.) The two chains together with the element $(s_j, [s_j - \sigma_j, s_j + \sigma_j]) \in \text{Sym} < H$ filling the gap between them extend our construction up to the point $u_{j+1} - h = \omega_{m_j+1} - 1$. Put $\tau_{m_j+1} = u_{j+1}$, $\omega_{m_j+1} = u_{j+1} + h$, $\tau_{m_j+1} = t_{j+1}$, $\omega_{m_j+1} = v_{j+1} - h$, $\tau_{m_j+1} = v_{j+1}$, $\omega_{m_j+1} = v_{j+1} + h$. Thus we have proceeded from step $j$ to step $j+1$ in our construction. Repeating the procedure, we extend the covering $\Omega$ to the whole interval $[a, b]$. It is seen directly from the construction that $\Omega$ has the required properties.

3.4. Proof of Saks–Henstock lemma. Let $\eta > 0$. Find gauges $\varphi_j$ on $[v_j, u_{j+1}]$ for $j = 0, 1, \ldots, k$ (denoting $v_0 = a$, $u_{k+1} = b$) such that

$$|(H) \int_{v_j}^{u_{j+1}} f \, dt - S(f, \Phi_j)| < \eta/(k + 1)$$

for every $\varphi_j$-fine $H$-covering $\Phi_j$ of $[v_j, u_{j+1}]$.

Let $\delta$ be a gauge on $[a, b]$ such that

$$\delta(t) \leq \delta(t) \quad \text{for} \quad t \in [a, b],$$

$$\delta(t) < |t - u_j| \quad \text{for} \quad t \neq u_j,$$

$$\delta(t) < |t - v_j| \quad \text{for} \quad t \neq v_j,$$

$$\delta(t) \leq \varphi_j(t) \quad \text{for} \quad t \in [v_j, u_{j+1}],$$

$$\delta(u_j) < \delta(t_j - t_j - u_j), \quad \delta(v_j) < \delta(t_j - (v_j - t_j));$$

$$\max(\delta(u_j), \delta(v_j)) < \xi = \xi(t_j, [u_j, v_j])$$

(with $\xi$ from formula (1)), and

$$\delta(u_j)|f(t_j)| < \eta/2(k + 1),$$

$$\delta(v_j)|f(t_j)| < \eta/2(k + 1).$$

Let $\Omega$ be the partition from Lemma 3.3 (corresponding to the gauge $\delta$ instead of $\delta$). Since $\Omega$ is a $\delta$-fine $H$-covering of $[a, b]$ we have

$$|(H) \int_a^b f(t) \, dt - S(f, \Omega)| < \varepsilon.$$

Further, denote

$$\Phi_j = \{(\tau_p, [\omega_{p-1}, \omega_p]); \quad p = m_j + 2, m_j + 3, \ldots, m_{j+1}\},$$

$$j = 0, 1, \ldots, k, \quad m_0 = -1, \quad m_{k+1} = l.$$

Then $\Phi_j$ is a $\varphi_j$-fine $H$-covering of $[v_j, u_{j+1}]$ and hence satisfies (10). Obviously we have
\[
\sum_{j=1}^{k} \left( (H) \int_{u_{j}}^{v_{j}} f(x) \, dx \right) = (H) \int_{a}^{b} f(x) \, dx - S(f, \Omega)
\]

\[
- \sum_{j=0}^{k} \left( (H) \int_{u_{j}}^{u_{j+1}} f(x) \, dx \right) - \sum_{j=1}^{k} f(t_{j})(v_{j} - \omega_{m_{j}+1} - u_{j} + \omega_{m_{j}}),
\]

and consequently,

\[
\left| \sum_{j=1}^{k} \left( (H) \int_{u_{j}}^{v_{j}} f(x) \, dx \right) \right| < \varepsilon + 2\eta.
\]

Since \( \eta > 0 \) has been arbitrary, the proof is complete.

**3.5. Remark.** The assertion of the Saks–Henstock lemma can be modified to

\[
\sum_{j=1}^{k} \left| F(v_{j}) - F(u_{j}) - f(t_{j})(v_{j} - u_{j}) \right| \leq 2\varepsilon.
\]

This is obtained by dividing the set of \((t_{j}, [u_{j}, v_{j}])\) in two groups according to the sign of the corresponding summand, and applying the lemma in the original form to each group separately.

Now we can prove a converse of Theorem 3.1.

**3.6. Theorem.** Let \( f: [a, b] \to \mathbb{R} \) be \( H \)-integrable, \( d \in (a, b) \). If there exists a finite limit \( \lim_{c \to d} F(c) = q \in \mathbb{R} \) for \( c \in (a, d) \setminus E \), then \( (H) \int_{a}^{b} f(x) \, dx \) exists and equals \( q \).

**Proof.** Let \( \varepsilon > 0 \) and let \( \delta \) be the gauge on \([a, b]\) corresponding to \( \varepsilon \) and \((H) \int_{a}^{b} f(x) \, dx \). Let \( \delta^{\prime} \) be a gauge on \([a, b]\) such that

\[
2|f(a)|\delta(d) < \varepsilon, \quad 2|f(d)|\delta(d) < \varepsilon,
\]

\[
|F(x) - q| < \varepsilon \quad \text{for any} \quad x \notin E, \quad d - \delta(d) < x < d,
\]

\[
|F(x)| < \varepsilon \quad \text{for any} \quad x \notin E, \quad a < x < a + \delta(a)
\]

(cf. Theorem 3.1),

\[
\delta^{\prime}(x) \leq \delta(x) \quad \text{for all} \quad x \in [a, b].
\]

Let \( \Lambda \) be a \( \delta^{\prime} \)-fine \( H \)-covering of \([a, d]\), where

\[
\Lambda = \{ (t_{j}, [x_{j-1}, x_{j}]) ; \ j = 1, \ldots, k \}.
\]

Find a \( \delta^{\prime} \)-fine modification \( \Lambda_{h} \) (cf. Remark 1.2) such that \( x_{j}^{\prime} = x_{j} + (-1)^{j-1} - jh \notin E \) for \( j = 1, \ldots, k, \ h > 0 \) and

\[
2h \sum_{j=1}^{k} |f(t_{j})| < \varepsilon.
\]
By the Saks-Henstock lemma we have

\[ \left| \sum_{j=2}^{k-1} (f(t_j)(x_j - x_{j-1}) - (H) \int_{x_{j-1}}^{x_j} f \, dt) \right| \leq \varepsilon. \tag{12} \]

Consequently, using (11), (12) and the properties of the gauge \( \tilde{\delta} \) we obtain

\[ \left| \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1}) - q \right| \leq \varepsilon + \left| \sum_{j=1}^{k-1} f(t_j)(x_j - x_{j-1}) - q \right| + \sum_{j=2}^{k-1} \left| f(t_j)(x_j - x_{j-1}) - (H) \int_{x_{j-1}}^{x_j} f \, dt \right| \]

\[ + 2\tilde{\delta}(d)|f(d)| + \left| (H) \int_{x_1}^{x_i} f \, dt - q \right| \]

\[ \leq 4\varepsilon + |\int_{a}^{x_i} - q| + \left| (H) \int_{a}^{x_i} f \, dt \right| \leq 6\varepsilon, \]

which proves the theorem.

The next theorem strengthens the result on continuity of the function \( F \), asserting that it has a derivative equal to \( f \) almost everywhere in \([a, b]\). The symbol \( m(M) \) stands for the Lebesgue measure of the set \( M \).

**3.7. Theorem.** There is a set \( M \subset [a, b] \), \( m(M) = 0 \), such that for every \( \varepsilon > 0 \) and \( t \in [a, b] \setminus M \) there is \( \delta = \delta(t) > 0 \) such that

\[ |F(y) - F(x) - f(t)(y - x)| < \varepsilon |y - x| \]

for every \( x, y \) such that

\[ (t, [x, y]) \in H, \quad [x, y] \subset B(t, \delta(t)), \quad x, y \notin E. \tag{14} \]

**Proof.** For \( \delta > 0 \), \( t \in [a, b] \setminus E \) define

\[ \Phi_\delta(t) = \inf_{s}(F(y) - F(x))/(y - x), \quad \Phi_\ast(t) = \sup_{s} \Phi_\delta(t), \]

\[ \Phi^\delta(t) = \sup_{s}(F(y) - F(x))/(y - x), \quad \Phi^\ast(t) = \inf_{s} \Phi^\delta(t), \]

where the infimum or supremum is taken over all \( x, y \) satisfying (14). Denote

\[ P_n = \{ t \in (a, b) ; \ \Phi_\ast(t) \leq f(t) - n^{-1} \}, \]

\[ Q_n = \{ t \in (a, b) ; \ \Phi^\ast(t) \geq f(t) + n^{-1} \}, \]

\[ M = \bigcup_{n=1}^{\infty} (P_n \cup Q_n). \]

If \( m(M) = 0 \), the theorem holds. Assume \( m_e(M) > 0 \). Then there exists, say, an index \( p \) such that \( m_e(P_p) = \sigma > 0 \) (\( m_e \) denotes the outer Lebesgue measure; the proof is analogous if \( m_e(Q_q) > 0 \) for some \( q \)).
Choose $0 < \varepsilon < \sigma/4p$ and find the gauge $\delta$ corresponding to $\varepsilon$ by Definition 2.1. For $t \in P_p$ set
\[ \mathcal{S}(t) = \{(x, y) \in [a, b]; \ x, y \notin E, (t, [x, y]) \in H, [x, y] \in B(t, \delta(t)), \]
\[ (F(y) - F(x))(y - x)^{-1} < f(t) - 1/2p \}. \]

Then $\bigcup_{t \in P_p} \mathcal{S}(t)$ covers $P_p$ in the sense of Vitali; hence there exists its finite disjoint subsystem of $(x_i, y_i), \ i = 1, \ldots, r$, such that
\[ \sum_{i=1}^{r} (y_i - x_i) \geq \sigma/2. \]

We may apply the Saks–Henstock lemma to this subsystem, which yields
\[ \left| \sum_{i=1}^{r} (F(y_i) - F(x_i) - f(t)(y_i - x_i)) \right| \leq \varepsilon < \sigma/4p; \]
on the other hand, from the definition of $\mathcal{S}(t)$ we have
\[ \left| \sum_{i=1}^{r} (F(y_i) - F(x_i) - f(t)(y_i - x_i)) \right| \geq \sum_{i=1}^{r} \frac{1}{2p} (y_i - x_i) \geq \frac{\sigma}{4p}, \]
a contradiction.

3.8. Theorem. Let $F$ be defined as above, put $F(t) = F(a)$ for $t < a$ and $F(t) = F(b)$ for $t > b$. Let $C \subseteq [a, b], \ m(C) = 0$. Then for every $\varepsilon > 0$ there is a gauge $\delta$ on $C$ such that for any finite system $\{(\tau_j, [\xi_j, \eta_j]); \ j = 1, \ldots, r\} \subset H$ such that $\tau_j \in C, \ [\xi_j, \eta_j] \subset B(\tau_j, \delta(\tau_j)); \ \xi_j, \eta_j \notin E$ and the intervals $[\xi_j, \eta_j]$ do not overlap, the inequality
\[ \sum_{j=1}^{r} |F(\eta_j) - F(\xi_j)| < \varepsilon \]
holds.

Proof. Denote
\[ C_n = \{t \in C; \ n - 1 \leq |f(t)| < n\}. \]

There exist open sets $G_n \supset C_n, \ m(G_n) < \varepsilon_n = \varepsilon(3 \cdot 2^n n)^{-1}$; for $t \in C_n$ choose $\delta_1(t)$ such that $B(t, \delta_1(t)) \subset G_n$. Since $[\xi_j, \eta_j]$ do not overlap, we have
\[ \sum_{j=1}^{r} |f(\tau_j)||\eta_j - \xi_j| = \sum_{n=1}^{\infty} \sum_{t \in C_n} |f(\tau_j)||\eta_j - \xi_j| \leq \sum_{n=1}^{\infty} nm(G_n) < \varepsilon/3. \]

For $\varepsilon/3$ find the gauge $\delta_2$ from Definition 2.1 and put $\delta(t) = \min(\delta_1(t), \delta_2(t))$ for $t \in C, \ \delta(t) = \delta_2(t)$ otherwise. Using the modified version of the Saks–Henstock lemma from Remark 3.5 we obtain
\[ \sum_{j=1}^{r} |F(\eta_j) - F(\xi_j) - f(\tau_j)(\eta_j - \xi_j)| < \frac{\varepsilon}{3}. \]
that is,
\[ \sum_{j=1}^{r} |F(\eta_j) - F(\xi_j)| < \frac{3}{2} \varepsilon + \sum_{j=1}^{r} |f(\tau_j)| (\eta_j - \xi_j) < \varepsilon \]
by inequality (16).

The next theorem shows that the properties from the two preceding theorems characterize the indefinite H-integral.

3.9. Theorem. Let \( E \) be an at most countable subset of \( [a, b] \), let \( f: [a, b] \rightarrow \mathbb{R}, \quad F: [a, b] \setminus E \rightarrow \mathbb{R} \). Extend \( F \) by \( F(t) = F(a) \) for \( t < a \), \( F(t) = F(b) \) for \( t > b \).

Assume that

(i) for almost all \( t \) and all \( \varepsilon > 0 \) there exists \( \delta(t) \) such that (13) holds for all \( x \), \( y \) satisfying (14);

(ii) if \( C \subset [a, b] \), \( m(C) = 0 \), then for every \( \varepsilon > 0 \) there is a gauge \( \delta \) on \( C \) such that (15) holds provided \( \tau_j, \xi_j, \eta_j \) satisfy the assumptions of Theorem 3.8.

Then \( (H) \int_{a}^{b} \frac{f}{a} \, dt \) exists and equals \( F(b) - F(a) \).

Proof. Let \( C_1 \) be the set of \( t \in [a, b] \) for which (i) is not fulfilled, \( C = C_1 \cup E \). Then \( m(C) = 0 \). Find \( \delta \) from (ii) and define \( \delta_1(t) = \min(\delta(t), \delta(t)) \) for \( t \in C \), \( \delta_1(t) = \delta(t) \) otherwise. Let \( \varepsilon > 0 \) and let \( \Delta \) be a \( \delta_1 \)-fine H-partition of \([a, b], \)
\[ \Delta = \{ (t_j, [x_j, y_j]) ; j = 1, \ldots, k \} \]

Similarly as in the proof of Theorem 3.6, modify \( \Delta \) so that \( F(x_j'), F(y_j') \) are defined for all \( j \)'s and \( S(f, \Delta) \) changes only by \( \varepsilon \) (cf. Remark 1.2 and inequality (11)). Then
\[ (H) \int_{a}^{b} \frac{f}{a} \, dt = \sum_{j=1}^{k} [F(y_j') - F(x_j')] \]
hence
\[ |(H) \int_{a}^{b} \frac{f}{a} \, dt - S(f, \Delta_b)| \leq \sum_{t \in C} |F(y_j') - F(x_j')| - f(t_j)(y_j' - x_j')| \]
\[ + \sum_{t \in C} |F(y_j') - F(x_j')| + \sum_{t \in C} |f(t_j)(y_j' - x_j')| \]
Using (i), (ii) and estimating the last sum as in the proof of the preceding theorem we conclude
\[ |\int_{a}^{b} \frac{f}{a} \, dt - S(f, \Delta)| \leq \text{const} \cdot \varepsilon \]
which completes the proof.
References


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