ANNALES
POLONICI MATHEMATICI
XLI (1983)

Coefficient inequalities and maximalization of some functionals for pairs of vector functions

by Kazimierz Włodarczyk (Łódź)

Abstract. The paper investigates pairs of vector functions [33] defined as a natural generalization of Aharonov's pairs [1].

Making use of the methods of Grunsky–Nehari ([13], [25]) type we obtain estimations of the first and third order functionals and also estimates of derivatives and Schwarz' derivatives for some pairs of vector functions.

For some cases a complete characterization of all extremal pairs is given.

Introduction. Aharonov [1] introduced the class A whose elements are pairs (F, G) of functions F and G, $F(0) = G(0) = 0$, analytic and univalent in $\Delta = \{z\colon |z| < 1\}$ such that $F(z)\,G(\zeta) \neq 1$ for any points $z, \zeta \in \Delta$. In the class A of Aharonov's pairs significant results have been obtained, especially by Aharonov [2], Hummel [14] and Jenkins [18].

The class in question has the following interesting property. For $G = F$, $G = \bar{F}$ and $G = -\bar{F}$, where $\bar{F}(z) = \overline{F(\bar{z})}$, it generates the classes of Bieberbach–Eilenberg's functions ([3], [4], [7]) of bounded functions and of Grunsky–Shah's functions ([12], [29]), respectively. These classes were studied in Hummel and Schiffer [15], Jenkins [17], Schiffer and Tammi [26], [27], Śladkowska [30], [31].

The following definition, which is a natural generalization of Aharonov's definition of a pair, was given in [33].

For any non-negative integers m, n such that $m+n \geqslant 2$ let

$$\begin{aligned} a_0 &= \{a_{01}, \ldots, a_{0m}\}, \quad m \geqslant 1, & b_0 &= \{b_{01}, \ldots, b_{0n}\}, \quad n \geqslant 1,\\ &= \emptyset, \qquad\qquad\qquad\quad m = 0, & &= \emptyset, \qquad\qquad\qquad\quad n = 0, \end{aligned}$$

be two sets of finite elements satisfying the conditions:

$$\begin{aligned} a_{0k} &\neq a_{0j}, \quad k \neq j,\ k, j = 1, \ldots, m,\ m > 1,\ n \geqslant 0,\\ b_{0k} &\neq b_{0j}, \quad k \neq j,\ k, j = 1, \ldots, n,\ m \geqslant 0,\ n > 1,\\ a_{0k}\, b_{0j} &\neq 1, \quad k = 1, \ldots, m,\ j = 1, \ldots, n,\ m, n \geqslant 1. \end{aligned}$$

Let C denote the complex plane.

The mappings F and G, $F=(F_1, \ldots, F_m)\colon \Delta \to C^m$ when $m \geqslant 1$, $G=(G_1, \ldots, G_n)\colon \Delta \to C^n$ when $n \geqslant 1$, $F=0$ or $G=0$ when $m=0$ or $n=0$, respectively, are said to form a pair (F, G) if $F_1, \ldots, F_m$ and $G_1, \ldots, G_n$ are functions of the form:

$$F_k(z) = a_{0k}+a_{1k}z+\ldots, \quad k=1, \ldots, m,\ n \geqslant 0,$$

$$G_k(z) = b_{0k}+b_{1k}z+\ldots, \quad k=1, \ldots, n,\ m \geqslant 0,$$

analytic and univalent in Δ, satisfying the conditions:

$$F_k(z) \neq F_j(\xi), \quad k \neq j,\ k, j = 1, \ldots, m,\ m>1,\ n \geqslant 0,$$

$$G_k(z) \neq G_j(\xi), \quad k \neq j,\ k, j = 1, \ldots, n,\ m \geqslant 0,\ n>1,$$

$$F_k(z)G_j(\xi) \neq 1, \quad k=1, \ldots, m,\ j=1, \ldots, n,\ m, n \geqslant 1,$$

for any points $z, \xi \in \Delta$.

The pair $(F, 0)$ is identified with the function F. Clearly, $(F, 0) = (0, F)$.

The class of all pairs (F, G) of vector functions F and G as defined above, for fixed admissible m, n, a_0, b_0 will be denoted by $C_{m,n}(a_0, b_0)$ ([33]). Accordingly, $C_{1,0}(a_0, b_0)$, $b_0 = \Phi$, denotes the class of functions F of the form

$$F(z) = a_{01}+a_{11}z+\ldots,$$

analytic and univalent in Δ.

Some subclasses of the class $C_{m,m}(a_0, b_0)$ have the property analogous to subclasses of A, the class of Aharonov's pairs; e.g., for $G=F$, $G=\bar{F}$ and $G=-\bar{F}$, where $\bar{F}=(\bar{F}_1, \ldots, \bar{F}_m)$, they generate the generalized classes of Bieberbach–Eilenberg's functions, of bounded functions, and Grunsky–Shah's functions, respectively. These classes were investigated in Gromova and Lebedev [11].

The classes $C_{m,0}(a_0, b_0)$, $m \geqslant 2$, generalize Gelfer's class [8], i.e. the class of the functions F analytic and univalent in Δ and satisfying the condition

$$F(z)+F(\xi) \neq 0$$

for all $z, \xi \in \Delta$, while the classes $C_{1,0}(a_0, b_0)$ are counterparts of the familiar class S, i.e. the class of functions F of the form

$$F(z) = z+a_2z^2+\ldots$$

analytic and univalent in Δ.

In [33] we proved three general inequalities of Grunsky–Nehari type ([13], [25]) for the classes $C_{m,n}(a_0, b_0)$, $m+n \geqslant 2$, and presented a thorough discussion of these inequalities.

The present paper is a continuation of the study initiated in [33] and deals with the problem of maximalization of certain functionals in the classes

$C_{m,m}(a_0, b_0)$ and $C_{m,0}(a_0, \Phi)$ for $m \geqslant 1$ with a_0, b_0 given by

(1) $$a_{0k} = a_{01} \exp[2\pi(k-1)i/m], \quad k = 1, \ldots, m,$$

(2) $$b_{0k} = b_{01} \exp[2\pi(k-1)i/m], \quad k = 1, \ldots, m,$$

and in their subclasses consisting of pairs (F, G) and of functions F, respectively, such that

(3) $$F_k = F_1 \exp[2\pi(k-1)i/m], \quad k = 1, \ldots, m,$$

(4) $$G_k = G_1 \exp[2\pi(k-1)i/m], \quad k = 1, \ldots, m.$$

The results obtained contain a complete, in some cases, characterization of all extremal pairs. The method employed thus proves to be workable and most promising.

1. Estimations of first-order functionals. In this section we estimate the functionals

$$\prod_{k=1}^{m} |F_k'(0)\, G_k'(0)| \quad \text{and} \quad \prod_{k=1}^{m} |F_k'(0)|,$$

$(F, G) \in C_{m,m}(a_0, b_0)$, $F \in C_{m,0}(a_0, \Phi)$.

Investigation of counterparts of the problem under consideration in the classes of systems of functions with disjoint images has been originated by Lavrentev [21] and Goluzin [9].

Let $(F, G) \in C_{m,m}(a_0, b_0)$, $m = 1$ or $m = 2l$, $l = 1, 2, \ldots$, and assume that a_0, b_0 satisfy conditions (1) and (2).

Let $r_0 = |z_0|$ when $G_k(z_0) = 0$ for some $z_0 \in \Delta$ and $k = 1, \ldots, m$, or $r_0 = 0$ when $G_k(z) \neq 0$ for all $z \in \Delta$ and $k = 1, \ldots, m$.

For any fixed $r \in (r_0; 1)$ let D_r be a $2m$-connected region bounded by some curves Γ_k, $k = 1, \ldots, 2m$, described parametrically by

$$\begin{aligned} \Gamma_k:\ w &= F_k(z), & k &= 1, \ldots, m, \\ &= 1/G_{k-m}(z), & k &= m+1, \ldots, 2m, \end{aligned}$$

where $z = r\exp(i\varphi)$ $(0 \leqslant \varphi \leqslant 2\pi)$, the orientation of the boundary ∂D_r being defined by the change of the parameter φ from 0 to 2π.

Write

$$w_k = \begin{cases} F_k(r), & k = 1, \ldots, m, \\ 1/G_{k-m}(r), & k = m+1, \ldots, 2m. \end{cases}$$

If $\infty \in D_r$, let $|w| = R$ be a circle of negative orientation will respect to its interior and such that the curves Γ_k, $k = 1, \ldots, 2m$, are inside it.

Let D_r^0 denote a one-connected region bounded by the curves Γ_k, $k = 1, \ldots, 2m$, the circle $|w| = R$ and some analytic arcs joining the point R of the circle $|w| = R$ with the points w_k, $k = 1, \ldots, 2m$.

If $\infty \notin D_r$, say, ∞ is inside the curve Γ_{2m}, then D_r^0 will denote a connected region bounded by the curves Γ_k, $k = 1, \ldots, 2m$, and some analytic arcs joining the point w_{2m} with the points w_k, $k = 1, \ldots, 2m-1$.

Let

$$A_{0k} = a_{0k}^{m/2}, \quad B_{0k} = b_{0k}^{m/2}, \quad k = 1, \ldots, m, \; m \geqslant 2. \tag{1.1}$$

In the closure $\bar{D}_r$ we define an analytic function g by

$$g(w) = \begin{cases} \log \dfrac{w-a_{01}}{1-wb_{01}}, & \text{when } m = 1, \\ \log \dfrac{(w^{m/2}-A_{01})(1+w^{m/2}B_{01})}{(w^{m/2}+A_{01})(1-w^{m/2}B_{01})}, & \text{when } m = 2l, \; l = 1, 2, \ldots \end{cases} \tag{1.2}$$

and assume

$$\begin{aligned} \sum_{q=0}^{\infty} c_q^k z^q + x_k \log z &= g[F_k(z)], \qquad k = 1, \ldots, m, \\ &= g[1/G_{k-m}(z)], \quad k = m+1, \ldots, 2m, \end{aligned} \qquad |z| = r. \tag{1.3}$$

Clearly, for $m = 1$ we have

$$x_1 = -x_2 = 1 \tag{1.4}$$

and for $m = 2l$, $l = 1, 2, \ldots$, we have

$$\begin{aligned} x_k = -x_{k+m} &= 1, \quad k = 2p+1, \\ &= -1, \quad k = 2p+2, \end{aligned} \tag{1.5}$$

where $p = 0, 1, \ldots, \frac{1}{2}(m-2)$.

As a simple consequence of the Green formula (Nehari [24]) we have the following inequality:

$$0 < \frac{1}{\pi} \iint_{D_r^0} |g'(w)|^2 d\tau = \frac{1}{2\pi i} \int_{D_r^0} \overline{g(w)}\, dg(w). \tag{1.6}$$

In view of the equality

$$\lim_{R \to \infty} \int_{|w|=R} \overline{g(w)}\, dg(w) = 0, \quad \text{when } \infty \in D_r, \tag{1.7}$$

from (1.6) we obtain

$$0 < -\frac{1}{2\pi i} \sum_{k=1}^{m} \left\{ \int_{\Gamma_k} \overline{\left[g(w) + 2\pi i \sum_{p=1}^{k} x_{p-1}\right] dg(w) + 2\pi i\, g[F_k(r)]} \right\} -$$

$$-\frac{1}{2\pi i} \sum_{k=m+1}^{2m} \left\{ \int_{\Gamma_k} \overline{\left[g(w) + 2\pi i \sum_{p=1}^{k} x_{p-1}\right] dg(w) + 2\pi i\, g[1/G_{k-m}(r)]} \right\}$$

and $x_0 = 0$. Thus, by (1.3)

$$0 > \int_0^{2\pi} \sum_{k=1}^{2m} \left[x_k(\log r - i\varphi) + \sum_{q=0}^{\infty} \bar{c}_q^k \bar{z}^q\right]\left[x_k + \sum_{q=0}^{\infty} q c_q^k z^q\right] d\varphi +$$
$$+ \sum_{k=1}^{2m} \left\{x_k \sum_{q=0}^{\infty} c_q^k r^q + |x_k|^2 \log r\right\}.$$

Consequently

$$\sum_{k=1}^{2m} \left[2x_k \operatorname{Re}\{c_0^k\} + \sum_{q=1}^{\infty} q|c_q^k|^2 r^{2q}\right] < \sum_{k=1}^{2m} 2 \log r^{-1}.$$

Passing to the limit as $r \to 1$, we obtain

THEOREM 1. *If $(F, G) \in C_{1,1}(a_0, b_0)$ or $(F, G) \in C_{m,m}(a_0, b_0)$, $m = 2l$, $l = 1, 2, \ldots$, and a_0, b_0 satisfy conditions* (1), (2), *then*

$$\sum_{k=1}^{2m} \left[2x_k \operatorname{Re}\{c_0^k\} + \sum_{q=1}^{\infty} q|c_q^k|^2\right] \leqslant 0, \tag{1.8}$$

where c_q^k and x_k, $k = 1, \ldots, 2m$, $q = 0, 1, \ldots$, are defined by (1.1)–(1.5).

From Theorem 1 we get

THEOREM 2. *If $(F, G) \in C_{1,1}(a_0, b_0)$, then*

$$|a_{11} b_{11}| \leqslant |1 - a_{01} b_{01}|^2 \tag{1.9}$$

and the equality holds only for pairs (F, G) in which the functions F_1, G_1 satisfy the equations

$$\frac{F_1 - a_{01}}{1 - b_{01} F_1} = \alpha z, \qquad \frac{G_1 - b_{01}}{1 - a_{01} G_1} = \beta z, \qquad |\alpha\beta| = 1. \tag{1.10}$$

Proof. Observe (cf. (1.8), (1.4)) that $\operatorname{Re}\{c_0^1 - c_0^2\} \leqslant 0$. Hence and from (1.2) and (1.3) we obtain (1.9).

Assuming $\operatorname{Re}\{c_0^1 - c_0^2\} = 0$ we conclude that

$$\operatorname{Re}\{g[F_1(z)]\} = \log|\alpha z|, \qquad -\operatorname{Re}\{g[1/G_1(z)]\} = \log|\beta z|, \qquad |\alpha\beta| = 1,$$

which consequently gives (1.10).

Remark 1. Lavrentev [21] proved an analogue of Theorem 2 for the class of systems of two functions analytic and univalent in Δ and having disjoint images, where the extremal regions are formed by two half-planes complementary for each other.

Let $(F, G) \in C_{m,m}(a_0, b_0)$, $m \geqslant 2$, $m = 2l+1$, $l = 1, 2, \ldots$; assume that $F_k(z) \neq 0$, $G_k(z) \neq 0$ for $z \in \Delta$ and $k = 1, \ldots, m$. Denote

$$F_k^{m/2}(z) = A_{0k} + A_{1k} z + \ldots, \qquad G_k^{m/2}(z) = B_{0k} + B_{1k} z + \ldots, \qquad k = 1, \ldots, m.$$

We shall now prove

THEOREM 3. *If* $(F, G) \in C_{m,m}(a_0, b_0)$, $m \geqslant 2$, a_0, b_0 *satisfy conditions* (1), (2) *and for* $z \in \Delta$ *and* $k = 1, \ldots, m$, $F_k(z) \neq 0$, $G_k(z) \neq 0$ *whenever* $m = 2l+1$, $l = 1, 2, \ldots$, *then*

$$\prod_{k=1}^{m} |A_{1k} B_{1k}| \leqslant 4^m \prod_{k=1}^{m} \left| A_{0k} B_{0k} \left(\frac{1 - A_{0k} B_{0k}}{1 + A_{0k} B_{0k}} \right)^2 \right|. \tag{1.11}$$

When $A_{01} B_{01} > 0$, *the equality holds only for pairs* (F, G) *in which the functions* F_k, G_k, $k = 1, \ldots, m$ (Fig. 1) *satisfy the equations:*

$$\sqrt{\frac{A_{0k}}{B_{0k}}} \frac{1}{F_k^{m/2}} - \sqrt{\frac{B_{0k}}{A_{0k}}} F_k^{m/2} = \left(\frac{1}{\sqrt{A_{0k} B_{0k}}} - \sqrt{A_{0k} B_{0k}} \right) \frac{1 - \varepsilon z}{1 + \varepsilon z}, \tag{1.12}$$

$$|\varepsilon| = 1, \ k = 1, \ldots, m,$$

$$\sqrt{\frac{B_{0k}}{A_{0k}}} \frac{1}{G_k^{m/2}} - \sqrt{\frac{A_{0k}}{B_{0k}}} G_k^{m/2} = \left(\frac{1}{\sqrt{A_{0k} B_{0k}}} - \sqrt{A_{0k} B_{0k}} \right) \frac{1 - \varepsilon z}{1 + \varepsilon z}, \tag{1.13}$$

$$|\varepsilon| = 1, \ k = 1, \ldots, m.$$

Proof. Assume that $m = 2l$, $l = 1, 2, \ldots$

From (1.8) it follows that

$$\mathrm{Re} \left\{ \sum_{p=0}^{(m-2)/2} (c_0^{2p+1} + c_0^{2p+2+m} - c_0^{2p+1+m} - c_0^{2p+2}) \right\} \leqslant 0 \tag{1.14}$$

and hence we obtain (1.11).

Assume that $A_{01} B_{01} > 0$ (i.e. $A_{0k} B_{0k} > 0$, $k = 1, \ldots, m$) and that the equality holds in (1.14).

Then

$$c_q^k = 0, \quad k = 1, \ldots, 2m, \ q = 1, 2, \ldots$$

and if

$$\mathrm{Re}\, \{c_0^k\} = 0, \quad k = 1, \ldots, 2m, \tag{1.15}$$

then, by (1.2), (1.3), and (1.5),

$$\begin{aligned} \mathrm{Re}\, \{g[F_k(z)]\} = -\mathrm{Re}\, \{g(1/G_k(z)]\} &= \log |z|, \qquad k = 2p+1, \\ &= -\log |z|, \quad k = 2p+2, \end{aligned}$$

where $p = 0, 1, \ldots, (m-2)/2$. Hence we obtain (1.12) and (1.15).

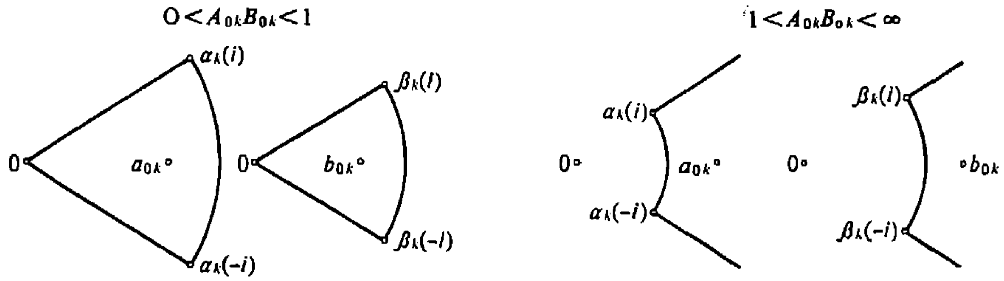
If conditions (1.15) were not fulfilled, then the images $F_k(\Delta)$ and $1/G_k(\Delta)$, $k = 1, \ldots, m$, together with their boundaries would not fill the whole plane C which is impossible in view of the obvious fact that $g'(w) \not\equiv 0$.

If $m = 2l+1$, $l = 1, 2, \ldots$, then $(\hat{F}, \hat{G}) \in C_{2m,2m}(\hat{a}_0, \hat{b}_0)$, where

$$\text{(1.16)} \qquad \begin{aligned} \hat{F}_k &= F_k^{1/2}, & k &= 1, \ldots, m, \\ &= -F_{k-m}^{1/2}, & k &= m+1, \ldots, 2m, \end{aligned}$$

$$\text{(1.16')} \qquad \begin{aligned} \hat{G}_k &= G_k^{1/2}, & k &= 1, \ldots, m, \\ &= -G_{k-m}^{1/2}, & k &= m+1, \ldots, 2m, \end{aligned}$$

and using the procedure described above we obtain (1.11)–(1.13).



$$\alpha_k(\tau) = \tau^{2/m} \left|\frac{a_{0k}}{b_{0k}}\right|^{1/2} \exp(i \arg a_{0k}), \quad \beta_k(\tau) = \tau^{2/m} \left|\frac{b_{0k}}{a_{0k}}\right|^{1/2} \exp(i \arg b_{0k})$$

Fig. 1

Remark 2. If the functions F satisfy condition (3), then from Theorem 3 one obtains Śladkowska's result for subclasses of the class $C_{2,2}(a_0, b_0)$ of pairs (F, F) (see [31]) and De Temple's result for subclasses of the classes $C_{m,m}(a_0, \bar{a}_0)$ of pairs $(F, \bar{F})$ (see [5], [6]).

Let $F \in C_{m,0}(a_0, \emptyset)$, $m \geqslant 2$. Moreover, if $m = 2l+1$, $l = 1, 2, \ldots$, we assume that $F_k(z) \neq 0$ for $z \in \Delta$ and $k = 1, \ldots, m$. Write

$$F_k^{m/2}(z) = A_{0k} + A_{1k} z + \ldots, \qquad k = 1, \ldots, m.$$

From Theorem 3 we obtain

THEOREM 4. *If $F \in C_{m,0}(a_0, \emptyset)$, $m \geqslant 2$, a_0 satisfies conditions* (1) *and for $z \in \Delta$ and $k = 1, \ldots, m$ we have $F_k(z) \neq 0$ whenever $m = 2l+1$, $l = 1, 2, \ldots$, then*

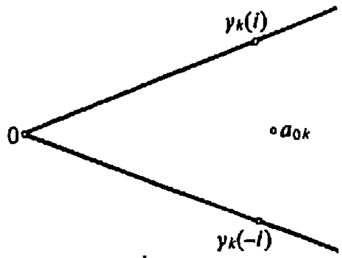
$$\prod_{k=1}^{m} |A_{1k}| \leqslant 2^m \prod_{k=1}^{m} |A_{0k}|.$$

The equality holds true only for functions F for which the functions F_k, $k = 1, \ldots, m$ (Fig. 2), satisfy the equations

$$F_k^{m/2} = A_{0k} \frac{1+\varepsilon z}{1-\varepsilon z}, \qquad |\varepsilon| = 1, \; k = 1, \ldots, m.$$

Remark 3. In the case $m = 3$, applying a homographic transformation to Theorem 4, one obtains Goluzin's result [9].

From Theorem 4 one can obtain Gelfer's result concerning subclasses of the class $C_{m,0}(a_0, \emptyset)$, $m \geq 2$, of functions F satisfying condition (3) (see [8], [22]).



$\gamma_k(\tau) = a_{0k}\tau^{2/m}$

Fig. 2

2. Estimations of third-order functionals. In the class S Bieberbach obtained the following estimation:

$$|a_3 - a_2^2| \leq 1.$$

Studies on homogeneous functionals of that type in the class of bounded functions have been originated by Jakubowski [16] and Zawadzki [34], while the first to investigate such functionals in the classes of Bieberbach–Eilenberg's functions and Grunsky–Shah's functions were Jenkins [17] and Śladkowska [30], [31].

In this section we shall present analogues of that estimation in the classes $C_{m,m}(a_0, b_0)$ and $C_{m,0}(a_0, \emptyset)$, where a_0, b_0 satisfy conditions (1), (2).

Let $(F, G) \in C_{m,m}(a_0, b_0)$, $m = 1$ or $m = 2l$, $l = 1, 2, \ldots$, and a_0, b_0 satisfy conditions (1), (2). Moreover, let

$$(2.1) \quad g(w) = -\frac{1}{b_{01}(w-a_{01})} + \frac{w}{a_{01}(1-wb_{01})}, \quad \text{when } m = 1,\ a_{01}b_{01} \neq 0,$$

$$= -\frac{1}{B_{01}}\left(\frac{1}{w^{m/2}-A_{01}} + \frac{1}{w^{m/2}+A_{01}}\right) + \frac{1}{A_{01}}\left(\frac{w^{m/2}}{1-w^{m/2}B_{01}} + \frac{w^{m/2}}{1+w^{m/2}B_{01}}\right),$$

when $m = 2l$, $l = 1, 2, \ldots$,

and

(2.2) $$\sum_{q=-\infty}^{\infty} c_q^k z^q = g\,[F_k(z)], \qquad k = 1, \ldots, m, \qquad |z| = r.$$
$$= g\,[1/G_{k-m}(z)], \qquad k = m+1, \ldots, 2m,$$

Hence, according to Green's formula,

$$0 > \tfrac{1}{2}\left\{ \sum_{k=1}^{m} \int_0^{2\pi} g\,[F_k(z)]\, \frac{d}{dz}\, g\,[F_k(z)]\, z d\varphi + \right.$$
$$\left. + \sum_{k=m+1}^{m+n} \int_0^{2\pi} g\,[1/G_{k-m}(z)]\, \frac{d}{dz}\, g\,[1/G_{k-m}(z)]\, z d\varphi \right\},$$

where $z = r \exp(i\varphi)$. Thus

$$\sum_{k=1}^{2m} \sum_{q=-\infty}^{\infty} q\,|c_q^k|^2 r^{2q} < 0$$

and consequently

(2.3) $$\sum_{k=1}^{2m} \sum_{q=-\infty}^{\infty} q\,|c_q^k|^2 \leqslant 0$$

as $r \to 1$.

We shall now represent inequality (2.3) in a bilinear form.

Putting $\sqrt{q}\, c^k = U_{qk}$, $-\sqrt{q}\, c_{-q}^k = V_{qk}$, $k = 1, \ldots, 2m$, $q = 1, 2, \ldots$ we have

$$\sum_{q=1}^{\infty} |U_q|^2 \leqslant \sum_{q=1}^{\infty} |V_q|^2.$$

Hence and from Schwarz' inequality it follows that

$$\operatorname{Re}\left\{ \sum_{q=1}^{\infty} U_q V_q \right\} \leqslant \sum_{q=1}^{\infty} |V_q|^2$$

and the equality holds if and only if

$$U_q = \bar{V}_q, \qquad q = 1, 2, \ldots$$

Thus we have proved

THEOREM 5. *If a_0, b_0 satisfy conditions* (1), (2), $(F, G) \in C_{1,1}(a_0, b_0)$, $a_{01} b_{01} \neq 0$, *or* $(F, G) \in C_{m,m}(a_0, b_0)$, $m = 2l$, $l = 1, 2, \ldots$, *then*

$$-\operatorname{Re}\left\{ \sum_{k=1}^{2m} \sum_{q=1}^{\infty} q c_q^k c_{-q}^k \right\} \leqslant \sum_{k=1}^{2m} \sum_{q=1}^{\infty} q\,|c_{-q}^k|^2$$

and the equality holds if and only if

$$c_q^k = -\bar{c}_{-q}^k, \quad k = 1, \ldots, m, \ q = 1, 2, \ldots,$$

where c_q^k, $k = 1, \ldots, 2m$, $q = \pm 1, 2, \ldots$, *are defined by* (2.2).

Before we pass to applications of Theorem 5 we shall present some simple properties of the classes under consideration which prove to be of special importance for investigations of homogeneous functionals.

Let $(F, G) \in C_{m,m}(a_0, b_0)$ or $F \in C_{m,0}(a_0, \emptyset)$. It can easily be verified that the following relations hold: $(F_\varepsilon, G_\tau) \in C_{m,m}(a_0, b_0)$, $F_\varepsilon \in C_{m,0}(a_0, \emptyset)$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, $\tau = (\tau_1, \ldots, \tau_m)$, $|\varepsilon_k| = |\tau_k| = 1$, $k = 1, \ldots, m$, $F_\varepsilon = (F_{1\varepsilon_1}, \ldots, F_{m\varepsilon_m})$, $G_\tau = (G_{1\tau_1}, \ldots, G_{m\tau_m})$, $F_{k\varepsilon_k}(z) = F_k(\varepsilon_k z)$, $G_{k\tau_k}(z) = G_k(\tau_k z)$, $k = 1, \ldots, m$.

We shall now prove the following

THEOREM 6. *If* $(F, G) \in C_{1,1}(a_0, b_0)$, $a_{01} b_{01} \neq 0$, *then*

$$\left|\frac{A}{(a_{11} b_{01})^2} + \frac{B}{(b_{11} a_{01})^2}\right| \leqslant \frac{1}{|a_{11} b_{01}|^2} + \frac{1}{|b_{11} a_{01}|^2}, \tag{2.4}$$

where

$$A = \frac{a_{31}}{a_{11}} - \frac{a_{21}^2}{a_{11}^2} + \frac{b_{01}}{a_{01}} \frac{a_{11}^2}{(1 - a_{01} b_{01})^2}, \quad B = \frac{b_{31}}{b_{11}} - \frac{b_{21}^2}{b_{01}^2} + \frac{a_{01}}{b_{01}} \frac{b_{11}^2}{(1 - a_{01} b_{01})^2}.$$

If $a_{01} b_{01} > 0$, *then the equality holds true only for the pairs* (F_ε, G_τ) *in which the functions* F_1 *and* G_1 (Fig. 3) *satisfy the equations*

$$\frac{1 + a_{01} b_{01} - \sqrt{a_{01} b_{01}} \left(\sqrt{\frac{b_{01}}{a_{01}}} F_1 + \sqrt{\frac{a_{01}}{b_{01}}} \frac{1}{F_1}\right)}{\frac{b_{01}}{a_{01}} F_1 - \frac{a_{01}}{b_{01}} \frac{1}{F_1}} = \frac{a_{11} b_{01}}{\sqrt{a_{01} b_{01}}} \frac{z}{(z - z_1)(z + \bar{z}_1)};$$

$$\frac{1 + a_{01} b_{01} - \sqrt{a_{01} b_{01}} \left(\sqrt{\frac{a_{01}}{b_{01}}} G_1 + \sqrt{\frac{b_{01}}{a_{01}}} \frac{1}{G_1}\right)}{\sqrt{\frac{a_{01}}{b_{01}}} G_1 - \sqrt{\frac{b_{01}}{a_{01}}} \frac{1}{G_1}} = \frac{b_{11} a_{01}}{\sqrt{a_{01} b_{01}}} \frac{z}{(z - z_2)(z + \bar{z}_2)},$$

respectively, and

$$\frac{a_{21}}{a_{11}}+\frac{a_{11}b_{01}}{1-a_{01}b_{01}} = -2\operatorname{Im}\{z_1\}i, \quad |z_1|=1, \quad \operatorname{Im}\{a_{11}b_{01}\}=0,$$

(2.5) $$|\operatorname{Im}\{z_1\}| \leqslant 1-\frac{1}{\sqrt{a_{01}b_{01}}}\left|\frac{a_{11}b_{01}}{1-a_{01}b_{01}}\right|;$$

$$\frac{b_{21}}{b_{11}}+\frac{b_{11}a_{01}}{1-a_{01}b_{01}} = -2\operatorname{Im}\{z_2\}i, \quad |z_2|=1, \quad \operatorname{Im}\{b_{11}a_{01}\}=0,$$

(2.6) $$|\operatorname{Im}\{z_2\}| \leqslant 1-\frac{1}{\sqrt{a_{01}b_{01}}}\left|\frac{b_{11}a_{01}}{1-a_{01}b_{01}}\right|.$$

Thus, for each a_{11}, b_{11}, the functions F_1, G_1 belong to one-parameter families, where the parameters are a_{21}, b_{21} or z_1, z_2, respectively.

Proof. Let $(F, G)\in C_{1,1}(a_0, b_0)$, $a_{01}b_{01}\neq 0$. In view of Theorem 5 we have

$$-\operatorname{Re}\left\{\sum_{k=1}^{2} c_1^k c_{-1}^k\right\} \leqslant \sum_{k=1}^{2} |c_{-1}^k|^2.$$

Since

$$c_{-1}^1 = -\frac{1}{a_{11}b_{01}}, \quad c_{-1}^2 = \frac{1}{b_{11}a_{01}}, \quad c_1^1 = \frac{A}{a_{11}b_{01}}, \quad c_1^2 = -\frac{B}{b_{11}a_{01}},$$

we have

(2.7) $$\operatorname{Re}\left\{\frac{A}{(a_{11}b_{01})^2}+\frac{B}{(b_{11}a_{01})^2}\right\} \leqslant \frac{1}{|a_{11}b_{01}|^2}+\frac{1}{|b_{11}a_{01}|^2}$$

and in case of equality

$$A = \frac{\overline{a_{11}b_{01}}}{a_{11}b_{01}}, \quad B = \frac{\overline{b_{11}a_{01}}}{b_{11}a_{01}}.$$

Let $a_{01}b_{01}>0$ and assume that in (2.7) equality holds. We can also assume that

$$A = B = 1,$$

which is equivalent to $\operatorname{Im}\{a_{11}b_{01}\} = \operatorname{Im}\{b_{11}a_{01}\} = 0$.

From (2.1) it follows that

$$g(w) = \frac{1}{\sqrt{a_{01}b_{01}}}\,\frac{\sqrt{\dfrac{b_{01}}{a_{01}}}\,w-\sqrt{\dfrac{a_{01}}{b_{01}}}\,\dfrac{1}{w}}{1+a_{01}b_{01}-\sqrt{a_{01}b_{01}}\left(\sqrt{\dfrac{b_{01}}{a_{01}}}\,w+\sqrt{\dfrac{a_{01}}{b_{01}}}\,\dfrac{1}{w}\right)}$$

and

$$c_0^1 = \frac{A_0}{a_{11} b_{01}}, \quad c_0^2 = -\frac{B_0}{b_{11} a_{01}},$$

where

$$A_0 = \frac{a_{21}}{a_{11}} + \frac{a_{11} b_{01}}{1 - a_{01} b_{01}}, \quad B_0 = \frac{b_{21}}{b_{11}} + \frac{b_{11} a_{01}}{1 - a_{01} b_{01}};$$

therefore

$$g[F_1] = \frac{1}{a_{11} b_{01}} \frac{Az^2 + A_0 z - 1}{z}, \quad g[1/G_1] = -\frac{1}{b_{11} a_{01}} \frac{Bz^2 + B_0 z - 1}{z}.$$

Consequently

$$(2.8) \quad \frac{1 + a_{01} b_{01} - \sqrt{a_{01} b_{01}} \left(\sqrt{\frac{b_{01}}{a_{01}}} F_1 + \sqrt{\frac{a_{01}}{b_{01}}} \frac{1}{F_1} \right)}{\sqrt{\frac{b_{01}}{a_{01}}} F_1 - \sqrt{\frac{a_{01}}{b_{01}}} \frac{1}{F_1}} = a_{11} \sqrt{\frac{b_{01}}{a_{01}}} \frac{z}{z^2 + A_0 z - 1},$$

$$(2.9) \quad \frac{1 + a_{01} b_{01} - \sqrt{a_{01} b_{01}} \left(\sqrt{\frac{a_{01}}{b_{01}}} G_1 + \sqrt{\frac{b_{01}}{a_{01}}} \frac{1}{G_1} \right)}{\sqrt{\frac{a_{01}}{b_{01}}} G_1 - \sqrt{\frac{b_{01}}{a_{01}}} \frac{1}{G_1}} = b_{11} \sqrt{\frac{a_{01}}{b_{01}}} \frac{z}{z^2 + B_0 z - 1}.$$

The points $\sqrt{a_{01}/b_{01}}$ and $\sqrt{b_{01}/a_{01}}$ being boundary, the left-hand sides of (2.8) and (2.9) are different from infinity for $z \in \Delta$, and thus we obtain

$$z^2 + A_0 z - 1 = (z - z_1)(z + \bar{z}_1), \quad z^2 + B_0 z - 1 = (z - z_2)(z + \bar{z}_2),$$

where $|z_k| = 1$, $k = 1, 2$, i.e.

$$A_0 = -(z_1 - \bar{z}_1) = -2 \operatorname{Im} \{z_1\} i, \quad B_0 = -(z_2 - \bar{z}_2) = -2 \operatorname{Im} \{z_2\} i.$$

The function

$$\zeta = z/(z - z_1)(z + \bar{z}_1)$$

transforms conformally Δ onto the set

$$C-\left(\left\{\zeta:\ -\infty < \operatorname{Im}\zeta \leqslant -\frac{1}{|z_1-i|^2},\ \operatorname{Re}\zeta = 0\right\}\cup\right.$$
$$\left.\cup\left\{\zeta:\ \frac{1}{|z_1+i|^2} \leqslant \operatorname{Im}\zeta < \infty,\ \operatorname{Re}\zeta = 0\right\}\right).$$

On the other hand, the function

$$\zeta = \frac{1+a_{01}b_{01}-\sqrt{a_{01}b_{01}}\,(w+1/w)}{w-1/w}$$

transforms conformally the set $\{w\colon |w| < 1\}$ whenever $0 < a_{01}b_{01} < 1$, and the set $\{w\colon |w| > 1\}$ whenever $1 < a_{01}b_{01} < \infty$, onto the set

$$C-\{\zeta\colon |\operatorname{Im}\zeta| \geqslant \tfrac{1}{2}|1-a_{01}b_{01}|,\ \operatorname{Re}\zeta = 0\}.$$

Thus

$$\max\left\{\frac{|a_{11}b_{01}|}{\sqrt{a_{01}b_{01}}}\frac{1}{|z_1+i|^2},\ \frac{|a_{11}b_{01}|}{\sqrt{a_{01}b_{01}}}\frac{1}{|z_1-i|^2}\right\} \leqslant \tfrac{1}{2}|1-a_{01}b_{01}|.$$

If we assume that

$$0 \leqslant \operatorname{Im}\{z_1\} \leqslant 1,$$

i.e.

$$2(1-\operatorname{Im}\{z_1\}) = |z_1-i|^2 \leqslant |z_1+1|^2,$$

we infer that

$$0 \leqslant \operatorname{Im}\{z_1\} \leqslant 1-\frac{1}{\sqrt{a_{01}b_{01}}}\left|\frac{a_{11}b_{01}}{1-a_{01}b_{01}}\right|.$$

On the other hand, if we assume that

$$-1 \leqslant \operatorname{Im}\{z_1\} \leqslant 0$$

then, by an analogous reasoning, we have

$$-\left(1-\frac{1}{\sqrt{a_{01}b_{01}}}\left|\frac{a_{11}b_{01}}{1-a_{01}b_{01}}\right|\right) \leqslant \operatorname{Im}\{z_1\} \leqslant 0.$$

Thus we have shown inequality (2.5).

Using a similar argument one can show inequality (2.6).

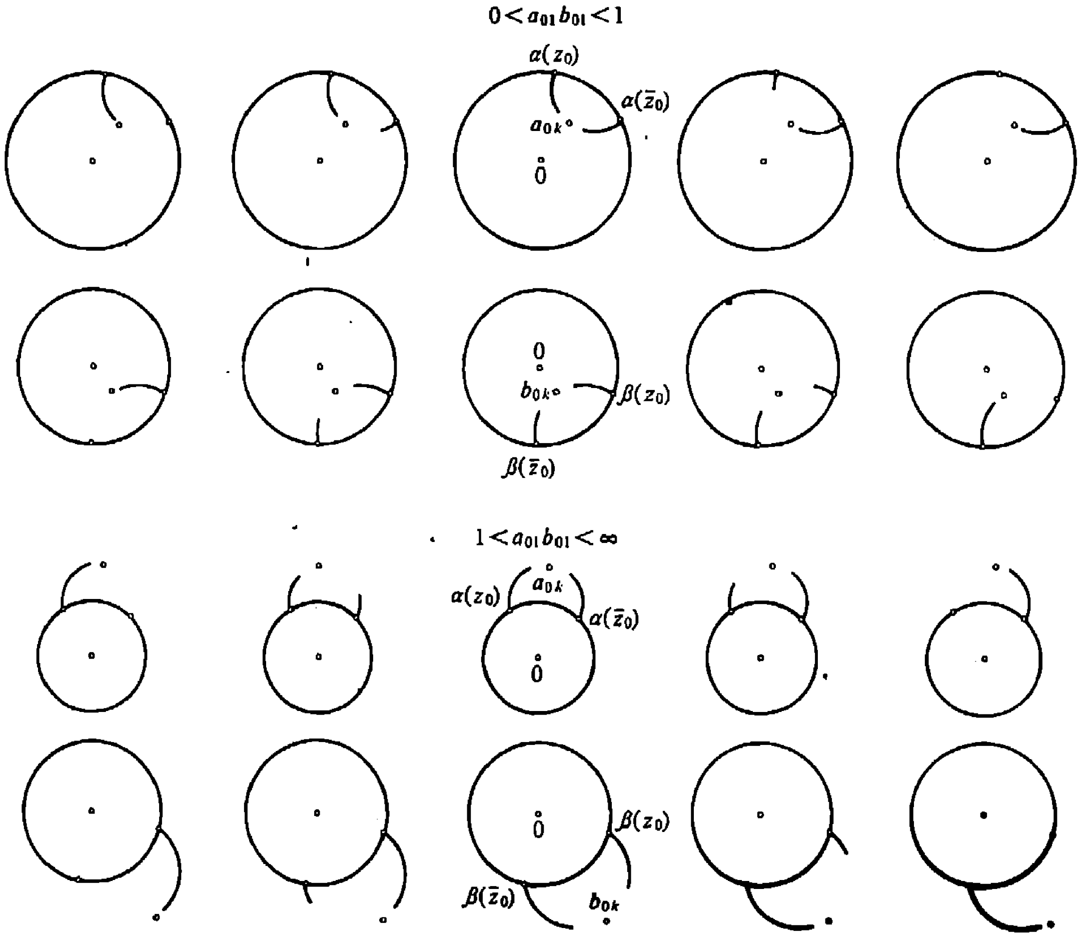
That concludes the proof of Theorem 6.

Remark 4. When $b_{01} = a_{01}$ and $a_{01} \to 0$, from Theorem 6 one directly obtains the corresponding result for the class A, and consequently the corresponding results for the classes of Bieberbach–Eilenberg's functions, of bounded functions, and of Grunsky–Shah's functions.

In the class of bounded functions, in case of $a_{11} = \bar{a}_{11}$, the result obtained coincides with Tammi's result [32].

The analogue of Theorem 6 concerning classes of extended bounded functions is due to Kortram [19].

A straightforward consequence of Theorem 6 is



$$\begin{array}{ccccc} \zeta = p & 0<\zeta<p & \zeta=\eta=0 & -p<\zeta<0 & \zeta=-p \\ \eta = q & 0<\eta<q & & -q<\eta<0 & \eta=-q \end{array}$$

$$\zeta = \operatorname{Im}\{z_1\}, \quad p = 1-\frac{1}{\sqrt{a_{01}b_{01}}}\left|\frac{a_{11}b_{01}}{1-a_{01}b_{01}}\right|, \quad q = 1-\frac{1}{\sqrt{a_{01}b_{01}}}\left|\frac{b_{11}a_{01}}{1-a_{01}b_{01}}\right|, \quad \eta = \operatorname{Im}\{z_2\}$$

$$\alpha(\tau) = \sqrt{\frac{a_{01}}{b_{01}}}\,\tau, \quad |z_0| = 1, \quad \operatorname{Re}\{z_0\} = \frac{2\sqrt{a_{01}b_{01}}}{1+a_{01}b_{01}}, \quad \beta(\tau) = \sqrt{\frac{b_{01}}{a_{01}}}\,\tau$$

Fig. 3

THEOREM 7. *If* $F \in C_{1,0}(a_0, \emptyset)$, *then*

$$\left|\frac{a_{31}}{a_{11}} - \frac{a_{21}^2}{a_{11}^2}\right| \leqslant 1.$$

The equality holds true only for functions F_ε *for which the functions* F_1 *satisfy the equations*

$$F_1 = a_{01} - a_{11}\frac{z}{(z-z_0)(z+\bar{z}_0)} \quad \text{and} \quad \frac{a_{21}}{a_{11}} = -2\,\mathrm{Im}\,\{z_0\}\,i, \quad |z_0| = 1.$$

Thus, for each a_{11}, *the functions* F_1 *belong to a one-parameter family, where the parameter is* a_{21} *or* z_0.

Remark 5. As an immediate consequence of Theorem 7 one gets Bieberbach's estimation.

We shall now prove a theorem which is a counterpart of Theorem 6.

THEOREM 8. *If* $(F, G) \in C_{m,m}(a_0, b_0)$, $m \geqslant 1$, a_0, b_0 *satisfy conditions* (1), (2) *and* $F_k(z) \neq 0$, $G_k(z) \neq 0$ *for* $z \in \Delta$ *and* $k = 1, \ldots, m$, *then*

$$\left|\sum_{k=1}^{m}\left\{\frac{A_k}{(A_{1k}B_{0k})^2} + \frac{B_k}{(B_{1k}A_{0k})^2}\right\}\right| \leqslant \sum_{k=1}^{m}\left\{\frac{1}{|A_{1k}B_{0k}|^2} + \frac{1}{|B_{1k}A_{0k}|^2}\right\},$$

where

$$A_k = \frac{A_{3k}}{A_{1k}} - \frac{A_{2k}^2}{A_{1k}^2} + \frac{B_{0k}}{A_{0k}}\frac{1+A_{0k}^2B_{0k}^2}{(1-A_{0k}^2B_{0k}^2)^2}A_{1k}^2 + \frac{A_{1k}^2}{4A_{0k}^2}, \quad k = 1, \ldots, m,$$

$$B_k = \frac{B_{3k}}{B_{1k}} - \frac{B_{2k}^2}{B_{1k}^2} + \frac{A_{0k}}{B_{0k}}\frac{1+A_{0k}^2B_{0k}^2}{(1-A_{0k}^2B_{0k}^2)^2}B_{1k}^2 + \frac{B_{1k}^2}{4B_{0k}^2}, \quad k = 1, \ldots, m.$$

When $A_{01}B_{01} > 0$, *the equality holds true only for pairs* (F_ε, G_τ) *in which the functions* F_k, G_k, $k = 1, \ldots, m$ (Fig. 4), *satisfy the equations*

$$\frac{1 + A_{0k}^2B_{0k}^2 - A_{0k}B_{0k}\left(\dfrac{B_{0k}}{A_{0k}}F_k^m + \dfrac{A_{0k}}{B_{0k}}\right)\dfrac{1}{F_k^m}}{\sqrt{\dfrac{B_{0k}}{A_{0k}}}F_k^{m/2} - \sqrt{\dfrac{A_{0k}}{B_{0k}}}\dfrac{1}{F_k^{m/2}}} = 2\frac{A_{1k}B_{0k}}{\sqrt{A_{0k}B_{0k}}}(1+A_{0k}B_{0k})\frac{z}{(z-z_{1k})(z+\bar{z}_{1k})}, \quad k = 1, \ldots, m;$$

$$\frac{1 + A_{0k}^2B_{0k}^2 - A_{0k}B_{0k}\left(\dfrac{A_{0k}}{B_{0k}}G_k^m + \dfrac{B_{0k}}{A_{0k}}\right)\dfrac{1}{G_k^m}}{\sqrt{\dfrac{A_{0k}}{B_{0k}}}G_k^{m/2} - \sqrt{\dfrac{B_{0k}}{A_{0k}}}\dfrac{1}{G_k^{m/2}}} = 2\frac{B_{1k}A_{0k}}{\sqrt{A_{0k}B_{0k}}}(1+A_{0k}B_{0k})\frac{z}{(z-z_{2k})(z+\bar{z}_{2k})}$$

respectively, and

$$\frac{A_{2k}}{A_{1k}}+\frac{A_{0k}^2B_{0k}^2+4A_{0k}B_{0k}-1}{2A_{0k}B_{0k}(1-A_{0k}^2B_{0k}^2)}A_{1k}B_{0k}=-2\operatorname{Im}\{z_{1k}\}i,\quad \operatorname{Im}\{A_{1k}B_{0k}\}=0,$$

$$(2.10)\qquad |z_{1k}|=1,\quad |\operatorname{Im}\{z_{1k}\}|\leqslant 1-\frac{1}{2}\frac{|A_{1k}B_{0k}|}{A_{0k}B_{0k}}\frac{1+A_{0k}B_{0k}}{|1-A_{0k}B_{0k}|},\qquad k=1,\ldots,m;$$

$$\frac{B_{2k}}{B_{1k}}+\frac{A_{0k}^2B_{0k}^2+4A_{0k}B_{0k}-1}{2A_{0k}B_{0k}(1-A_{0k}^2B_{0k}^2)}B_{1k}A_{0k}=-2\operatorname{Im}\{z_{2k}\}i,\quad \operatorname{Im}\{B_{1k}A_{0k}\}=0,$$

$$(2.11)\qquad |z_{2k}|=1,\quad |\operatorname{Im}\{z_{2k}\}|\leqslant 1-\frac{1}{2}\frac{|B_{1k}A_{0k}|}{A_{0k}B_{0k}}\frac{1+A_{0k}B_{0k}}{|1-A_{0k}B_{0k}|},\qquad k=1,\ldots,m.$$

Thus, for each a_{1k}, b_{1k}, *the functions* F_k, G_k *belong to one-parameter families, where the parameters are* a_{2k}, b_{2k} *or* z_{1k}, z_{2k}, $k=1,\ldots,m$, *respectively.*

Proof. In accordance with (2.1),

$$g(w)=\frac{2(1+A_{0k}B_{0k})}{\sqrt{A_{0k}B_{0k}}}\frac{\sqrt{B_{0k}/A_{0k}}\,w^{m/2}-\sqrt{A_{0k}/B_{0k}}/w^{m/2}}{1+A_{0k}^2B_{0k}^2-A_{0k}B_{0k}[(B_{0k}/A_{0k})w^m+(A_{0k}/B_{0k})/w^m]}$$

for any $k=1,\ldots,m$. This means that

$$c_{-1}^k=-\frac{1}{A_{1k}B_{0k}},\quad c_{-1}^{k+m}=\frac{1}{B_{1k}A_{0k}},\quad c_1^k\,\frac{A_k}{A_{1k}B_{0k}},\quad c_0^{k+m}=-\frac{B_k}{B_{1k}A_{0k}},$$
$$k=1,\ldots,m,$$

and

$$c_0^k=\frac{A_k^0}{A_{1k}B_{0k}},\quad c_0^{k+m}=-\frac{B_k^0}{B_{1k}A_{0k}},\quad k=1,\ldots,m,$$

where

$$A_k^0=\frac{A_{2k}}{A_{1k}}+\frac{A_{0k}^2B_{0k}^2+4A_{0k}B_{0k}-1}{2A_{0k}B_{0k}(1-A_{0k}^2B_{0k}^2)}A_{1k}B_{0k},\quad k=1,\ldots,m,$$
$$B_k^0=\frac{B_{2k}}{B_{1k}}+\frac{A_{0k}^2B_{0k}^2+4A_{0k}B_{0k}-1}{2A_{0k}B_{0k}(1-A_{0k}^2B_{0k}^2)}B_{1k}A_{0k},\quad k=1,\ldots,m.$$

On applying Theorem 5 we get

$$-\operatorname{Re}\Big\{\sum_{k=1}^{2m}c_1^kc_{-1}^k\Big\}\leqslant\sum_{k=1}^{2m}|c_{-1}^k|^2$$

and the equality holds true if and only if

$$c_1^k = -\bar{c}_{-1}^k, \quad k = 1, \ldots, 2m,$$
$$c_q^k = 0, \quad k = 1, \ldots, 2m, \; q = 2, 3, \ldots.$$

Consequently, we have

$$\operatorname{Re}\left\{\sum_{k=1}^{m}\left[\frac{A_k}{(A_{1k}B_{0k})^2}+\frac{B_k}{(B_{1k}A_{0k})^2}\right]\right\} \leqslant \sum_{k=1}^{m}\left[\frac{1}{|A_{1k}B_{0k}|^2}+\frac{1}{|B_{1k}A_{0k}|^2}\right], \tag{2.12}$$

and if in (2.12) equality holds, then

$$A_k = \frac{A_{1k}B_{0k}}{\bar{A}_{1k}\bar{B}_{0k}}, \quad B_k = \frac{B_{1k}A_{0k}}{\bar{B}_{1k}\bar{A}_{0k}}, \quad k = 1, \ldots, m.$$

Let $A_{01}B_{01} > 0$ (which means that also $A_{0k}B_{0k} > 0$, $k = 1, \ldots, m$) and assume that in (2.12) equality holds. We can also assume that, possibly after a suitable rotation, we have

$$A_k = B_k = 1, \quad k = 1, \ldots, m,$$

i.e.

$$\operatorname{Im}\{A_{1k}B_{0k}\} = \operatorname{Im}\{B_{1k}A_{0k}\} = 0, \quad k = 1, \ldots, m.$$

One can easily see that

$$\frac{1+A_{0k}^2B_{0k}^2-A_{0k}B_{0k}\left(\dfrac{B_{0k}}{A_{0k}}F_k^m+\dfrac{A_{0k}}{B_{0k}}\dfrac{1}{F_k^m}\right)}{\sqrt{\dfrac{B_{0k}}{A_{0k}}}F_k^{m/2}-\sqrt{\dfrac{A_{0k}}{B_{0k}}}\dfrac{1}{F_k^{m/2}}} \tag{2.13}$$
$$= 2\frac{A_{1k}B_{0k}}{\sqrt{A_{0k}B_{0k}}}(1+A_{0k}B_{0k})\frac{z}{z^2+A_k^0 z-1}, \quad k = 1, \ldots, m,$$

and

$$\frac{1+A_{0k}^2B_{0k}^2-A_{0k}B_{0k}\left(\dfrac{A_{0k}}{B_{0k}}G_k^m+\dfrac{B_{0k}}{A_{0k}}\dfrac{1}{G_k^m}\right)}{\sqrt{\dfrac{A_{0k}}{B_{0k}}}G_k^{m/2}-\sqrt{\dfrac{B_{0k}}{A_{0k}}}\dfrac{1}{G_k^{m/2}}} \tag{2.14}$$
$$= 2\frac{B_{1k}A_{0k}}{\sqrt{A_{0k}B_{0k}}}(1+A_{0k}B_{0k})\frac{z}{z^2+B_k^0 z-1}, \quad k = 1, \ldots, m.$$

It is also worth noticing that

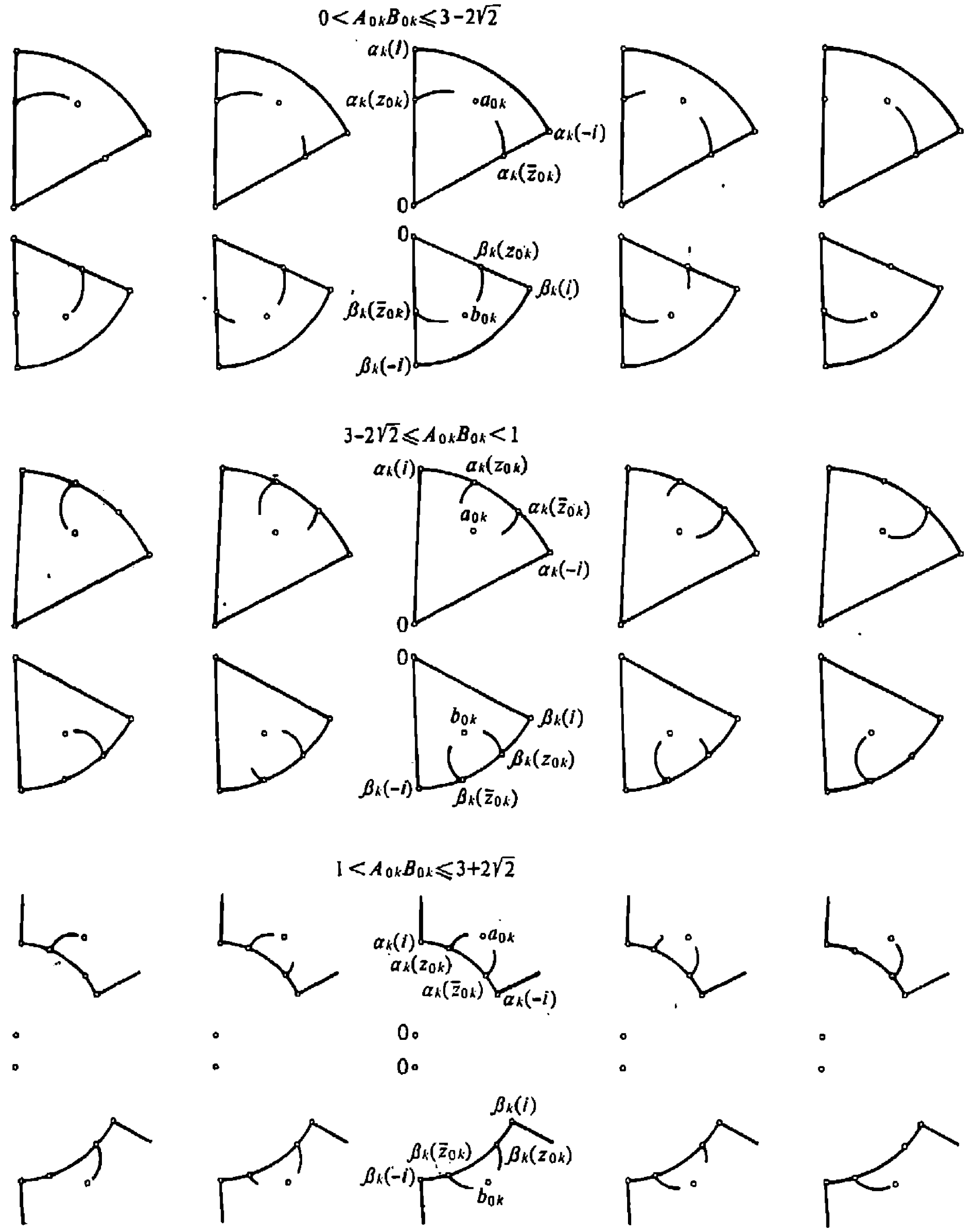
$$z^2+A_k^0 z-1 = (z-z_{1k})(z+\bar{z}_{1k}), \quad z^2+B_k^0 z-1 = (z-z_{2k})(z+\bar{z}_{2k}),$$

where $|z_{nk}| = 1$, $n = 1, 2$, and $k = 1, \ldots, m$. Indeed, we have assumed that

$F_k(z) \neq 0$, $G_k(z) \neq 0$ for $z \in \Delta$ and $k = 1, \ldots, m$. Also, from the definition of the classes $C_{m,m}(a_0, b_0)$ under consideration it follows that the points $\sqrt{a_{0k}/b_{0k}}$ and $\sqrt{b_{0k}/a_{0k}}$, $k = 1, \ldots, m$, are boundary. Consequently, the left-hand sides of (2.13) and (2.14) are different from infinity.

Thus

$$A_k^0 = -(z_{1k} - \bar{z}_{1k}) = -2 \operatorname{Im} \{z_{1k}\} i, \quad B_k^0 = -(z_{2k} - \bar{z}_{2k}) = -2 \operatorname{Im} \{z_{2k}\} i,$$
$$k = 1, \ldots, m.$$



Since the function

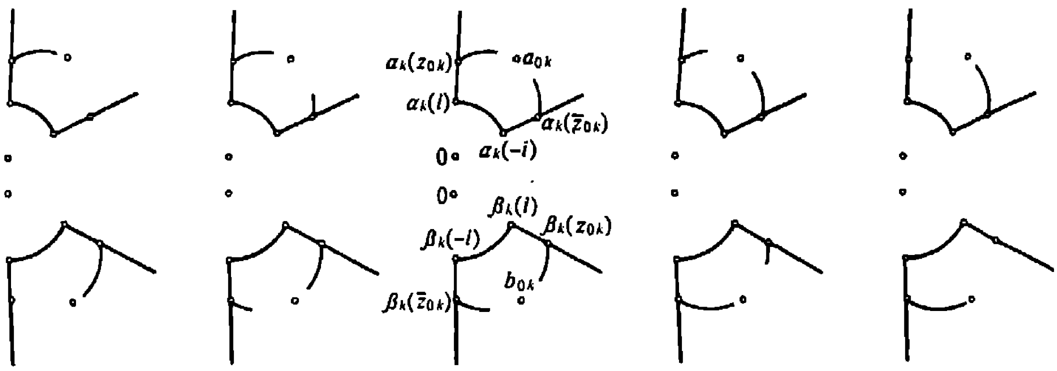
$$\zeta_k = \frac{1+A_{0k}^2 B_{0k}^2 - A_{0k} B_{0k}(w^m + 1/w^m)}{w^{m/2} - 1/w^{m/2}}$$

transforms conformally the set $\{w\colon\ |w| < 1,\ \pi(2k-3)/m < \arg w < \pi(2k-1)/m\}$, whenever $0 < A_{0k} B_{0k} < 1$, or the set $\{w\colon\ |w| > 1,\ \pi(2k-3)/m < \arg w < \pi(2k-1)/m\}$, whenever $1 < A_{0k} B_{0k} < \infty$, onto the set

$$C - \{\zeta\colon\ |\mathrm{Im}\,\zeta| \geqslant 2\sqrt{A_{0k} B_{0k}}\,|1 - A_{0k} B_{0k}|,\ \mathrm{Re}\,\zeta = 0\}, \qquad k = 1, \ldots, m,$$

it follows, on applying an argument similar to that used in the proof of

$3+2\sqrt{2} \leqslant A_{0k} B_{0k} < \infty$



$$\begin{array}{lllll} \zeta_k = p_k & 0 < \zeta_k < p_k & \zeta_k = \eta_k = 0 & -p_k < \zeta_k < 0 & \zeta_k = -p_k \\ \eta_k = q_k & 0 < \eta_k < q_k & & -q_k < \eta_k < 0 & \eta_k = -q_k \end{array}$$

$$p_k = 1 - \frac{1}{2}\frac{|A_{1k} B_{0k}|}{A_{0k} B_{0k}} \cdot \frac{1 + A_{0k} B_{0k}}{|1 - A_{0k} B_{0k}|}, \qquad q_k = 1 - \frac{1}{2}\frac{|B_{1k} A_{0k}|}{A_{0k} B_{0k}} \frac{1 + A_{0k} B_{0k}}{|1 - A_{0k} B_{0k}|}$$

$$z_{0k} = \sqrt{\frac{A_{0k}^2 B_{0k}^2 - 4A_{0k} B_{0k} + 1 - |1 - A_{0k} B_{0k}|\sqrt{A_{0k}^2 B_{0k}^2 - 6A_{0k} B_{0k} + 1}}{2A_{0k} B_{0k}}}\, i,$$

$$A_{0k} B_{0k} \in (0;\ 3 - 2\sqrt{2}]$$

$$|z_{0k}| = 1,\ \mathrm{Im}\,\{z_{0k}\} = \frac{1}{2}\frac{|1 - A_{0k} B_{0k}|}{\sqrt{A_{0k} B_{0k}}}, \quad A_{0k} B_{0k} \in [3 - 2\sqrt{2};\ 1) \cup (1;\ 3 + 2\sqrt{2}]$$

$$\dot{z}_{0k} = \sqrt{\frac{A_{0k}^2 B_{0k}^2 - 4A_{0k} B_{0k} + 1 + |1 - A_{0k} B_{0k}|\sqrt{A_{0k}^2 B_{0k} - 6A_{0k} B_{0k} + 1}}{2A_{0k} B_{0k}}}\, i$$

$$A_{0k} B_{0k} \in [3 + 2\sqrt{2};\ \infty)$$

$$\alpha_k(\tau) = \tau^{2/m}\left|\frac{a_{0k}}{b_{0k}}\right|^{1/2} \exp(i \arg a_{0k}), \qquad \beta_k(\tau) = \tau^{2/m}\left|\frac{b_{0k}}{a_{0k}}\right|^{1/2} \exp(i \arg b_{0k})$$

$\zeta_k = \mathrm{Im}\,\{z_{1k}\}$ $\qquad\qquad$ $\eta_k = \mathrm{Im}\,\{z_{2k}\}$

Fig. 4

Theorem 6, that

$$\max\left\{2\frac{|A_{1k}B_{0k}|}{\sqrt{A_{0k}B_{0k}}}\frac{1+A_{0k}B_{0k}}{|z_{1k}+i|^2},\ 2\frac{|A_{1k}B_{0k}|}{\sqrt{A_{0k}B_{0k}}}\frac{1+A_{0k}B_{0k}}{|z_{1k}-i|^2}\right\}$$
$$\leqslant 2\sqrt{A_{0k}B_{0k}}\,|1-A_{0k}B_{0k}|, \qquad k=1,\ldots,m.$$

Hence (2.10) follows.

Exactly in the same way one can prove (2.11).

Remark 6. For $G=F$ or $G=\bar{F}$, from Theorem 6 and 8 one obtains the corresponding results for the classes of generalized Bieberbach–Eilenberg's functions and of generalized bounded functions.

As a simple corollary of Theorem 8 we have

THEOREM 9. *If $F\in C_{m,0}(a_0,\emptyset)$, $m\geqslant 2$, a_0 satisfies condition* (1), *and $F_k(z)\neq 0$ for $z\in\Delta$ and $k=1,\ldots,m$, then*

$$\left|\sum_{k=1}^{m}\frac{A_k}{A_{1k}^2}\right|\leqslant\sum_{k=1}^{m}\frac{1}{|A_{1k}|^2}, \tag{2.15}$$

where

$$A_k=\frac{A_{3k}}{A_{1k}}-\frac{A_{2k}^2}{A_{1k}^2}+\frac{A_{1k}^2}{4A_{0k}^2}, \qquad k=1,\ldots,m.$$

The equality holds true only for functions F_ε for which the functions F_k, $k=1,\ldots,m$ (Fig. 5), *satisfy the equations*

$$\frac{A_{0k}}{F_k^{m/2}}-\frac{F_k^{m/2}}{A_{0k}}=2\frac{A_{1k}}{A_{0k}}\frac{z}{(z-z_k)(z+\bar{z}_k)}, \qquad k=1,\ldots,m,$$

and

$$\frac{A_{2k}}{A_{1k}}-\frac{A_{1k}}{2A_{0k}}=-2\,\mathrm{Im}\,\{z_k\}\,i, \quad \mathrm{Im}\left\{\frac{A_{1k}}{A_{0k}}\right\}=0,$$

$$|z_k|=1, \quad |\mathrm{Im}\,\{z_k\}|\leqslant 1-\frac{1}{2}\left|\frac{A_{1k}}{A_{0k}}\right|, \qquad k=1,\ldots,m.$$

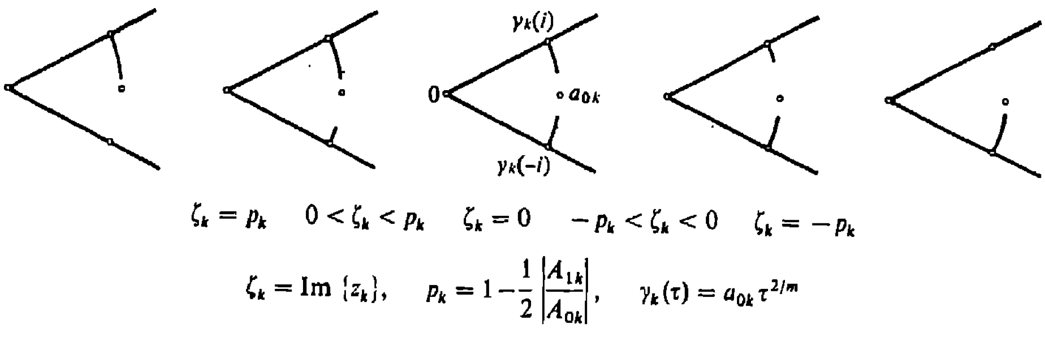


Fig. 5

Thus, for each a_{1k}, the functions F_k belong to a one-parameter family, where the parameter is a_{2k} or z_k, $k = 1, \ldots, m$.

Remark 7. Making use of Lebedev and Mamaj's inequality [23], Grinšpan and Kolomojceva [10] obtained the counterpart of inequality (2.15) in subclasses of the classes $C_{2,0}(a_0, \emptyset)$ of functions F satisfying condition (3).

3. Estimations of derivatives and Schwarz' derivatives. Let $(F, G) \in C_{m,m}(a_0, b_0)$ and let the functions F, G satisfy conditions (3), (4).

We define

$$\sum_{q,p=0}^{\infty} \alpha_{qp} z^q \zeta^p = \log \frac{F_1(z) - F_1(\zeta)}{(z-\zeta)\,[1 - F_1(z) G_1(\zeta)]}, \quad \text{when } m = 1,$$

$$= \log \frac{[F_1^{m/2}(z) - F_1^{m/2}(\zeta)]\,[1 + F_1^{m/2}(z) G_1^{m/2}(\zeta)]}{(z-\zeta)\,[F_1^{m/2}(z) + F_1^{m/2}(\zeta)]\,[1 - F_1^{m/2}(z) G_1^{m/2}(\zeta)]}, \quad \text{when } m \geqslant 2, \tag{3.1}$$

$$\sum_{q,p=0}^{\infty} \beta_{qp} z^q \zeta^p = \log \frac{G_1(z) - G_1(\zeta)}{(z-\zeta)\,[1 - G_1(z) F_1(\zeta)]}, \quad \text{when } m = 1,$$

$$= \log \frac{[G_1^{m/2}(z) - G_1^{m/2}(\zeta)]\,[1 + G_1^{m/2}(z) F_1^{m/2}(\zeta)]}{(z-\zeta)\,[G_1^{m/2}(z) + G_1^{m/2}(\zeta)]\,[1 - G_1^{m/2}(z) F_1^{m/2}(\zeta)}, \quad \text{when } m \geqslant 2.$$

For any ϱ, $r_0 < \varrho < r < 1$, for any real number x_0, and for any complex numbers $x_1, \ldots, x_N$, let

$$g(w) = x_0 \log \frac{w - a_{01}}{1 - w b_{01}} + P_1(w), \qquad m = 1,$$

$$= x_0 \log \frac{w^{m/2} - A_{01}}{w^{m/2} + A_{01}} \frac{1 + w^{m/2} B_{01}}{1 - w^{m/2} B_{01}} + P_m(w), \qquad m \geqslant 2, \tag{3.2}$$

where

$$P_1(w) = \frac{1}{2\pi i} \sum_{p=1}^{N} \int_{|\zeta| = \varrho} \frac{x_p}{\zeta^{p+1}} \log \frac{w - F_1(\zeta)}{w - a_{01}} \frac{1 - w b_{01}}{1 - w G_1(\zeta)}\, d\zeta, \tag{3.3}$$

$$P_m(w)$$
$$= \frac{1}{2\pi i} \sum_{p=1}^{N} \int_{|\zeta| = \varrho} \frac{x_p}{\zeta^{p+1}} \log \frac{(w^{\frac{m}{2}} - F_1^{\frac{m}{2}}(\zeta))(w^{\frac{m}{2}} + A_{01})(1 + w^{\frac{m}{2}} G_1^{\frac{m}{2}}(\zeta))(1 - w^{\frac{m}{2}} B_{01})}{(w^{\frac{m}{2}} - A_{01})(w^{\frac{m}{2}} + F_1^{\frac{m}{2}}(\zeta))(1 + w^{\frac{m}{2}} B_{01})(1 - w^{\frac{m}{2}} G_1^{\frac{m}{2}}(\zeta))}\, d\zeta$$
$$m \geqslant 2.$$

Then

$$(3.4)\qquad \begin{aligned} x_0 \log z + \sum_{q=-\infty}^{\infty} c_q^1 z^q &= g[F_1(z)], \quad m = 1,\\ &= g[F_1^{m/2}(z)], \quad m \geqslant 2,\\ -x_0 \log z - \sum_{q=-\infty}^{\infty} c_q^2 z^q &= g[1/G_1(z)], \quad m = 1,\\ &= g[1/G_1^{m/2}(z)], \quad m \geqslant 2 \qquad (|z| = r) \end{aligned}$$

and

$$(3.5)\qquad \begin{aligned} c_q^1 &= \sum_{p=0}^{N} \alpha_{qp} x_p, \quad c_q^2 = \sum_{p=0}^{N} \beta_{qp} x_p, \quad q = 0, 1, \ldots,\\ c_{-q}^1 &= c_{-q}^2 = -x_q/q, \quad q = 1, \ldots, N,\\ &= 0, \qquad q = N+1, N+2, \ldots \end{aligned}$$

Thus D_r being defined as before, by the same argument as that used in the proof of Theorem 1, we get

$$(3.6)\qquad \sum_{k=1}^{2} \{2x_0 \operatorname{Re}\{c_0^k\} + \sum_{q=-\infty}^{\infty} q|c_{-q}^k|^2\} \leqslant 0.$$

Consequently, putting

$$U_0 = x_0 \sum_{k=1}^{2} \operatorname{Re}\{c_0^k\}, \quad U_{kq} = \sqrt{q} c_q^k, \quad V_{kq} = -\sqrt{q} c_{-q}^k,$$

$$k = 1, 2, \ q = 1, 2, \ldots$$

and applying a bilinearization method, we obtain

$$U_0 + \operatorname{Re}\{\sum_{k=1}^{\infty} U_k V_k\} \leqslant U_0 + [(\sum_{k=1}^{\infty} |V_k|^2 - 2U_0) \sum_{k=1}^{\infty} |V_k|^2]^{1/2} \leqslant \sum_{k=1}^{\infty} |V_k|^2$$

and the equality holds true if and only if $U_0 = 0$, $U_k = \bar{V}_k$, $k = 1, \ldots, m$.

Thus we have proved

THEOREM 10. *If* $(F, G) \in C_{m,m}(a_0, b_0)$, *and the functions* F, G *satisfy conditions* (3), (4), *then*

$$(3.7)\qquad \operatorname{Re}\{\sum_{k=1}^{2} [x_0 \operatorname{Re}\{c_0^k\} - \sum_{q=1}^{\infty} q c_q^k c_{-q}^k]\} \leqslant \sum_{k=1}^{2} \sum_{q=1}^{\infty} q|c_{-q}^k|^2.$$

The equality holds true if and only if

$$(3.8)\qquad x_0 \sum_{k=1}^{2} \operatorname{Re}\{c_0^k\} = 0, \quad c_q^k = -\bar{c}_{-q}^k, \quad k = 1, 2, \ q = 1, 2, \ldots,$$

where c_q^k, $k = 1, 2$, $q = 0, \pm 1, \pm 2, \ldots$, *are defined in* (3.1)–(3.5).

Theorem 1 can be generalized as follows:

THEOREM 11. *If* $(F, G) \in C_{1,1}(a_0, b_0)$, *then*

$$|F_1'(z)G_1'(z)| \leqslant \left(\frac{|1-F_1(z)G_1(z)|}{1-|z|^2}\right)^2 \tag{3.9}$$

and the equality at a point $\zeta \in \Delta$ *holds true only for pairs* (F, G) *in which the functions* F_1, G_1 *satisfy the equations*

$$\frac{F_1-F_1(\zeta)}{1-F_1G_1(\zeta)} = \alpha\frac{z-\zeta}{1-\bar{\zeta}z}, \quad \frac{G_1-G_1(\zeta)}{1-G_1F_1(\zeta)} = \beta\frac{z-\zeta}{1-\bar{\zeta}z}, \quad |\alpha\beta| = 1. \tag{3.10}$$

Proof. Putting in (3.7) $x_q = z^q$, $z \in \Delta$, $q = 0, 1, \ldots$, and letting $N \to \infty$, we see that

$$\operatorname{Re}\Big\{\sum_{q,p=0}^{\infty}(\alpha_{qp}+\beta_{qp})z^{q+p}\Big\} \leqslant -\log(1-|z|^2)^2,$$

and hence we immediately obtain (3.9).

Putting in (3.8) $x_q = \zeta^q$, $\zeta \in \Delta$, $q = 0, 1, \ldots$, and letting $N \to \infty$, we infer that

$$\operatorname{Re}\Big\{\sum_{p=0}^{\infty}(\alpha_{0p}+\beta_{0p})\zeta^p\Big\} = 0,$$

$$\sum_{q=1,p=0}^{\infty}\alpha_{qp}z^q\zeta^p = \sum_{q=1,p=0}^{\infty}\beta_{qp}z^q\zeta^p = -\log(1-\bar{\zeta}z),$$

and hence we get (3.10).

Analogously we can prove

THEOREM 12. *If* $(F, G) \in C_{m,m}(a_0, b_0)$, $m \geqslant 2$, *the functions* F, G *satisfy conditions* (3), (4), *then*

$$|F_1'(z)G_1'(z)| \leqslant \frac{|F_1(z)G_1(z)|}{(1-|z|^2)^2}\left|\frac{4}{m}\frac{1-F_1^{m/2}(z)G_1^{m/2}(z)}{1+F_1^{m/2}(z)G_1^{m/2}(z)}\right|^2.$$

For $G = F$ *and* $A_{01} > 0$, *the equality at a point* $\zeta \in \Delta$ *holds true if*

$$\frac{1}{F_1^{m/2}} - F_1^{m/2} = \left(\frac{1}{A_{01}} - A_{01}\right)\frac{1-\varepsilon\zeta}{1+\varepsilon\zeta}\,\frac{1-\varepsilon\eta}{1+\varepsilon\eta}, \quad \eta = \frac{z-\zeta}{1-\bar{\zeta}z},$$

where $\operatorname{Im}\{\varepsilon\zeta\} = 0$, $|\varepsilon| = 1$.

Remark 8. If $m = 2$ and $G = F$, then from Theorem 12 one obtains Śladkowska's result [31], and for $m = 2$ and $G = \bar{F}$ De Temple's result [5] follows.

Denote by $\hat{c}_q^k$ the quantities defined in (3.5), where $x_0, x_1, \ldots, x_N$ are replaced by $y_0 = x_0, y_1, \ldots, y_N$, respectively. Thus, by (3.6)

$$\sum_{k=1}^{2}[2y_0 \operatorname{Re}\{\hat{c}_0^k\} + \sum_{q=-\infty}^{\infty} q|\hat{c}_q^k|^2] \leqslant 0. \tag{3.11}$$

Adding the corresponding sides of inequalities (3.6) and (3.11) and applying a bilinearization method to the inequality obtained, we get

$$\operatorname{Re}\{\sum_{k=1}^{2}[y_0\hat{c}_0^k + x_0 c_0^k - \sum_{q=1}^{\infty} q(c_q^k \hat{c}_{-q}^k + \hat{c}_q^k c_{-q}^k)]\} \leqslant \sum_{k=1}^{2}\sum_{q=1}^{\infty} q(|\hat{c}_{-q}^k|^2 + |c_{-q}^k|^2). \tag{3.12}$$

Schwarz' derivative $\{h; z\}$ of a function h is defined as follows:

$$\{h; z\} = \left(\frac{h''(z)}{h'(z)}\right)' - \frac{1}{2}\left(\frac{h''(z)}{h'(z)}\right)^2.$$

Put $x_q = qz^{q-1}$, $y_q = q\zeta^{q-1}$, $q = 1, 2, \ldots$, and let $N \to \infty$ and $\zeta \to z$. Then from (3.12) the following two theorems follow:

THEOREM 13. *If* $(F, G) \in C_{q,1}(a_0, b_0)$, *then*

$$\left|\{F_1; z\} + \{G_1; z\} + \frac{12F_1'(z)G_1'(z)}{[1 - F_1(z)G_1(z)]^2}\right| \leqslant \frac{12}{(1-|z|^2)^2}.$$

THEOREM 14. *If* $(F, G) \in C_{m,m}(a_0, b_0)$, $m \geqslant 2$, *and the functions* F_k, G_k, $k = 1, \ldots, m$, *satisfy conditions* (3), (4), *then*

$$\left|\{F_1^{m/2}; z\} + \{G_1^{m/2}; z\} + 12F_1^{m/2\prime}(z)G_1^{m/2\prime}(z)\frac{1 + F_1^{m/2}(z)G_1^{m/2}(z)}{[1 - F_1^m(z)G_1^m(z)]^2} + \frac{3}{2}\left[\frac{F_1^{m/2\prime}(z)}{F_1^{m/2}(z)}\right]^2 + \frac{3}{2}\left[\frac{G_1^{m/2\prime}(z)}{G_1^{m/2}(z)}\right]^2\right| \leqslant \frac{12}{(1-|z|^2)^2}. \tag{3.13}$$

Remark 9. From Theorem 13 one easily obtains the corresponding results concerning the classes of Bieberbach–Eilenberg's functions and of bounded functions.

In particular, from Theorem 13 Krause's [20] and Schober's [28] result for the class S follows.

From inequality (3.13) one can derive the corresponding result for subclasses of the classes $C_{m,0}(a_0, \emptyset)$ of functions F satisfying condition (3).

For $m = 2$ the result coincides with Grinšpan and Kolomojceva's result [10].

References

[1] D. Aharonov, *A generalization of a theorem of J. A. Jenkins*, Math. Z. 110 (1969), p. 218–222.

[2] –, *On pairs of functions and related classes*, Duke Math. J. 40 (1973), p. 669–676.

[3] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. 138 (1916), p. 940–955.

[4] –, *Über einige Extremalprobleme im Gebiet der konformen Abbildung*, Math. Ann. 77 (1916), p. 153–172.

[5] D. W. De Temple, *Grunsky–Nehari inequalities for a subclass of bounded univalent functions*, Trans. Amer. Math. Soc. 159 (1971), p 317–328.

[6] –, *An area method for systems of univalent functions whose ranges do not overlap*, Math. Z. 128 (1972), p. 23–33.

[7] S. Eilenberg, *Sur quelques propriétés topologiques de la surface de sphère*, Fund. Math. 25 (1935), p. 267–272.

[8] S. A. Gel'fer, *On a class of the regular functions which assume no pair of the values w and −w*, Math. Sb. 19 (1946), p. 33–46 (in Russian).

[9] G. M. Goluzin, *The variation method in conformal mappings* IV, ibidem 29 (1951), p. 455–468 (in Russian).

[10] A. Z. Grinšpan and Z. V. Kolomojceva, *Some estimations in the class of the functions which assume no pair of the values w and −w*, Vestn. Leningr. Univ. 19 (1973), p. 28–34 (in Russian).

[11] L. L. Gromova and N. A. Lebedev, *On non-insistent regions in a circle* II, ibidem 13 (1973), p. 25–36 (in Russian).

[12] H. Grunsky, *Einige Analoga zum Schwazschen Lemma*, Math. Ann. 108 (1933), p. 190–196.

[13] –, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. 45 (1939), p. 29–61.

[14] J. A. Hummel, *Inequalities of Grunsky type for Aharonov pairs*, J. Analyse Math. 25 (1972), p. 217–257.

[15] –, and M. Schiffer, *Coefficient inequalities for Bieberbach–Eilenberg functions*, Arch. Rational Mech. Anal. 32 (1969), p. 87–99.

[16] Z. J. Jakubowski, *Sur le maximum de la fonctionnelle* $|A_3-\alpha A_2|^2$ $(0 \leqslant \alpha < 1)$ *dans la famille de fonctions* F_M, Bull. Soc. Sci. Lett. 13 (1962), p. 1–19.

[17] J. A. Jenkins, *On Bieberbach–Eilenberg functions* III, Trans. Amer. Math. Soc. 119 (1965), p. 195–215.

[18] –, *A remark on "pairs" of regular functions*, Proc. Amer. Math. Soc. 31 (1972), p. 119–121.

[19] R. Kortram, *On an extended class of bounded univalent functions*, Ann. Acad. Sci. Fenn. 578 (1974), p. 1–16.

[20] W. Kraus, *Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung*, Mitt. Math. Sem. Giessen 21 (1932), p. 1–28.

[21] M. A. Lavrent'ev, *Concerning conformal mappings*, Trudy Matem. Inst. AN SSSR 5 (1934), p. 195–246 (in Russian).

[22] N. A. Lebedev, *The principle of areas in the theory of schlicht functions*, Naŭka, Moskva 1975 (in Russian).

[23] N. A. Lebedev and L. V. Mamaj, *A generalization of an inequality of P. Garabedian and V. Shiffer*, Vestn. Leningr. Univ. 19 (1970), p. 44–45 (in Russian).

[24] Z. Nehari, *Conformal mappings*, McGraw-Hill, New York–Toronto–London 1952.

[25] –, *Some inequalities in the theory of functions*, Trans. Amer. Math. Soc. 75 (1953), p. 256–286.

[26] M. Schiffer and O. Tammi, *On the coefficient problem for bounded univalent functions*, Trans. Amer. Math. Soc. 140 (1969), p. 461–474.

[27] —, —, *A Gree's inequality for the power matrix*, Ann. Acad. Sci. Fenn. 501 (1971), p. 3–15.

[28] G. Schober, *Univalent functions – selected topics*, Springer-Verlag Lecture Notes in Math. 478, Berlin–Heidelberg–New York 1975.

[29] Tao-Shing-Shah, *On the moduli of some classes of analytic functions*, Acta Math. Sinica 5 (1955), p. 439–454.

[30] J. Śladkowska, *Coefficient inequalities for Shah's functions*, Demonstr. Math. 5 (1973), p. 171–192.

[31] —, *Coefficient inequalities for Bieberbach–Eilenberg functions*, Zesz. Nauk. Politechn. Śląskiej 25 (1974), p. 11–46 (in Polish).

[32] O. Tammi, *Grunsky type of inequalities and determination of the totality of the extremal functions*, Ann. Acad. Sci. Fenn. 443 (1969), p. 1–20.

[33] K. Włodarczyk, *Inequalities of Grunsky–Nehari type for pairs of vector functions*, Ann. Polon. Math. 37 (1980), p. 179–198.

[34] R. Zawadzki, *Sur les modules des coefficients B_0 et B_1 de fonctions holomorphes univalentes bornées inférieurement*, Bull. Soc. Sci. Lett. 12 (1961), p. 1–9.

Reçu par la Rédaction le 28. 4. 1977