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On a problem of P. Montel

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1. Preliminary remarks. In [5], p. 66, P. Montel proposed to find the precise bounds for the modulus of a function $F(z)$ regular and univalent in the unit circle $|z| < 1$ which satisfies

$$F(0) = 0\,, \quad F(z_0) = 1 \quad (0 < |z_0| < 1)\,. \tag{1.1}$$

Montel's problem is connected with the interpolation problem for univalent functions as considered in [6], when two preassigned values of F at two points $0, z_0$ of the unit circle are given. It is equivalent to the determination of the least upper bound

$$k = k(z_1, z_2) = \sup_f |f(z_1)/f(z_2)|$$

where z_1, z_2 are two different fixed points of the unit circle and f ranges over the class of regular and univalent functions vanishing at the origin. Then evidently

$$k^{-1}(z_0, z) \leqslant |F(z)| \leqslant k(z, z_0)\,,$$

both bounds being sharp.

The solution of Montel's problem, as given in [3], is rather complicated. The precise upper bound $k(z_1, z_2)$ involves Weierstrass's elliptic $\wp$-function with periods equal to those of a certain hyperelliptic integral depending on z_1, z_2 unless the points $0, z_1, z_2$ are collinear. In the latter case $k(z_1, z_2)$ can be brought to a more convenient form since in this case (and only in this case, see [3]) the extremal function is the Koebe function. Using the well-known inequalities

$$\frac{|f'(0)|\,|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|f'(0)|\,|z|}{(1-|z|)^2} \tag{1.2}$$

valid for any function regular and univalent in $|z| < 1$ and satisfying $f(0) = 0$, we immediately obtain

$$\left|\frac{f(z_1)}{f(z_2)}\right| \leqslant \left|\frac{z_1}{z_2}\right| \left(\frac{1+|z_2|}{1-|z_1|}\right)^2 \tag{1.3}$$

and this can be considered as an approximate formula for $k(z_1, z_2)$. The sign of equality in (1.3) is attained only when $f(z)$ is the Koebe function

$Az(1-e^{-i\theta}z)^{-2}$ and $z_1 = |z_1|e^{i\theta}$, $z_2 = -|z_2|e^{i\theta}$, i.e. when z_1, z_2 lie on the opposite radii of the unit circle.

The aim of this paper is to give another approximate estimation of $k(z_1, z_2)$ in terms of elementary functions:

$$\left|\frac{f(z_1)}{f(z_2)}\right| \leqslant \left|\frac{z_1}{z_2}\right| \left(\frac{|1-z_1\bar{z}_2|+|z_1-z_2|}{1-|z_1|^2}\right)^2 \left(\frac{|1-z_1\bar{z}_2|+|z_1-z_2|}{|1-z_1\bar{z}_2|-|z_1-z_2|}\right)^{2|z_1z_2|/(1-|z_1z_2|)} \tag{1.4}$$

for any function $f(z)$ regular and univalent in $|z|<1$, satisfying $f(0)=0$.

According to a result due to Z. Lewandowski [4], for functions f^* starlike with respect to the origin we have a sharp estimation

$$\left|\frac{f^*(z_1)}{f^*(z_2)}\right| \leqslant \left|\frac{z_1}{z_2}\right| \left(\frac{|1-z_1\bar{z}_2|+|z_1-z_2|}{1-|z_1|^2}\right)^2$$

so that it is only the last factor in (1.4) that could be improved. Making z_1 (resp. z_2) tend to 0 in (1.4) we obtain the left-hand side (resp. the right-hand side) inequality in (1.2) so that (1.4) is a generalization of (1.2). At the same time, for $z_2 \to z_1$ we obtain $|f(z_1)/f(z_2)| \to 1$. It is worth-while to mention that the determination of precise bounds for the ratio of derivatives is much less troublesome. Put $\zeta = (z-z_2)/(1-z\bar{z}_2)$ and consider $\varphi(\zeta) = f(z)$. Then

$$f'(z_2) = \varphi'(0)\left(\frac{d\zeta}{dz}\right)_{z=z_2} = \frac{\varphi'(0)}{1-|z_2|^2},$$

$$f'(z_1) = \varphi'(\zeta_1)\left(\frac{d\zeta}{dz}\right)_{z=z_1} = \frac{1-|z_2|^2}{(1-z_1\bar{z}_2)^2}\varphi'(\zeta_1),$$

where $\zeta_1 = (z_1-z_2)/(1-z_1\bar{z}_2)$. Hence

$$\frac{f'(z_1)}{f'(z_2)} = \left(\frac{1-|z_2|^2}{1-z_1\bar{z}_2}\right)^2 \frac{\varphi'(\zeta_1)}{\varphi'(0)} \tag{1.5}$$

and in view of the well-known inequality

$$\left|\frac{\varphi'(\zeta_1)}{\varphi'(0)}\right| \leqslant \frac{1+|\zeta_1|}{(1-|\zeta_1|)^3}$$

we obtain

$$\left|\frac{f'(z_1)}{f'(z_2)}\right| \leqslant \frac{(1-|z_2|^2)^2(|1-z_1\bar{z}_2|+|z_1-z_2|)}{(|1-z_1\bar{z}_2|-|z_1-z_2|)^3}. \tag{1.6}$$

Inequality (1.6) can be brought to a more symmetric form (1.8) by using the identity

$$(1-|z_1|^2)(1-|z_2|^2) = |1-z_1\bar{z}_2|^2 - |z_1-z_2|^2. \tag{1.7}$$

Thus

$$\left|\frac{f'(z_1)}{f'(z_2)}\right| \leqslant \frac{1-|z_2|^2}{1-|z_1|^2}\left(\frac{|1-z_1\bar{z}_2|+|z_1-z_2|}{|1-z_1\bar{z}_2|-|z_1-z_2|}\right)^2. \tag{1.8}$$

The bound in (1.8) is the best possible, with the sign of equality for $\varphi(\zeta)$ being the Koebe function. The greatest lower bound for $|f'(z_1)/f'(z_2)|$ can be obtained from (1.8) at once by interchanging the indices.

At the same time we immediately deduce from (1.5) that the domain of variability of $f'(z_1)/f'(z_2)$, when z_1, z_2 are fixed and f ranges over the class (S) of univalent functions with $f(0) = 0$, $f'(0) = 1$, can be obtained from the well-known domain of variability of $f'(\zeta_1)$ as determined by A. Grad [7], by a suitable homothety and rotation.

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2. Derivation of the formula (1.4). Suppose that $k(t)$ is an arbitrary continuous complex-valued function of a real variable t, $0 \leqslant t < +\infty$, satisfying $|k(t)| = 1$. Let $w = f(z, t)$ be the solution of the differential equation

(2.1) $$\frac{\partial w}{\partial t} = -w\,\frac{1+k(t)w}{1-k(t)w},$$

such that $f(z, 0) = z$. Then, according to Loewner (cf. [1], or [2]), the univalent functions $f \in (S)$ which can be represented as a limit $\lim\limits_{t\to+\infty} e^t f(z, t)$ form for varying $k(t)$ a dense subclass (S′) of the class (S). Besides, for $f(z, t)$ we have $f(z, t) = e^{-t}(z + a_2(t)z^2 + \ldots)$ and $|f(z, t)| < 1$ for any $|z| < 1$ and $0 \leqslant t < +\infty$. Since the class (S′) is dense in (S), the bounds for $|f(z_1)/f(z_2)|$, $f \in (S')$ will remain true for the wider class (S).

Let z_1, z_2 be two fixed points of the unit circle different from each other and from the origin. Put $f_1 = f(z_1, t)$, $f_2 = f(z_2, t)$; z_1, z_2 being fixed, we can write d/dt instead of $\partial/\partial t$. From (2.1) it follows that

(2.2a) $$\frac{d}{dt}\log(f_1 - f_2) = \frac{-1 - k(f_1+f_2) + k^2 f_1 f_2}{(1-kf_1)(1-kf_2)},$$

(2.2b) $$\frac{d}{dt}\log\frac{1}{f_1 f_2} = \frac{2 - 2k^2 f_1 f_2}{(1-kf_1)(1-kf_2)},$$

(2.2c) $$\frac{d}{dt}\log e^{-t} = -1 = \frac{-1 + k(f_1+f_2) - k^2 f_1 f_2}{(1-kf_1)(1-kf_2)}.$$

Adding (2.2a)-(2.2c) we obtain

(2.2) $$\frac{d}{dt}\log\frac{e^{-t}(f_1 - f_2)}{f_1 f_2} = -2\,\frac{kf_1}{1-kf_1}\cdot\frac{kf_2}{1-kf_2}.$$

Besides,

$$\frac{d}{dt}\log\frac{1}{1-f_1\bar{f}_2} = -2\frac{kf_1}{1-kf_1}\left(\overline{\frac{kf_2}{1-kf_2}}\right). \tag{2.3}$$

By adding (2.2) and (2.3) we have

$$\frac{d}{dt}\log\frac{e^{-t}(f_1-f_2)}{f_1f_2(1-f_1\bar{f}_2)} = -2\frac{kf_1}{1-kf_1}\left[\frac{kf_2}{1-kf_2}+\left(\overline{\frac{kf_2}{1-kf_2}}\right)\right], \tag{2.4}$$

and this gives

$$\frac{d}{dt}\log\left|\frac{e^{-t}(f_1-f_2)}{f_1f_2(1-f_1\bar{f}_2)}\right| = -X_1X_2, \tag{2.5}$$

where

$$X_j = \frac{kf_j}{1-kf_j}+\left(\overline{\frac{kf_j}{1-kf_j}}\right) = \operatorname{Re}\frac{2kf_j}{1-kf_j}, \qquad j=1,2.$$

From (2.5) it follows that

$$\frac{d}{dt}\log\left|\frac{f_1-f_2}{1-f_1\bar{f}_2}\right| = 1-X_1X_2+\frac{d}{dt}\log|f_1f_2|. \tag{2.6}$$

However, we have from (2.1)

$$\frac{d}{dt}\log|f_j| = \operatorname{Re}\left(-\frac{1+kf_j}{1-kf_j}\right) = -1-X_j, \qquad j=1,2, \tag{2.7}$$

so that (2.6) takes the form

$$\begin{aligned}\frac{d}{dt}\log\left|\frac{f_1-f_2}{1-f_1\bar{f}_2}\right| &= 1-X_1X_2-1-X_1-1-X_2 \\ &= -(1+X_1)(1+X_2),\end{aligned} \tag{2.8}$$

and in view of (2.7) we finally obtain

$$\begin{aligned}\frac{d}{dt}\log\left|\frac{f_1-f_2}{1-f_1\bar{f}_2}\right| &= -\left(\frac{d}{dt}\log|f_1|\right)\left(\frac{d}{dt}\log|f_2|\right) \\ &= -\frac{1-|f_1|^2}{|1-kf_1|^2}\cdot\frac{1-|f_2|^2}{|1-kf_2|^2}.\end{aligned} \tag{2.9}$$

(2.9) means that the hyperbolic distance of points f_1, f_2 with respect to the unit circle is a strictly decreasing function of t. Therefore we can introduce a new parameter $u = |(f_1-f_2)/(1-f_1\bar{f}_2)|$, connected with t by the relation

$$dt = -\frac{|1-kf_1|^2|1-kf_2|^2}{(1-|f_1|^2)(1-|f_2|^2)}\cdot\frac{du}{u}. \tag{2.10}$$

Putting $f_1 = f_2$ in (2.3) we obtain

$$\frac{d}{dt}\log\frac{1}{1-|f_j|^2} = -\frac{2|f_j|^2}{|1-kf_j|^2}, \qquad j = 1, 2\,. \tag{2.11}$$

Now (2.7) gives

$$\frac{d}{dt}\log|f_j| = -\frac{1-|f_j|^2}{|1-kf_j|^2}, \qquad j = 1, 2\,, \tag{2.12}$$

and using the equality $|1-kf_j|^2 = 1-kf_j-\bar{k}\bar{f}_j+|f_j|^2$ we obtain from (2.11) and (2.12)

$$\frac{d}{dt}\log\frac{|f_1|(1-|f_2|^2)}{|f_2|(1-|f_1|^2)} = \frac{1}{|1-kf_1|^2|1-kf_2|^2}\{k[(f_2-f_1)- -f_1f_2(\bar{f}_2-\bar{f}_1)]+\bar{k}[(\bar{f}_2-\bar{f}_1)-\bar{f}_1\bar{f}_2(f_2-f_1)]\}\,,$$

or, using (2.10)

$$u\frac{d}{du}\log\frac{|f_2|(1-|f_1|^2)}{|f_1|(1-|f_2|^2)} = \frac{1}{(1-|f_1|^2)(1-|f_2|^2)}\{k[(f_2-f_1)- -f_1f_2(\bar{f}_2-\bar{f}_1)]+\bar{k}[(\bar{f}_2-\bar{f}_1)-\bar{f}_1\bar{f}_2(f_2-f_1)]\}\,. \tag{2.13}$$

Putting $f_2-f_1 = \tau|f_2-f_1|$ we have

$$\begin{aligned}|k[(f_2-f_1)-f_1f_2(\bar{f}_2-\bar{f}_1)]+\bar{k}[(\bar{f}_2-\bar{f}_1)-\bar{f}_1\bar{f}_2(f_2-f_1)]| &\leqslant |f_2-f_1|\,|k\tau(1-f_1f_2\bar{\tau}^2)+\bar{k}\bar{\tau}(1-\bar{f}_1\bar{f}_2\tau^2)| \\ &\leqslant 2|f_2-f_1|\,|1-f_1f_2\bar{\tau}^2| \leqslant 2|f_2-f_1|\,|1-f_1\bar{f}_2|\frac{1+|f_1f_2|}{1-|f_1f_2|}\,.\end{aligned}$$

Hence (2.13) takes the form

$$\begin{aligned}u\frac{d}{du}\log\frac{|f_2|(1-|f_1|^2)}{|f_1|(1-|f_2|^2)} &\leqslant \frac{1+|f_1f_2|}{1-|f_1f_2|}\cdot\frac{2|f_2-f_1|\,|1-f_1\bar{f}_2|}{(1-|f_1|^2)(1-|f_2|^2)} \\ &\leqslant \frac{1+|z_1z_2|}{1-|z_1z_2|}\cdot\frac{2u}{1-u^2}\,,\end{aligned} \tag{2.14}$$

since $|f_1f_2|$ decreases strictly by (2.12) and since (1.7) holds for any complex z_1, z_2.

After dropping the common factor u in (2.14) and integrating both sides from 0 to $u_0 = |(z_1-z_2)/(1-z_1\bar{z}_2)|$ with respect to u, we obtain

$$\begin{aligned}&\log\frac{|z_2|(1-|z_1|^2)}{|z_1|(1-|z_2|^2)} - \log\left|\frac{f(z_2)}{f(z_1)}\right| \\ &\qquad\leqslant \frac{1+|z_1z_2|}{1-|z_1z_2|}\log\frac{1+u_0}{1-u_0} = \log\left(\frac{|1-z_1\bar{z}_2|+|z_2-z_1|}{|1-z_1\bar{z}_2|-|z_2-z_1|}\right)^{(1+|z_1z_2|)/(1-|z_1z_2|)}\end{aligned}$$

or

$$\left|\frac{f(z_1)}{f(z_2)}\right| \leqslant \left|\frac{z_1}{z_2}\right|\frac{1-|z_2|^2}{1-|z_1|^2}\cdot\frac{|1-z_1\bar{z}_2|+|z_2-z_1|}{|1-z_1\bar{z}_2|-|z_2-z_1|}\times \times\left(\frac{|1-z_1\bar{z}_2|+|z_2-z_1|}{|1-z_1\bar{z}_2|-|z_2-z_1|}\right)^{2|z_1z_2|/(1-|z_1z_2|)} \tag{2.15}$$

which becomes (1.4) after using (1.7).

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