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Models for doubly commuting contractions

by Marek Słociński (Cracow)

Abstract. The idea of Sz.-Nagy and Foiaş model for a contraction is generalized for the pair of doubly commuting contraction. The operators are described as backward shifts on vector valued L^2 and H^2 spaces on bidisc restricted to invariant subspace defined by some bounded analytic operator functions.

In what follows H is a separable Hilbert space with inner product (x, y); $x, y \in H$ and norm $\|x\| = \sqrt{(x, x)}$; $x \in H \cdot L(H)$ stands for the algebra of all bounded linear operators on H. For $T \in L(H)$, T^* denotes the adjoint of T. The operator $T \in L(H)$ is called *contraction* if $\|T\| \leqslant 1$. Operators T_1 and T_2 doubly commute if $T_1 T_2 = T_2 T_1$ and $T_1^* T_2 = T_2 T_1^*$. In the present paper we give, using the dilation theory, a functional model for the pair of doubly commuting contractions on a separable Hilbert space.

First we give the decomposition of the pair of doubly commuting contraction. Let T_1 and T_2 be a pair of doubly commuting contractions on the space H. If the decomposition $H = H_1 \oplus H_2$ is the canonical decomposition of T_1 (see Theorem I.3, 2 of [1]) on the unitary and completely nonunitary parts, then $H_1 = \bigcap_{n \neq 0} \operatorname{Ker} D_{T_1^n}$ where for $n \geqslant 0$ $D_{T_1^n} = (I - T_1^{*n} T_1^n)^{1/2}$ and for $n < 0$ $D_{T_1^n} = (I - T_1^{|n|} T_1^{*|n|})^{1/2}$. Since T_1 and T_2 doubly commute, T_2 and T_2^* commute with $(I - T_1^{*n} T_1^n)$ and $(I - T_1^{|n|} T_1^{*|n|})$ for every $n \geqslant 0$. Consequently T_2 and T_2^* commute with $D_{T_1^n}$ for every $n \neq 0$. If $x \in \operatorname{Ker} D_{T_1^n}$ then $D_{T_1^n} T_2 x = T_2 D_{T_1^n} x = 0$, hence $T_2 x \in \operatorname{Ker} D_{T_1^n}$. Analogically $T_2^* x \in \operatorname{Ker} D_{T_1^n}$ for $x \in \operatorname{Ker} D_{T_1^n}$. This implies that, for every $n \neq 0$, $\operatorname{Ker} D_{T_1^n}$ is reducing subspace for T_2 and consequently H_1 reduces T_2. The same consideration applied to the pairs $T_{1|H_1}$, $T_{2|H_1}$ and $T_{1|H_2}$, $T_{2|H_2}$, which evidently are doubly commutative, proves the following:

Proposition 1. *Suppose that T_1 and T_2 are doubly commuting contractions on the space. Then there is a decompostion $H = \bigoplus_{i=1}^{4} H_i$ such that H_i reduces T_1 and T_2 $(i = 1, \ldots, 4)$ and the following conditions hold:*

$T_{1|H_1}$ and $T_{2|H_1}$ are unitary operators,

$T_{1|H_2}$ *is completely nonunitary and* $T_{2|H_2}$ *is unitary,*
$T_{1|H_3}$ *is unitary and* $T_{2|H_3}$ *is completely nonunitary,*
$T_{1|H_4}$ *and* $T_{2|H_4}$ *are completely nonunitary.*

Now it is easy to see that to find the model of doubly commuting contractions it is sufficient to solve this problem in two cases:

(I) T_1 and T_2 are completely nonunitary.

(II) T_1 is completely nonunitary and T_2 is unitary.

We begin with case (I). The following result, which is the consequence of Theorem 2 of [2], will be useful in our work:

THEOREM 0. *Suppose that T_1 and T_2 are doubly commuting contractions on the space H. Then there is the minimal unitary dilation U_1, U_2 of the pair T_1, T_2 such that*

(0.1) *the dilation space K has the form $K = \bigoplus_{i,j=-\infty}^{\infty} H_{i,j}$ where*

$$H_{0,0} = H, \qquad H_{1,0} = \overline{(U_1 - T_1)H}, \qquad H_{0,1} = \overline{(U_2 - T_2)H},$$
$$H_{-1,0} = \overline{(U_1^* - T_1^*)H}, \qquad H_{0,-1} = \overline{(U_2^* - T_2^*)H},$$
$$H_{1,1} = \overline{(U_1 U_2 - U_1 T_2 - U_2 T_1 + T_1 T_2)H},$$
$$H_{1,-1} = \overline{(U_1^* U_2 - U_1 T_2^* - U_2 T_1^* + T_1 T_2^*)H},$$
$$H_{-1,1} = \overline{(U_1^* U_2 - U_1^* T_2 - U_2 T_1^* + T_1^* T_2)H},$$
$$H_{-1,-1} = \overline{(U_1^* U_2^* - U_1^* T_2^* - U_2^* T_1^* + T_1^* T_2^*)H},$$

and for $n, m \geqslant 1$

$$H_{n,0} = U_1^{n-1} H_{1,0}, \qquad H_{-n,0} = U_1^{*(n-1)} H_{-1,0};$$
$$H_{0,m} = U_2^{m-1} H_{0,1}, \qquad H_{0,-m} = U_2^{*(m-1)} H_{0,-1},$$
$$H_{n,m} = U_1^{n-1} U_2^{m-1} H_{1,1}; \qquad H_{-n,-m} = U_1^{*(n-1)} U_2^{*(m-1)} H_{-1,-1},$$
$$H_{n,-m} = U_1^{n-1} U_2^{*(m-1)} H_{1,-1}; \qquad H_{-n,m} = U_1^{*(n-1)} U_2^{m-1} H_{-1,1},$$

(0.2) $U_1 H_{-1,j} \oplus H_{0,j} = H_{0,j} \oplus H_{1,j}, \quad U_2 H_{j,-1} \oplus H_{j,0} = H_{j,0} \oplus H_{j,1}$

for every integer number j.

Now, we can define the following spaces:

$$H_{\cdot j} = \bigoplus_{i=-\infty}^{\infty} H_{i,j}, \qquad H_{\cdot j}^{+} = \bigoplus_{i=0}^{\infty} H_{i,j} \qquad (j = 0, 1, 2, \ldots),$$

$$H_{i\cdot} = \bigoplus_{j=-\infty}^{\infty} H_{i,j}, \qquad H_{i\cdot}^{+} = \bigoplus_{j=0}^{\infty} H_{i,j} \qquad (i = 0, 1, 2, \ldots),$$

$$K_1^+ = \bigoplus_{i=0}^{\infty} H_{i.} = \bigoplus_{j=-\infty}^{\infty} H_{.j}^+, \quad K_2^+ = \bigoplus_{j=0}^{\infty} H_{.j} = \bigoplus_{i=-\infty}^{\infty} H_{i}^+$$

and

$$K^+ = \bigoplus_{i,j=0}^{\infty} H_{i,j} = \bigoplus_{i=0}^{\infty} H_{i.}^+ = \bigoplus_{j=0}^{\infty} H_{.j}^+.$$

A subspace $L \subset H$ is called a *wandering subspace for the unitary operator U on H* if $U^n L \perp L$ for every $n \neq 0$. L is called *wandering for the pair of unitary operators U_1, U_2 on H* if $U_1^n U_2^m L \perp L$ for every n, m such that $|m|+|n| > 0$. If L is wandering subspace for U_i, then we can define the spaces $M_i(L)$ and $M_i^+(L)$ as follows $M_i(L) = \bigoplus_{n=-\infty}^{\infty} U_i^n L$ and $M_i^+(L) = \bigoplus_{n=0}^{\infty} U_i^n L$. If L is wandering for the pair U_1, U_2, then $M(L)$ and $M^+(L)$ denote the spaces $M(L) = \bigoplus_{n,m=-\infty}^{\infty} U_1^n U_2^m L$ and $M^+(L) = \bigoplus_{n,m=0}^{\infty} U_1^n U_2^m L$.

Now we prove the following:

LEMMA 1. *Suppose that T_1 and T_2 are doubly commuting contractions on H. Let U_1, U_2 be a unitary dilation of T_1, T_2 as in Theorem 0. Then with notations as above the following conditions hold true:*

(1.1) *the spaces $H_{-1,-1}$, $H_{-1,1}$, $H_{1,-1}$, $H_{1,1}$ are wandering subspaces for the pair U_1, U_2,*

(1.2) *$H_{0,-1}$, $H_{0,1}$ and $H_{-1,0}$, $H_{1,0}$ are wandering subspaces for U_2 and U_1 respectively,*

(1.3) *$H_{.i}^+$ and $H_{i.}^+$ are invariant subspaces for U_1 and U_2 respectively* ($i = 0, \pm 1, \pm 2, \ldots$),

(1.4) *$H_{.i}$ and $H_{i.}$ reduce U_1 and U_2 respectively* ($i = 0, \pm 1, \pm 2, \ldots$).

Proof. Theorem 0 implies that for $n, m \geq 0$, $n+m > 0$ we have $U_1^n U_2^m H_{1,1} = H_{1+n,1+m} \perp H_{1,1}$ and consequently $U_1^n U_2^m H_{1,1} \perp H_{1,1}$ for $n, m \leq 0$ and $|n|+|m| > 0$. If $n \geq 0$, $m \leq 0$ and $n+|m| > 0$ and $x, y \in H_{1,1}$, then $U_1^n x \in H_{1+n,1}$ and $U_2^{-m} y = U_2^{|m|} y \in H_{1,|m|+1}$, hence $(U_1^n U_2^m x, y) = (U_1^n x, U_2^{-m} y) = 0$. Thus $U_1^n U_2^m H_{1,1} \perp H_{1,1}$ for $n \geq 0$, $m < 0$ and $n+|m| > 0$. Consequently $U_1^n U_2^m H_{1,1} \perp H_{1,1}$ for $n \leq 0$, $m \geq 0$ and $|n|+m > 0$. So we have proved that $H_{1,1}$ is the wandering subspace for the pair U_1, U_2. The rest of (1.1) we prove analogically.

Since for $n \geq 0$, $U_2^n H_{0,1} = H_{0,n+1}$, $H_{0,1}$ is a wandering subspace for U_2. This gives (1.2).

Now we shall show that $H_{.i}^+$ is invariant for U_1. It is sufficient to show that for $j \geq 0$, $U_1 H_{j,i} \subset H_{.i}^+$. If $j > 0$ then $U_1 H_{j,i} = H_{j+1,i} \subset H_{.i}^+$. For $j = 0$, by (0.2) we have $U_1 H_{0,i} \subset H_{0,i} \oplus H_{1,i} \subset H_{.i}^+$ which completes the proof of (1.3).

To prove (1.4) it is sufficient to show that for every j, $U_1 H_{j,i} \subset H_{.i}$ and $U_1^* H_{j,i} \subset H_{.i}$. If $j \geqslant 1$ then $U_1 H_{j,i} = H_{j+1,i} \subset H_{.i}$. If $j = 0, -1$ then by (0.2) we get $U_1 H_{j,i} \subset H_{0,i} \oplus H_{1,i} \subset H_{.i}$. For $j \leqslant -1$ we have $U_1 H_{j,i} = H_{j+1,i} \subset H_{.i}$. Thus $U_1 H_{.i} \subset H_{.i}$. It is easy to see that (0.2) implies that $U_1^* H_{1,i} \oplus H_{0,i} = H_{0,i} \oplus H_{-1,i}$. By using this, similar proof as above shows that $U_1^* H_{.i} \subset H_{.i}$, which completes the proof of our lemma.

If L is a subspace of the Hilbert space K we denote by P_L the orthogonal projection of K onto L. Our second result will be the following:

LEMMA 2. *Suppose that T_1 and T_2 are doubly commuting contractions on H. Let U_1 and U_2 be a unitary dilation of T_1, T_2 as in Theorem 0. If $T_{1,i} = P_{H_{0,i}} U_{1|H_{0,i}}$ and $T_{2,i} = P_{H_{i,0}} U_{2|H_{i,0}}$, $P_{1,i}$ is the orthogonal projection of $H_{.i}$ onto $M_1(H_{-1,i})$ and $P_{2,i}$ is the orthogonal projection of $H_{i.}$ onto $M_2(H_{i,-1})$ ($i = 0, \pm 1, \pm 2, \ldots$), then the following conditions hold true:*

(2.1) *$U_{1|H_{.i}}$ and $U_{2|H_{i.}}$ are minimal unitary dilations of $T_{1,i}$ and $T_{2,i}$ respectively ($i = 0, \pm 1, \pm 2, \ldots$),*

(2.2) *if T_i is completely nonunitary, then $T_{i,j}$ is completely nonunitary ($i = 1, 2$, $j = 0, \pm 1, \pm 2, \ldots$),*

(2.3) *if for the contraction T on H and its minimal unitary dilation U, L_T^+ and L_T^- denote the spaces $L_T^+ = \overline{(U-T)H}$ and $L_T^- = \overline{(U^*-T^*)H}$, then $L_{T_{1,i}}^+ = H_{1,i}$, $L_{T_{1,i}}^- = H_{-1,i}$, $L_{T_{2,i}}^+ = H_{i,1}$, $L_{T_{2,i}}^- = H_{i,-1}$ ($i = 0, \pm 1, \pm 2, \ldots$).*

Proof. If $x \in H_{0,i}$ then by (0.2) we get that $U_1 x = x_0 + x_1$ where $x_k \in H_{k,i}$. Thus $T_{1,i} x = P_{H_{0,i}} U_1 x = x_0$. If follows that $U_1 x = T_{1,i} x + x_1$ where $x_1 \perp H_{0,i}$. Using mathematical induction one can prove that $U_1^n x = T_{1,i}^n x + x_1$ where $x_1 \perp H_{0,i}$. This implies that $U_{1|H_{.i}}$ is a unitary dilation of $T_{1,i}$. To finish the proof of (2.1) we have to show that $H_{.i} = \bigvee_{n=-\infty}^{\infty} U_1^n H_{0,i}$. Since $H_{0,i} \subset H_{.i}$, by (1.4) we get $\bigvee_{n=-\infty}^{\infty} U_1^n H_{0,i} \subset H_{.i}$. Suppose now, that $y \in H_{.i}$ and $y \perp \bigvee_{n=-\infty}^{\infty} U_1^n H_{0,i}$. Since $K = \bigvee_{n,m=-\infty}^{\infty} U_1^n U_2^m H$, we are to show that $(U_1^n U_2^m x, y) = 0$ for every integer numbers n, m and $x \in H = H_{0,0}$. Fix n, m and x. It is easy to see, by (0.2) and definition of $H_{0,j}$ that $U_2^m x = \sum_{j=0}^{m} x_j$ where $x_j \in H_{0,j}$. Since $x_j \in H_{0j}$, by (1.4) we get that $U_1^n x_j \in H_{.j}$ which for $j \neq i$ is orthogonal to $H_{.i} \supset H_{0,i}$. Consequently, for $j \neq i$, we have $(U_1^n x_j, y) = 0$. If $j = i$ then $(U_1^n x_i, y) = 0$, by our assumption that $y \perp \bigvee_{n=-\infty}^{\infty} U_1^n H_{0,i}$. Finally $(U_1^n U_2^m x, y) = \sum_{j=0}^{m} (U_1^n x_j, y) = 0$ which completes the proof of (2.1).

Suppose now, that $x \in H_{0,i}$ has the following property: $\|T_{1,i}^n x\| = \|T_{1,i}^{*n} x\| = \|x\|$ for every $n \geqslant 0$. To prove (2.2) it is sufficient to show that $x = 0$ if T_1 is completely nonunitary. For $i = 0$ this is obvious because $T_{1,0} = T_1$. Consider now the case $i > 0$. If $j = -i$ then by (0.2) we infer that $U_2^j x = x_0 + x_1$ where $x_k \in H_{0,k}$. Since $H_{0.}$ reduces U_2 (see (1.4)) we have the following equalities:

$$\begin{aligned}\|T_{1,i}^n x\|^2 &= \|U_2^{-i} T_{1,i}^n x\|^2 \\ &= \|U_2^{-i} P_{H_{0,i}} U_1^n x\|^2 = \|P_{H_{0.}} U_2^{-i} U_1^n x\|^2 = \|P_{H_{0.}} U_1^n x_0 + P_{H_{0.}} U_1^n x_1\|^2 \\ &= \|P_{H_{0,0}} U_1^n x_0 + P_{H_{0,1}} U_1^n x_1\|^2 = \|T_{1,0}^n x_0\|^2 + \|T_{1,1}^n x_1\|^2.\end{aligned}$$

In fact, to prove the third and the sixth equality it is sufficient to show that $P_{H_{0.}} y = P_{H_{0,i}} y$ for $y \in H_{.i}$. Let $y \in H_{.i}$. Then $y = \sum_{j=-\infty}^{\infty} y_j$ where $y_j \in H_{j,i}$. If $j \neq 0$ then $H_{j,i} \perp H_{0,k}$ for every k, hence $H_{j,i} \perp H_{0.}$. It follows that if $j \neq 0$ then $P_{H_{0.}} y_j = 0$. If $j = 0$ then $P_{H_{0.}} y_0 = y_0$ because $y_0 \in H_{0,i} \subset H_{0.}$. Consequently $P_{H_{0.}} y = y_0$. On the other hand, if $j \neq 0$ then $H_{j,i} \perp H_{0,i}$, hence $P_{H_{0,i}} y_j = 0$. For $j = 0$ we have $P_{H_{0,i}} y_0 = y_0$. Finally we have $P_{H_{0,i}} y = y_0$ which proves that $P_{H_{0.}} y = P_{H_{0,i}} y$. So we have proved that $\|T_{1,i}^n x\|^2 = \|T_{1,0}^n x_0\|^2 + \|T_{1,1}^n x_1\|^2$. On the other hand, $\|x\|^2 = \|U_2^{-i} x\|^2 = \|x_0\|^2 + \|x_1\|^2$. Our assumption about x implies that $\|x_0\|^2 + \|x_1\|^2 = \|T_1^n x_0\|^2 + \|T_{1,1}^n x_1\|^2 \leqslant \|T_1^n x_0\|^2 + \|x_1\|^2$. Consequently $\|x_0\|^2 \leqslant \|T_1^n x_0\|^2 \leqslant \|x_0\|^2$ which proves that $\|T_1^n x_0\| = \|x_0\|$ for every $n > 0$. Analogically we prove that $\|T_1^n x_0\| = \|x_0\|$ for every $n > 0$. Since T_1 is completely nonunitary, these equalities imply that $x_0 = 0$. Consequently $U_2^{-i} x = x_1 \in H_{1,0} \perp H_{0,0}$. For $j \leqslant -i$ we have $U_2^j x = U_2^{j+i} U_2^{-1} x \subset U_2^{j+i} H_{0,1} = H_{0,1+j+i} \perp H_{0,0}$. We have proved that $U_2^j x H_{0,0}$ for every j. Consequently $x \perp \bigvee_{n=-\infty}^{\infty} U_2^n H_{0,0}$. Now, (2.1) implies that $x \perp H_{0.}$. But $x \in H_{0.}$, hence $x = 0$ which ends the proof of (2.2) for $i < 0$. For $i > 0$, the proof is similar.

Now we shall prove that $\overline{(U_1 - T_{1,i}) H_{0,i}} = H_{1,i}$. If $x \in H_{0,i}$ then $U_1 x = T_{1,i} x + x_1$ where $x_1 \in H_{1,i}$ (see the proof of (2.1)). It follows that $(U_1 - T_{1,i}) x = x_1 \in H_{1,i}$, hence $L_{T_{1,i}}^+ \subset H_{1,i}$. Suppose now, that $y \in H_{1,i}$ and $y \perp L_{T_{1,i}}^+$. Now, if $x \in H_{0,i}$ then $0 = (y, (U_1 - T_{1,i}) x) = (y, U_1 x) - (y, T_{1,i} x) = (y, U_1 x)$. The last equality holds because $T_{1,i} x \in H_{0,i}$ and $y \in H_{1,i}$. Let $x \in H_{0,i}$. Then it is easy to see that for $n > 1$, $U_1^{n-1} x = x_0 + x_1$ where $x_1 \in H_{0,i}$ and $x_1 \in \bigoplus_{j=1}^{\infty} H_{j,i}$. It follows that for $n > 1$, $U_1^n x = U_1 x_0 + U_1 x_1$ where $x_0 \in H_{0,i}$ and $U_1 x_1 \in \bigoplus_{j=2}^{\infty} H_{j,i} H_{1,i}$. Consequently $(y, U_1^n x) = (y, U_1 x_0) + (y, U_1 x_1) = 0$ for every $n > 1$. Since for $n \geqslant 0$, $H_{0,i} \perp H_{1+n,i} = U_1^n H_{1,i}$, we have $U_1^{-n} H_{0,i} \perp H_{1,i}$ for every $n \geqslant 0$. This implies that for $n \leqslant 0$, $(y, U_1^n x)$

$= 0$. We have proved that for every integer number n and every $x \in H_{0,i}$, $(y, U_1^n x) = 0$, hence $y \perp \bigvee_{n=-\infty}^{\infty} U_1^n H_{0,i}$. It follows by (2.1) that $y \in H_{\cdot i}$. But $y \in H_{\cdot i}$, so $y = 0$. This proves that $L_{T_{1,i}}^+ = H_{1,i}$. The rest of (2.3) we prove analogically, which completes the proof of Lemma 2.

Now, applying Theorems I.1.4 and I.2.1 of [1] and Lemma 2 to the operators $T_{1,i}$ and $T_{2,i}$ we get the following conditions:

(2.4) $\quad H_{\cdot i} = M_1(U_1 H_{-1,i}) \oplus R_{1,i}, \quad H_{i\cdot} = M_2(U_2 H_{i,-1}) \oplus R_{2,i},$

(2.5) $\quad H_{\cdot i}^+ = M_1^+(U_1 H_{-1,i}) \oplus R_{1,i}, \quad H_{i\cdot}^+ = M_2^+(U_2 H_{i,-1}) \oplus R_{2,i},$

(2.6) $\quad H_{\cdot i}^+ = H_{0,i} \oplus M_1^+(H_{1,i}), \quad H_{i\cdot}^+ = H_{i,0} \oplus M_2^+(H_{i,1}),$

(2.7) $\quad P_{1,i} M_1^+(H_{1,i}) \subset M_1^+(U_1 H_{-1,i}), \quad P_{2,i} M_2^+(H_{i,1}) \subset M_2^+(U_2 H_{i,-1}),$

(2.8) if T_1 is completely nonunitary then $H_{\cdot i} = M_1(H_{-1,i}) \vee M_1(H_{1,i})$,
if T_2 is completely nonunitary then $H_{i\cdot} = M_2(H_{i,-1}) \vee M_2(H_{i,1})$,

(2.9) if T_1 is completely nonunitary then $R_{1,i} = \overline{(I - P_{1,i}) M_1(H_{1,i})}$,
if T_2 is completely nonunitary then $R_{2,i} = \overline{(I - P_{2,i}) M_2(H_{i,1})}$.

Now we need some definitions. Let L_{--}, L_{-+}, L_{+-} and L_{++} be defined as follows: $L_{--} = U_1 U_2 H_{-1,-1}$, $L_{-+} = U_1 H_{-1,1}$, $L_{+-} = U_2 H_{1,-1}$, $L_{++} = H_{1,1}$ and let Φ_{1-} be a unitary map of $M_1(L_{-+})$ onto

$$L_1^2(L_{-+}) = \{f\colon \Gamma \to L_{-+} \mid \int_\Gamma \|f(z_1)\|^2 \, dm(z_1) < \infty\}$$

such that $\Phi_{1-}(\sum_{n=-\infty}^{\infty} U_1^n x_n) = \sum_{n=-\infty}^{\infty} z_1^n x_n$ where $x_n \in L_{-+}$, Φ_{2-} be a unitary map of $M_2(L_{+-})$ onto $L_2^2(L_{+-})$ such that $\Phi_{2-}(\sum_{n=-\infty}^{\infty} U_2^n x_n) = \sum_{n=-\infty}^{\infty} z_2^n x_n$ where $x_n \in L_{+-}$, Φ_{i+} be a unitary map of $M_i(L_{++})$ onto $L_i^2(L_{++})$ such that $\Phi_{i+}(\sum_{n=-\infty}^{\infty} U_i^n x_n) = \sum_{n=-\infty}^{\infty} z_i^n x_n$ where $x_n \in L_{++}$ $(i = 1, 2)$.

Since $P_{1,1}$ commutes with U_1 and $P_{2,1}$ commutes with U_2, then by applying (2.7) and Lemma V.3.1 of [1] to $U_{1|M_1(L_{-+})}$, $U_{1|M_1(L_{++})}$, $P_{1,1|M_1(L_{++})}$ and $U_{2|M_2(L_{+-})}$, $U_{2|M_2(L_{++})}$, $P_{2,1|M_2(L_{++})}$ respectively we conclude that there are bounded analytic functions $\{Q_1^+(z_1), L_{++}, L_{-+}\}$ and $\{Q_2^+(z_2), L_{++}, L_{+-}\}$ such that

(2.10) $\quad \Phi_{i-} P_{i,1} x = Q_i^+ \Phi_{i+} x$ for every $x \in M_i(L_{++})$ where for every $f \in L_i^2(L_{++})$, $(Q_i^+ f)(z_i) = Q_i^+(z_i) f(z_i)$ $(i = 1, 2)$.

Define now $T_1 = P_{H_{0\cdot}} U_{1|H_{0\cdot}}$, $T_2 = P_{H_{\cdot 0}} U_{2|H_{\cdot 0}}$, $P_1 = \bigoplus_{i=-\infty}^{\infty} P_{1,i}$ and $P_2 = \bigoplus_{i=-\infty}^{\infty} P_{2,i}$. Then we can prove the following:

LEMMA 3. *Suppose that T_1 and T_2 are doubly commuting contractions on H. Let U_1, U_2 be a unitary dilation of T_1, T_2 as in Theorem 0. Then the following conditions are fulfilled:*

(3.1) *$H_{\cdot -1}$, $H_{\cdot 1}$ and $H_{-1\cdot}$, $H_{1\cdot}$ are wandering subspaces for U_2 and U_1 respectively,*

(3.2) *K_1^+ is invariant subspace for U_1 and reduces U_2,*
K_2^+ is invariant subspace for U_2 and reduces U_1,

(3.3) $\tilde{T}_1 = \bigoplus_{i=-\infty}^{\infty} T_{1,i}$ *and* $\tilde{T}_2 = \bigoplus_{i=-\infty}^{\infty} T_{2,i}$,

(3.4) *U_i is the minimal unitary dilation of $\tilde{T}_i$ ($i = 1, 2$),*

(3.5) $L_{\tilde{T}_1}^+ = H_{1\cdot}$, $L_{\tilde{T}_1}^- = H_{-1\cdot}$, $L_{\tilde{T}_2}^+ = H_{\cdot 1}$, $L_{\tilde{T}_2}^- = H_{\cdot -1}$,

(3.6) *P_1 is the orthogonal projection of K onto $M_1(H_{-1\cdot})$,*
P_2 is the orthogonal projection of K onto $M_2(H_{\cdot -1})$.

Proof. It is a consequence of (1.1) that for $i \neq 0$, $H_{i,1}$ is a wandering subspace for U_2. Combining this with (1.2) we get that H_i is a wandering subspace for U_2. The rest of (3.1) we obtain analogically.

Since $K_1^+ = \bigoplus_{j=-\infty}^{\infty} H_j^+ = \bigoplus_{i=0}^{\infty} H_i$ and $K_2^+ = \bigoplus_{i=-\infty}^{\infty} H_i^+ = \bigoplus_{j=0}^{\infty} H_{\cdot j}$, then (3,2) follows immediately by (1.3) and (1.4).

Since $P_{H_0} y = P_{H_{0,i}} y$ for $y \in H_{\cdot i}$, and for $x \in H_{0,i}$, $U_1 x \in H_{\cdot i}$ (see (1.4)) we have $T_1 x = P_{H_{0\cdot}} U_1 x = P_{H_{0,i}} U_1 x = T_{1,i} x$ for $x \in H_{0,i}$ which proves (3.3).

Since minimal unitary dilation of the orthogonal sum of contractions is the orthogonal sum of their minimal unitary dilations (see [1]) then (2.1) and (3.3) imply (3.4).

It is easy to see that (3.5) is a consequence of (2.1) (2.3) and (3.3).

To prove (3.6) we have to show that $x \in M_1(H_{-1\cdot})$ if and only if $P_1 x = x$. Let $x \in M_1(H_{-1\cdot})$. This implies that $x = \sum_{n=-\infty}^{\infty} U_1^n x_n$ where $x_n \in H_{-1\cdot}$. It follows that $x_n = \sum_{m=-\infty}^{\infty} x_{n,m}$ where $x_{n,m} \in H_{-1,m}$ and consequently $x = \sum_{n,m=-\infty}^{\infty} U_1^n x_{n,m}$. Let $y_m = \sum_{n=-\infty}^{\infty} U_1^n x_{n,m}$. Since $x_{n,m} \in H_{-1,m}$, $y_m \in M_1(H_{-1,m}) \subset H_{\cdot m}$. Now $P_1 y_m = P_{1|H_{\cdot m}} y_m = P_{1,m} y_m = y_m$, by definition of $P_{1,m}$. Thus $P_1 x = P_1(\sum_{m=-\infty}^{\infty} y_m) = \sum_{m=-\infty}^{\infty} P_1 y_m = \sum_{m=-\infty}^{\infty} y_m = x$. Suppose now that $P_1 x = x$. Every $x \in K$ is of the form $x = \sum_{m=-\infty}^{\infty} y_m$ where $y_m \in H_{\cdot m}$. Our assumption implies that $p_{1,m} y_m = y_m$. It follows, by definition of $P_{1,m}$

that $y_m \in M_1(H_{-1,m})$, hence $y_m = \sum_{n=-\infty}^{\infty} U_1^n x_{n,m}$ where $x_{n,m} \in H_{-1,m}$. Consequently, $x = \sum_{n,m=-\infty}^{\infty} U_1^n x_{n,m}$. Since for every n the sum $\sum_{m=-\infty}^{\infty} x_{n,m} \in H_{-1,\cdot}$, we get that $x \in M_1(H_{-1,\cdot})$ and the proof of Lemma 3 is complete.

Now, applying Theorems I.1.4 and I.2.1 of [1] by Lemma 3 to the operators $\tilde{T}_1$ and $\tilde{T}_2$ we get the following conditions.

(3.7) $K = M_1(U_1 H_{-1,\cdot}) \oplus R_1, \quad K = M_2(U_2 H_{\cdot,-1}) \oplus R_2,$

(3.8) $K_1^+ = M_1^+(U_1 H_{-1,\cdot}) \oplus R_1, \quad K_2^+ = M_2^+(U_2 H_{\cdot,-1}) \oplus R_2,$

(3.9) $K_1^+ = H_{0,\cdot} \oplus M_1^+(H_{1,\cdot}), \quad K_2^+ = H_{\cdot,0} \oplus M_2^+(H_{\cdot,1}),$

(3.10) $P_1 M_1^+(H_1) \subset M_1^+(U_1 H_{-1,\cdot}), \quad P_2 M_2^+(H_{\cdot,1}) \subset M_2^+(U_2 H_{\cdot,-1}),$

(3.11) if T_1 is completely nonunitary then $K = M_1(U_1 H_{-1,\cdot}) \vee M_1(H_{1,\cdot})$,
if T_2 is completely nonunitary then $K = M_2(U_2 H_{\cdot,-1}) \vee M_2(H_{\cdot,1})$,

(3.12) if T_1 is completely nonunitary then $R_1 = \overline{(I-P_1) M_1(H_{1,\cdot})}$,
if T_2 is completely nonunitary then $R_2 = \overline{(I-P_2) M_2(H_{\cdot,1})}$.

Now we define the following operators $\tilde{T}_{1,0} = U_2 T_{1,-1} U^{-1}_{2|U_2 H_{0,-1}}$ and $T_{2,0} = U_1 T_{2,-1} U^{-1}_{1|U_1 H_{-1,0}}$. We prove the lemma:

LEMMA 4. *Suppose that T_1 and T_2 are doubly commuting contractions on H. Let U_1, U_2 be a unitary dilation of T_1, T_2 as in Theorem 0. Then with notations as above the following conditions hold*:

(4.1) *$U_2 H_{\cdot,-1}$ reduces U_1 and P_1, $U_1 H_{-1,\cdot}$ reduces U_2 and P_2,*

(4.2) *$U_{1|U_2 H_{\cdot,-1}}$ is the minimal unitary dilation of $\tilde{T}_{1,0}$,*
$U_{1|U_1 H_{-1,\cdot}}$ is the minimal unitary dilation of $\tilde{T}_{2,0}$,

(4.3) *L_{--}, L_{+-} and L_{--}, L_{-+} are wandering subspaces for U_1 and U_2 respectively,*

(4.4) *$P_{1|U_1 H_{\cdot,-1}}$ is the orthogonal projection of $U_2 H_{\cdot,-1}$ onto $M_1(L_{--})$,*
$P_{2|U_2 H_{-1,\cdot}}$ is the orthogonal projection of $U_1 H_{-1,\cdot}$ onto $M_2(L_{--})$,

(4.5) *if T_1 is completely nonunitary then $\tilde{T}_{i,0}$ is completely nonunitary $(i = 1, 2)$,*

(4.6)
$$L^+_{\tilde{T}_{1,0}} = L_{+-}, \quad L^+_{\tilde{T}_{1,0}} = U_2 H_{-1,-1},$$
$$L^+_{\tilde{T}_{2,0}} = L_{-+}, \quad L^-_{\tilde{T}_{2,0}} = U_1 H_{-1,-1}.$$

Proof. Since $H_{\cdot,-1}$ reduces U_1 and P_1 (see (1.4) and definition of P_1) we have $U_1 U_2 H_{\cdot,-1} = U_2 U_1 H_{\cdot,-1} = U_2 H_{\cdot,-1}$ and $P_1 U_2 H_{\cdot,-1}$

$= U_2 P_1 H_{\cdot,-1} \subset U_2 H_{\cdot,-1}$. The last equality is true because P_1 is the orthogonal projection onto $M_1(H_{-1,\cdot})$ and $M_1(H_{-1,\cdot})$ reduces U_2. Since U_2 is unitary and P_1 is selfadjoint, the proof of (4.1) is complete.

Equality $P_1 U_2 H_{\cdot,-1} = U_2 P_1 H_{\cdot,-1}$ gives $P_1 U_2 H_{\cdot,-1} = U_2 M_1(H_{-1,-1}) = M_1(U_2 H_{-1,-1})$. This implies (4.4).

Let $P_{H_{0,-1}}$ be the projection of $H_{\cdot,-1}$ onto $U_1 H_{0,-1}$ and let $P_{U_2 H_{0,-1}}$ be the projection of $U_2 H_{\cdot,-1}$ onto $U_2 H_{0,-1}$. It is easy to see that for $x \in U_2 H_{0,-1}$ we have $P_{U_2 H_{0,-1}} x = U_2 P_{H_{0,-1}} U_2^{-1} x$. Suppose now that $x \in U_2 H_{0,-1}$. Then $x = U_2 y$ where $y \in H_{0,-1}$ and for every $n \geqslant 0$ $U_1^n x \in U_2 H_{-1}$ (see (1.4)). Now, by (2.1) we get $\tilde{T}_{1,0}^n x = U_2 T_{1,-1}^n U_2^{-1} x = U_2 T_{1,-1}^n y = U_2 P_{H_{0,-1}} U_1^n y = U_2 P_{H_{0,-1}} U_1^n U_2^{-1} x = U_2 P_{H_{0,-1}} U_2^{-1} U_1^n x = P_{U_2 H_{0,-1}} U_1^n x$ which proves that $U_{1|U_2 H_{0,-1}}$ is a dilation of $\tilde{T}_{1,0}$. Since $\bigvee_{n=-\infty}^{\infty} U_1^n U_2 H_{0,-1} = U_2\left(\bigvee_{n=-\infty}^{\infty} U_1^n H_{0,-1}\right) = U_2 H_{\cdot,-1}$ (by (2.1)), (4.2) is proved.

As an immediate consequence of (1.1) we get (4.3).

If T_i is completely nonunitary then by (2.2) $T_{i,-1}$ is completely nonunitary. Since $\tilde{T}_{i,0}$ is unitarily equivalent to $T_{i,-1}$ we get (4.5).

It is easy to see that $L_{\tilde{T}_{1,0}}^{+} = U_2 L_{T_{1,-1}}^{+}$. Then by (2.3) we have $L_{\tilde{T}_{1,0}}^{+} = L_{+-}$. The rest of (4.6) we prove analogically. This completes the proof of Lemma 4.

Now applying Theorems I.1.4 and I.2.1 of [1] to the operators $\tilde{T}_{1,0}$ and $\tilde{T}_{2,0}$, we get by Lemma 4 the following conditions:

(4.7) $\quad U_2 H_{\cdot,-1} = M_1(L_{--}) \oplus \tilde{R}_1, \quad U_1 H_{-1,\cdot} = M_2(L_{--}) \oplus \tilde{R}_2,$

(4.8) $\quad U_2 H_{\cdot,-1}^{+} = M_1^{+}(L_{--}) \oplus \tilde{R}_1, \quad U_1 H_{-1,\cdot}^{+} = M_2^{+}(L_{--}) \oplus \tilde{R}_2,$

(4.9)
$$U_2 H_{\cdot,-1}^{+} = U_2 H_{0,-1} \oplus M_1^{+}(L_{+-}),$$
$$U_1 H_{-1,\cdot}^{+} = U_1 H_{-1,0} \oplus M_2^{+}(L_{-+}),$$

(4.10)
$$P_{1|H_2 H_{\cdot,-1}} M_1^{+}(L_{+-}) \subset M_1^{+}(L_{--}),$$
$$P_{2|U_1 H_{-1,\cdot}} M_2^{+}(L_{-+}) \oplus M_2^{+}(L_{--}),$$

(4.11) if T_1 is completely nonunitary then $U_2 H_{\cdot,-1} = M_1(L_{--}) \vee M_1(L_{+-})$,
if T_2 is completely nonunitary then $U_1 H_{-1,\cdot} = M_2(L_{--}) \vee M_2(L_{-+})$,

(4.12) if T_1 is completely nonunitary then $\tilde{R}_1 = \overline{(I-P_1) M_1(L_{+-})}$,
if T_2 is completely nonunitary then $\tilde{R}_2 = \overline{(I-P_2) M_2(L_{-+})}$.

Therefore we obtain that there are bounded analytic functions $\{Q_1^-(z_1), L_{+-}, L_{--}\}$ and $\{Q_2^-(z_2), L_{-+}, L_{--}\}$ such that

(4.13) $\tilde{\Phi}_{1-} P_1 x = Q_1^- \tilde{\Phi}_{1+} x$ for every $x \in M_1(L_{+-})$ where for every $f \in L_1^2(L_{+-})$ we have $(Q_1^- f)(z_1) = Q_1^-(z_1) f(z_1)$,
$\tilde{\Phi}_{2-} P_2 x = Q_2^- \tilde{\Phi}_{2+} x$ for every $x \in M_2(L_{-+})$ where for every $f \in L_2^2(L_{-+})$ we have $(Q_2^- f)(z_2) = Q_2^-(z_2) f(z_2)$.

$\tilde{\Phi}_{i-}$ and $\tilde{\Phi}_{i+}$ are defined as follows:

$\tilde{\Phi}_{i-}$ is a unitary map of $M_i(L_{--})$ onto $L_i^2(L_{--})$ such that $\tilde{\Phi}_{i-}\left(\sum_{n=-\infty}^{\infty} U_i^n x_n\right) = \sum_{n=-\infty}^{\infty} z_i^n x_n$ with $x_n \in L_{--}$ $(i = 1, 2)$;

$\tilde{\Phi}_{1+}$ is a unitary map of $M_1(L_{+-})$ onto $L_1^2(L_{+-})$ such that $\tilde{\Phi}_{1+}\left(\sum_{n=-\infty}^{\infty} U_1^n x_n\right) = \sum_{n=-\infty}^{\infty} z_1^n x_n$ with $x_n \in L_{+-}$;

Φ_{2+} is a unitary map of $M_2(L_{-+})$ onto $L_2^2(L_{-+})$ such that $\tilde{\Phi}_{2+}\left(\sum_{n=-\infty}^{\infty} U_2^n x_n\right) = \sum_{n=-\infty}^{\infty} z_2^n x_n$ with $x_n \in L_{-+}$.

Our next result is the following:

LEMMA 5. *Suppose that T_1 and T_2 are doubly commuting contractions on H. Let U_1, U_2 be unitary dilations of T_1, T_2 as in Theorem 0. Then the following conditions hold true:*

(5.1) *if L is a wandering subspace for the pair U_1, U_2 then $M_1(L)$ and $M_2(L)$ are wandering subspaces for U_2 and U_1 respectively and $M(L) = M_1 M_2(L) = M_2 M_1(L)$,*

(5.2) *if T_1 and T_2 are completely nonunitary then*

$$K = M(H_{-1,-1}) \vee M(H_{-1,1}) \vee M(H_{1,-1}) \vee M(H_{1,1}),$$

(5.3) *P_1 commutes with U_2 and P_1; P_2 commutes with U_1,*

(5.4) $$(I-P_1)K_2^+ \subset K^+, \quad (I-P_2)K_1^+ \subset K^+,$$

(5.5) $$(I-P_1)M_1(H_{1.}^+) \subset K^+, \quad (I-P_2)M_2(H_{.1}^+) \subset K^+,$$

(5.6) $$(I-P_1)(I-P_2)K \subset K^+,$$

(5.7) $$P_1 P_2 K = M(L_{--}),$$

(5.8) $$P_1(I-P_2)K = \overline{(I-P_2)M_1 M_2(L_{-+})},$$
$$P_2(I-P_1)K = \overline{(I-P_1)M_2 M_1(L_{+-})},$$

(5.9) $$(I-P_1)(I-P_2)K = \overline{(I-P_1)(I-P_2)M(L_{++})},$$

$$(5.10)\quad K = M(L_{--})\oplus\overline{(I-P_1)M_1 M_2(L_{-+})}\oplus$$
$$\oplus\overline{(I-P_2)M_1 M_2(L_{+-})}\oplus\overline{(I-P_1)(I-P_2)M(L_{++})},$$

$$(5.11)\quad P_i K^+ \subset K^+, \quad (I-P_i)K^+ \subset K^+ \quad (i = 1, 2),$$

$$(5.12)\quad K^+ = M^+(L_{--})\oplus\overline{(I-P_1)M_1 M_2^+(L_{-+})}\oplus$$
$$\oplus\overline{(I-P_2)M_1^+ M_2(L_{+-})}\oplus\overline{(I-P_1)(I-P_2)M(L_{++})},$$

$$(5.13)\quad K^+\ominus H = M^+(L_{-+}) \vee M^+(L_{+-}) \vee \overline{(I-P_1)M_1 M_2^+(L_{++})} \vee$$
$$\vee\overline{(I-P_2)M_1^+ M_2(L_{++})}.$$

Proof. It is easy to see that to prove (5.1) it is sufficient to show that $M_1(L)$ is wandering for U_2. Let $n \neq 0$ and $x, y \in M_1(L)$. The vectors x and y have the form $x = \sum_{m=-\infty}^{\infty} U_1^m x_m$, $y = \sum_{k=-\infty}^{\infty} U_1^k y_k$ where $x_m, y_k \in L$. It follows that $(U_2^n x, y) = \sum_{k,m=-\infty}^{\infty} (U_2^n U_1^m x_m, U_1^k y_k) = \sum_{k,m=-\infty}^{\infty} (U_2^n U_1^{m-k} x_m, y_k)$. Evidently for every m, k, $|n|+|m-k| > 0$. Since L is a wandering subspace for the pair U_1, U_2, we have $(U_2^n U_2^{m-k} x_m, y_k) = 0$ and consequently $(U_2^n x, y) = 0$ which completes the proof of (5.1).

If T_1 is completely nonunitary then by (2.8) we have $H_{\cdot 1} = M_1(H_{-1,1}) \vee M_1(H_{1,1})$ and $H_{\cdot -1} = M_1(H_{-1,-1}) \vee M_1(H_{1,-1})$. Then, if, additionally, T_2 is completely nonunitary, by (3.11) we have

$$K = M_2\big(M_1(H_{-1,-1}) \vee M_1(H_{1,-1})\big) \vee M_2\big(M_1(H_{-1,1}) \vee M_1(H_{1,1})\big).$$

Now by (5.1) we get (5.2).

Since $H_{-1\cdot}$ reduces U_2 (see (1.4)) and U_2 doubly commutes with U_1, we see that $M_1(H_{-1\cdot})$ reduces U_2. Thus, by (3.6) we may conclude that P_1 commutes with U_2. Analogically we prove that P_2 commutes with U_1. Now by definition of P_1 we have that $H_{-1\cdot}$ reduces P_1. Since U_2 commutes with P_1, we have that $M_2(H_{\cdot -1})$ reduces P_1. Hence (3.6) gives us that P_1 and P_2 commute which completes the proof of (5.3).

Since $(I-P_1) = \bigoplus_{i=-\infty}^{\infty} (I-P_{1,i})$ and $K_2^+ = \bigoplus_{i=0}^{\infty} H_{\cdot i}$, by (2.4) and (2.5) we have

$$(I-P_1)K_2^+ = \bigoplus_{i=0}^{\infty} (I-P_{1,i})H_{\cdot i} = \bigoplus_{i=0}^{\infty} R_{1,i} \subset \bigoplus_{i=0}^{\infty} H_{\cdot i}^+ = K^+$$

which proves (5.3). Since $H_{1\cdot}^+ \subset K^+ \subset K_2^+$ and K_2^+ reduces U_1, $M_1(H_1^+) \subset K_2^+$. Now (5.4) implies (5.5).

It is a consequence of (5.4) that to prove (5.6) it is sufficient to show that $(I-P_2)K \subset K_2^+$. But this follows from (3.6), (3.7), and (3.8).

By definition of P_2 we have that $P_2 H_{-1\cdot} = P_{2,-1} H_{-1\cdot} = M_2(H_{-1,-1})$.

It follows, by the commutativity of P_2 and U_1 that $P_2 M_1(H_{-1.})$ $= M_1(P_2 H_{-1.}) = M_1 M_2(H_{-1,-1}) = M(H_{-1,-1}) = M(L_{--})$. Since $P_2 K$ $= M_1(H_{-1.})$, see (3.6), we have $P_1 P_2 K = P_2 P_1 K = P_2 M_1(H_{-1}) = M(L_{--})$ which proves (5.7).

By (3.6) we have $P_1 K = M_1(H_{-1.})$. Thus

$$P_1(I-P_2)K = (I-P_2)P_1 K = M_1((I-P_2)H_{-1.}).$$

If T_2 is completely nonunitary, then by (2.9) we have

$$(I-P_2)H_{-1.} = \overline{(I-P_2)M(H_{-1,1})}.$$

Consequently

$$P_1(I-P_2)K = \overline{(I-P_2)M_1 M_2(H_{-1,1})} = \overline{(I-P_2)M_1 M_2(L_{-+})}$$

which ends the proof of (5.8). If T_2 is completely nonunitary, by (3.12) we have

$$(I-P_1)(I-P_2)K = \overline{(I-P_1)(I-P_2)M_2(H_{.1})}.$$

If additionally T_1 is completely nonunitary then by (2.9),

$$(I-P_1)H_{.1} = \overline{(I-P_1)M_1(H_{1,1})}.$$

Consequently

$$(I-P_1)(I-P_2)K = \overline{(I-P_1)(I-P_2)M(L_{++})}$$

which proves (5.9).

Now, by (5.7), (5.8), (5.9), the evident equality

$$K = P_1 P_2 K \oplus P_1(I-P_2)K \oplus P_2(I-P_1)K \oplus (I-P_1)(I-P_2)K$$

implies (5.10).

Since $K^+ \subset K_2^+$, by (5.4) we have $(I-P_1)K^+ \subset K^+$. To prove inclusion $P_1 K^+ \subset K^+ = \bigoplus_{i=0}^{\infty} H_{.i}^+$ it is sufficient to show that $P_1 H_{.i}^+ \subset K^+$ for every $i \geq 0$. But $H_{.i}^+ = M_1^+(U_1 H_{-1,i}) \oplus (I-P_{1,i})H_{.1}$ (see (2.5) and (2.4)). Hence $P_1 H_{.i}^+ = M_1^+(U_1 H_{-1,i}) \subset H_{.i}^+ \subset K^+$. So we have proved (5.11). Since $H_{.i}^+$ is invariant for U_1 and $H_{i.}^+$ is invariant for U_2 (see (1.3)), $K^+ = \bigoplus_{i=0}^{\infty} H_{.i}^+$ $= \bigoplus_{i=0}^{\infty} H_{i.}^+$ is invariant for U_1 and U_2. By (0.2) we have that L_{--} $= U_1 U_2 H_{-1,-1} \subset K^+$. Consequently for every $n, m \geq 0$, $U_1^n U_2^m L_{--} \subset K^+$ which proves that $M^+(L_{--}) \subset K^+$. Since $M_1 M_2^+(L_{+-}) = M_2^+ M_1(L_{+-})$ $\subset M_2^+(H_{.-1}) \subset K_2^+$ (see (3.8)), by (5.4) we get that $(I-P_1)M_1 M_2^+(L_{+-})$ $\subset K^+$. Analogically we prove that $(I-P_2)M_1^+ M_2(L_{-+}) \subset K^+$. Finally by (5.6) we get that

$$M^+(L_{--}) \vee (I-P_1)M_1 M_2^+(L_{+-}) \vee (I-P_2)M_1^+ M_2(L_{-+}) \vee \\ \vee (I-P_1)(I-P_2)M(L_{++}) \subset K^+.$$

Suppose now that $x \in K^+$. Then $x = x_1+x_2+x_3+x_4$ where $x_1 = P_1 P_2 x$, $x_2 = P_1(I-P_2)x$, $x_3 = P_2(I-P_1)x$ and $x_4 = (I-P_1)(I-P_2)x$. It follows, by (5.11), that $x_i \in K^+$ $(i = 1, \ldots, 4)$. This implies that $x_1 \perp U_1^n U_2^m L_{--}$ if $n<0$ or $m<0$. But $x_1 = P_1 P_2 x \in P_1 P_2 K = M(L_{--})$ (see (5.7)). Consequently $x_1 \in M^+(L_{--})$. Since $x_2 \in K^+$, $(I-P_2)x_2 = x_2 \in U_1^n U_2^m L_{++}$ if $n<0$. It follows that $x_2 \perp (I-P_2) U_1^n U_2^m L_{++}$ if $m<0$. But $x_2 = P_1(I-P_2)x \in P_1(I-P_2)$, $K = (I-P_2)M(L_{++})$. Hence $x_2 \in (I-P_2)M_1 M_2^+(L_{++})$. Analogically we prove that $x_3 \in (I-P_1)M_2^+ M_1(L_{+-})$. Finally

$$x_4 = (I-P_1)(I-P_2)x \in (I-P_1)(I-P_2)K = (I-P_1)(I-P_2)M(L_{++}).$$

We have proved that

$$K^+ = M^+(L_{--}) \vee (I-P_1)M_1 M_2^+(L_{+-}) \vee (I-P_2)M_1^+ M_2(L_{-+}) \vee \\ \vee (I-P_1)(I-P_2)M(L_{++}).$$

Now, (5.12) is an easy consequence of (5.10).

Since $M_1 M_2^+(L_{++}) = M_2^+ M_1(L_{++}) \subset M_2^+(H_{.1}) = \bigoplus H_{.i} \subset K_2^+$, by (5.4) we get that $(I-P_1)M_1 M_2^+(L_{++}) \subset K^+$. Moreover, $(I-P_1)M_1 M_2^+(L_{++}) \subset (I-P_1)\bigoplus H_{.i} \subset \bigoplus H_{.i}$. If follows that $H \perp (I-P_1)M_1 M_2^+(L_{++})$. Analogically we prove that $(I-P_2)M_1^+ M_2(L_{++}) \subset K^+$ and $H \perp (I-P_2)M_1^+ M_2(L_{++})$. By (0.2) we have that $L_{-+} \subset K^+$ and $L_{+-} \subset K^+$. But K^+ is invariant for U_1 and U_2, hence $M^+(L_{-+}) \subset K^+$ and $M^+(L_{+-}) \subset K^+$. Since $M^+(L_{-+}) = M_2^+ M_1^+(L_{-+}) \subset M_2^+(H_{.1}) = \bigoplus H_{.i}$ and $M^+(L_{+-}) \subset M_1^+(H_{1.}) = \bigoplus H_{i.}$, we have that $H \perp M^+(L_{-+})$ and $H \perp M^+(L_{+-})$. We have proved that

$$M^+(L_{-+}) \vee M^+(L_{+-}) \vee (I-P_1)M_1 M_2^+(L_{++}) \vee \\ \vee (I-P_2)M_1^+ M_2(L_{++}) \subset K^+ \ominus H.$$

To finish the proof, it is sufficient to show that for every $i>1$, $H_{.i}^+$ and $H_{i.}^+$ are included in

$$M^+(L_{+-}) \vee M^+(L_{-+}) \vee (I-P_1)M_1 M_2^+(L_+ L_+) \vee (I-P_2)M_1^+ M_2(L_{++}).$$

Since for $i>1$, $H_{.i}^+ = U_2^{i-1} H_{.1}^+$ and the right-hand of inclusion which we have to prove is invariant for U_2, it is sufficient to show that

$$H_{.1}^+ \subset M^+(L_{-+}) \vee M^+(L_{+-}) \vee (I-P_1)M_1 M_2^+(L_{++}) \vee \\ \vee (I-P_2)M_1^+ M_2(L_{++}).$$

Since T_1 is completely nonunitary, by (2.5) and (2.9) we have

$$H_{.1}^{+} = M_1^{+}(L_{-+}) \oplus (I-P_1) M_1(L_{++})$$
$$\subset M_2^{+}(M_1^{+}(L_{-+})) \vee M_2^{+}(I-P_1) M_1(L_{++})$$
$$= M^{+}(L_{-+}) \vee (I-P_1) M_1 M_2^{+}(L_{++}).$$

Analogically we prove that $H_{i.}^{+} \subset M^{+}(L_{+-}) \vee (I-P_2) M_1^{+} M_2(L_{++})$ which finishes the proof of Lemma 5.

Now we need the following definitions: Let L be a wandering subspace for the pair U_1, U_2. Then: $\Phi_{i,L}$ denotes the unitary map of $M(L)$ onto $\oplus M_i(L)$ (the orthogonal sum of countably many copies of $M_i(L)$) such that $\Phi_{i,L}(\sum_{n=-\infty}^{\infty} U_j^n x_n) = \{x_n\}$, where $x_n \in M_i(L)$ $(i = 1, 2)$ $(j \in \{1, 2\} \setminus \{i\})$, $\tilde{\Phi}_{i,L}$ denotes the unitary map of $\oplus M_i(L)$ onto $L^2(L)$ $(i = 1, 2)$ such that $\Phi_{1,L}(\{\sum_{n=-\infty}^{\infty} U_1^n x_{n,m}\}) = \sum_{n,m=-\infty}^{\infty} z_1^n z_2^m x_{n,m}$ and ' $\Phi_{2,L}(\{\sum_{m=-\infty}^{\infty} U_2^m x_{nm}\}) = \sum_{n,m=-\infty}^{\infty} z_1^n z_2^m x_{n,m}$ where $x_{n,m} L$, $L^2(L) = \{f\colon\ \Gamma^2 \to L$ such that $\int_{\Gamma^2} \|f(z_1, z_2)\|^2 dm(z_1, z_2) < \infty\}$. Φ_{ab} denotes the unitary map of $M(L_{ab})$ onto $L^2(L_{ab})$ such that $\Phi_{ab}(\sum_{n,m=-\infty}^{\infty} U_1^n U_2^m x_{n,m}) = \sum_{n,m=-\infty}^{\infty} z_1^n z_2^m x_{n,m}$ where $x_{nm} \in L_{ab}$ $(a, b = +, -)$.

If Φ is a unitary map L onto L', then $\oplus\Phi$ denotes the unitary map of $\oplus L$ onto $\oplus L'$ such that $(\oplus\Phi)(\{x_n\}) = \{\Phi x_n\}$ where $x_n \in L$. With these definitions the following lemma is not difficult to prove:

LEMMA 6. *With the notations as above the following equalities hold true*:

$$\Phi_{--} = \tilde{\Phi}_{2,L_{--}} \circ \oplus\Phi_{2-} \circ \Phi_{2,L_{--}} = \tilde{\Phi}_{2,L_{--}} \circ \oplus\Phi_{1-} \circ \Phi_{1,L_{--}}, \tag{6.1}$$

$$\tilde{\Phi}_{-+} = \tilde{\Phi}_{2,L_{-+}} \circ \oplus\Phi_{2+} \circ \Phi_{2,L_{-+}} = \tilde{\Phi}_{1,L_{-+}} \circ \oplus\Phi_{1-} \circ \Phi_{1,L_{-+}}, \tag{6.2}$$

$$\Phi_{+-} = \tilde{\Phi}_{2,L_{+-}} \circ \oplus\Phi_{2-} \circ \Phi_{2,L_{+-}} = \tilde{\Phi}_{1,L_{+-}} \circ \oplus\Phi_{1+} \circ \Phi_{1,L_{+-}}, \tag{6.3}$$

$$\Phi_{++} = \tilde{\Phi}_{2,L_{++}} \circ \oplus\Phi_{2+} \circ \Phi_{2,L_{++}} = \tilde{\Phi}_{1,L_{++}} \circ \oplus\Phi_{1+} \circ \Phi_{1,L_{++}}. \tag{6.4}$$

Suppose now that T_1 and T_2 are completely nonunitary doubly commuting contractions on H. Let U_1, U_2 be their unitary dilation as in Theorem 0. Let $x \in M(L_{+-})$. Then $\Phi_{2,L_{--}} P_1 x = \{P_1 x_n\}_{n=-\infty}^{\infty}$ if $x = \sum_{n=-\infty}^{\infty} U_2^n x_n$ where $x_n = \sum_{m=-\infty}^{\infty} U_1^m x_{m,n} \in M_1(L_{+-}) \subset U_2 H_{-1}$. Thus, by (4.13)

$$(\oplus\Phi_{1-})(\Phi_{2,L_{--}}(P_1 x))(z) = \{(\Phi_{1-} P_1 x_{nm})(z)\}_{n=-\infty}^{\infty}$$
$$= \{Q_1^{-}(z)(\Phi_{1+} x_n)(z)\}_{n=-\infty}^{\infty}.$$

Consequently, by (6.1) we have

$$(\Phi_{--}P_1x)(z_1,z_2)=\sum_{n=-\infty}^{\infty} z_2^n Q_1^-(z_1)\sum_{n=-\infty}^{\infty} z_1^m x_{m,n}$$
$$=Q_1^-(z_1)\sum_{n,m=-\infty}^{\infty} z_1^m z_2^n x_{n,m}=Q_1^-(z_1)(\Phi_{+-}x)(z_1,z_2).$$

We have proved that:

(7.1) if $x\in M(L_{+-})$ then $\Phi_{--}P_1x=Q_1^-\Phi_{+-}x$ where for $f\in L^2(L_{+-})$,
$$(Q_1^-f)(z_1,z_2)=Q_1^-(z_1)f(z_1,z_2).$$

Analogically we prove:

(7.2) if $x\in M(L_{-+})$ then $\Phi_{--}P_2x=Q_2^-\Phi_{-+}x$ where for $f\in L^2(L_{-+})$
$$(Q_2^-f)(z_1,z_2)=Q_2^-(z_2)f(z_1,z_2),$$

(7.3) if $x\in M(L_{++})$ then $\Phi_{+-}P_2x=Q_2^+\Phi_{++}x$ where for $f\in L^2(L_{++})$,
$$(Q_2^+f)(z_1,z_2)=Q_2^+(z_2)f(z_1,z_2),$$

(7.4) if $x\in M(L_{++})$ then $\Phi_{-+}P_1x=Q_1^+\Phi_{++}x$ where for $f\in L^2(L_{++})$,
$$Q_1^+f(z_1,z_1)=Q_1^+(z_1)f(z_1,z_2).$$

Now, if $x\in M(L_{++})$ then $\Phi_{--}P_1P_2x=Q_1^-\Phi_{+-}P_2x=Q_1^-Q_2^+\Phi_{++}x$. Suppose now, that $x\in M(L_{+-})$. Then we have

$$\begin{aligned}\|(I-P_1)x\|^2&=\|x\|^2-\|P_1x\|^2=\|\Phi_{+-}x\|^2-\|\Phi_{--}P_1x\|^2\\&=\|\Phi_{+-}x\|^2-\|Q_1^-\Phi_{+-}x\|^2\\&=(\Phi_{+-}x,\Phi_{+-}x)-(Q_1^-\Phi_{+-}x,Q_1^-\Phi_{+-}x)\\&=\int\big((I_{L_{+-}}-Q_1^-(z_1)^*Q_1^-(z_1))(\Phi_{+-}x)(z_1,z_2),\\&\qquad\qquad(\Phi_{+-}x)(z_1,z_2)\big)dm(z_1,z_2).\end{aligned}$$

It follows, by putting $x=(\Phi_{+-})^{-1}\chi_E$ where χ_E is the charactersitic function of the measurable set E, that for almost all (z_1,z_2), we have $\big(I_{L_{+-}}-Q_1^-(z_1)^*Q_1^-(z_1)\big)\geqslant 0$. Consequently we can define $\Delta_1(z)=\big(I_{L_{+-}}-Q_1^-(z_1)^*Q_1^-(z_1)\big)^{1/2}$ and operator $\Delta_1\colon L^2(L_{+-})\to L^2(L_{+-})$ such that $(\Delta_1f)(z_1,z_2)=\Delta_1(z_1)f(z_1,z_2)$ for every $f\in L^2(L_{+-})$.

Now we have that for $x\in M(L_{+-})$, $\|(I-P_1x)\|=\|\Delta_1\Phi_{+-}x\|$. Consequently we can define a unitary map Φ_{us} of $(I-P_1)M(L_{+-})$ onto $\overline{\Delta_1L^2(L_{+-})}$ such that $\Phi_{us}(I-P_1)x=\Delta_1\Phi_{+-}x$ for every $x\in M(L_{+-})$. Analogically we can define $\Delta_2(z)=\big(I_{L_{-+}}-Q_2^+(z)^*Q_2^+(z)\big)^{1/2}$ for almost every $z\in\Gamma$, the operator $\Delta_2\colon L^2(L_{-+})\to L^2(L_{-+})$ such that $(\Delta_2f)(z_1,z_2)=\Delta_2(z_2)f(z_1,z_2)$ for every $f\in L^2(L_{-+})$ and the unitary map Φ_{su}: $(I-P_2)M(L_{-+})$ onto $\overline{\Delta_2L^2(L_{-+})}$ such that $\Phi_{su}(I-P_2)x=\Delta_2\Phi_{-+}x$ for every $x \in M(L_{-+})$. Suppose now that $x \in M(L_{++})$. Then we have

$$\begin{aligned}
&\|(I-P_1)(I-P_2)x\|^2\\
&\quad = \|x\|^2 - \|P_1 x\|^2 - \|P_2 x\|^2 + \|P_1 P_2 x\|^2\\
&\quad = \|\Phi_{++} x\|^2 - \|\Phi_{-+} P_1 x\|^2 - \|\Phi_{+-} P_2 x\|^2 + \|\Phi_{--} P_1 P_2 x\|^2\\
&\quad = \|\Phi_{++} x\|^2 - \|Q_1^+ \Phi_{++} x\|^2 - \|Q_2^+ \Phi_{++} x\|^2 + \|Q_1^- Q_2^+ \Phi_{++} x\|^2\\
&\quad = \int_{\Gamma^2} \big((I_L - Q_1^+(z_1)^* Q_1^+(z_1) - Q_2^+(z_2)^* Q_2^+(z_2) +\\
&\qquad + Q_2^+(z_2)^* Q_2^-(z_2)^* Q_1^-(z_1) Q_2^+(z_2))(\Phi_{++} x)(z_1, z_2),\\
&\qquad\qquad (\Phi_{++} x)(z_1, z_2)\big)\, dm(z_1, z_2).
\end{aligned}$$

It follows, by putting $x = (\Phi_{++})^{-1} \chi_E$ that we can define an operator function

$$\begin{aligned}
\Delta(z_1, z_2) = \big(I_L - Q_1^+(z_1)^* Q_1^+(z_1) - Q_2^+(z_2)^* Q_2^+(z_2) +\\
+ Q_2^+(z_2)^* Q_1^-(z_1)^* Q_2^-(z_1) Q_2^+(z_2)\big)^{1/2}
\end{aligned}$$

and an operator $\Delta\colon L^2(L_{++}) \to L^2(L_{++})$ such that $(\Delta f)(z_1, z_2) = \Delta(z_1, z_2) f(z_1, z_2)$ for every $f \in L^2(L_{++})$. Now, for $x \in M(L_{++})$ we have $\|(I-P_1)(I-P_2)x\| = \|\Delta\Phi_{++} x\|$, hence we can define the unitary map Φ_{uu} of $(I-P_1)(I-P_2)M(L_{++})$ onto $\overline{\Delta L^2(L_{++})}$ such that $\Phi_{uu}(I-P_1)(I-P_2)x = \Delta\Phi_{++} x$ for every $x \in M(L_{++})$.

Let $\Phi = \Phi_{--} \oplus \Phi_{us} \oplus \Phi_{su} \oplus \Phi_{uu}$. It is easy to see, by (5.10), that Φ is unitary map of K onto $L^2(L_{--}) \oplus \overline{\Delta_1 L^2(L_{+-})} \oplus \overline{\Delta_2 L^2(L_{-+})} \oplus \overline{\Delta L^2(L_{++})}$. Since there is a unitary equivalence between the spaces $H^2(L_{--}) = \{f\colon D^2 x \to L_{--}$ such that f is analytic and $\sup \int_{\Gamma^2} \|f(r_1 z_1, r_2 z_2)\|^2\, dm(z_1 z_2) < \infty,\ 0 + r_1, r_2 < 1\}$ and

$$\begin{aligned}
\{f \in L^2(L_{--}) \text{ such that } x_{nm} = 0 \text{ if } n < 0 \text{ or}\\
m < 0 \text{ where } x_{nm} \text{ are Fourier coefficients of } f\}
\end{aligned}$$

we can say that $H^2(L_{--})$ is a subspace of $L^2(L_{--})$. Analogically we can treat the space

$$\begin{aligned}
L^2 \otimes H^2(L) = \{f\colon \Gamma \times D \to L \text{ such that, for almost all } z_1 \in \Gamma,\ f(z_1, \cdot) \in H_1^2(L)\\
\text{and for all } z_2 \in D,\ f(\cdot, z_2) \in L_1^2(L)\}
\end{aligned}$$

as the subspace of $L^2(L)$ and $H^2 \otimes L^2(L)$ as the subspace of $L^2(L)$. Now it is easy to see by (5.12) that $\Phi K^+ = H^2(L_{--}) \oplus \Delta_1 L^2 \otimes H^2(L_{+-}) \oplus \Delta_2 H^2 \otimes \otimes L^2(L_{-+}) \oplus \Delta L^2(L_{++})$. Consequently, by (5.13) we get that

$$\Phi H = H^2(L_{--}) \oplus \Delta_1 L^2 \otimes H^2(L_{+-}) \oplus \Delta_2 H^2 \otimes L^2(L_{-+}) \oplus \Delta L^2(L_{++}) \ominus$$

$$\ominus\{\Phi x \text{ where } x\in M^+(L_{-+}) \vee M^+(L_{+-})\overline{(I-P_1)M_1M_2^+(L_{++})} \vee \vee \overline{(I-P_2)M_1^+M_2(L_{++})}\}.$$

Suppose now that x has the form $x = x_1\oplus x_2\oplus(I-P_1)x_3+(I-P_2)x_4$ where $x_1\in M^+(L_{+-})$, $x_2\in M^+(L_{-+})$, $x_3\in M_1M_2^+(L_{++})$ and $x_4\in M_1^+M_2(L_{++})$. Then

$$\begin{aligned} x &= P_1P_2x_1\oplus P_1P_2x_2\oplus P_1(I-P_2)x_1+P_1(I-P_2)x_3+P_2(I-P_1)x_2+\\ &\quad +P_2(I-P_1)x_4+(I-P_1)(I-P_2)x_3+(I-P_1)(I-P_2)x_4\\ &= P_1x_1+P_2x_2+(I-P_1)x_1+P_2(I-P_1)x_3+(I-P_2)x_2+P_1(I-P_2)x_4+\\ &\quad +(I-P_1)(I-P_2)x_3+(I-P_1)(I-P_2)x_4. \end{aligned}$$

Consequently

$$\begin{aligned} \Phi x &= \Phi_{--}P_1x_1+\Phi_{--}P_2x_2+\Phi_{us}(I-P_1)x_1+\\ &\quad +\Phi_{us}(I-P_1)P_2x_3+\Phi_{su}(I-P_2)x_2+\Phi_{su}(I-P_2)P_1x_4+\\ &\quad +\Phi_{uu}(I-P_1)(I-P_2)(x_3+x_4)\\ &= Q_1^-\Phi_{+-}x_1+Q_2^-\Phi_{-+}x_2+\Delta_1\Phi_{++}x_2+\Delta_1Q_2^+\Phi_{++}x_3+\\ &\quad +\Delta_2\Phi_{-+}x_2+\Delta_2Q_1^+\Phi_{++}x+\Delta\Phi_{++}(x_3+x_4). \end{aligned}$$

Now it is easy to see that Φ_x, where $x\in M^+(L_{+-})\vee M^+(L_{-+})$, $(M-P_1)M_1M_2^+(L_{++})\vee(I-P_2)M_1^+M_2(L_{++})$ has the form

$$\Phi x = (Q_1^-f_1+Q_2^-f_2)\oplus(\Delta_1f_1+\Delta_1Q_2^+f_3)\oplus(\Delta_2f_2+\Delta_2Q_1^+f_4)\oplus\Delta(f_3+f_4)$$

where $f_1\in H^2(L_{+-})$, $f_2\in H^2(L_{-+})$, $f_3\in L^2H^2(L_{++})$ and $f_4\in H^2L^2(L_{++})$. Let $V_i = U_i|_{K^+}$. It is easy to see that $K^+\ominus H$ is invariant for V_1 and V_2. Consequently, H is invariant for V_1^* and V_2^*. Since $U_1|_{H_{.0}}$ and $U_2|_{H_{0.}}$ are minimal unitary dilations of T_1 and T_2 respectively, $V_1|_{H_{.0}^+} = U_1|_{H_{.0}^+}$ and $V_1|_{H_0^+} = U_1|_{H_0^+}$ are minimal isometric dilations of T_1 and T_2 respectively. Since $H_{.0}^+$ reduces V_1 and $H_{0.}^+$ reduces V_2 we have $T_1^*x = (V_1|_{H_{.0}^+})^*x = V_1^*x$ and $T_2^* = (V_2|_{H_0^+})^*x = V_2^*x$ for every $x\in H$. So we have to find $\Phi V_1^*\Phi^{-1}$ and $\Phi V_2^*\Phi^{-1}$. First we shall show that $M^+(L_{--})$, $(I-P_1)M_1M_2^+(L_{+-})$, $(I-P_2)M_1^+M_2(L_{-+})$ and $(I-P_1)(I-P_2)M(L_{++})$ reduce V_1 and V_2. Since these spaces span K^+ (see (5.12)) it is sufficient to show that they are invariant for V_1 and V_2. It is easy to see that $M^+(L_{--})$ is invariant for U_1 and $U_2|_{M^+(L_{--})} = V_1|_{M^+(L_{--})}$ is a unilateral shift. Since the space $M_1M_2^+(L_{+-})$ reduces U_1 by the commutativity of U_1 and P_1 we get that $(I-P_1)M_1M_2^+(L_{+-})$ reduces U_1. Consequently $V_1|_{(I-P_1)M_1M_2^+(L_{+-})} = U_1|_{(I-P_1)M_1M_2^+(L_{+-})}$. By commutativity of U_1 and P_2 we get that $(I-P_2)M_1^+M_2(L_{-+})$ is an invariant subspace for U_1 and consequently $V_1|_{(I-P_2)M_1^+M_2(L_{-+})}$ is a unilateral shift. Since $M(L_{++})$ reduce U_1 we may conclude that $V_1|_{(I-P_1)(I-P_2)M(L_{++})} = U_1|_{(I-P_1)(I-P_2)M(L_{++})}$. It is easy to see that for $V_1|_{M^+(L_{--})}$, the space $M_2^+(L_{--})$ is the complete wandering subspace (a wandering subspace L for the isometry V on H is called *complete* if $H = \bigoplus_{n=0}^{\infty} V^n L$). Consequently, $V_1^*(\sum_{n=0}^{\infty} U_1^n x_n) = U_1^*(\sum_{n=0}^{\infty} U_1^n x_n - x_0)$ where $x_n \in M_2^+(L_{--})$. Similarly if $x_n \in M_2(L_{-+})$ then

$$V_1^*(I-P_1)(\sum_{n=-\infty}^{\infty} U_1^n x_n) = U_1^*(I-P_1)(\sum_{n=-\infty}^{\infty} U_1^n x_n - x_0).$$

If $x = \sum_{n,m=-\infty}^{\infty} U_1^n U_2^m x_{nm}$ where $x_{nm} \in L_{--}$ and $f = \Phi_{--} x$ then $f(z_1, z_2) = \sum_{n,m=-\infty}^{\infty} z_1^n z_2^m x_{nm}$. It follows that

$$(\Phi U_1^* x)(z_1, z_2) = (\Phi_{--} U_1^* x)(z_1, z_2) = \Phi(\sum_{n,m=-\infty}^{\infty} U_1^{n-1} U_2^m x_{nm})(z_1, z_2)$$

$$= \sum_{n,m=-\infty}^{\infty} z_1^{n-1} z_2^m x_{nm} = 1/z_1(\sum_{n,m=-\infty}^{\infty} z_1^n z_2^m x_{nm}) = 1/z_1 f(z_1, z_2).$$

Evidently $\Phi(\sum_{n=-\infty}^{\infty} U_2^m x_{0m}) = \sum_{m=-\infty}^{\infty} z_2^m x_{0m} = f(0, z_2)$. Consequently, we have that $(\Phi V_1^* \Phi^* f)(z_1, z_2) = 1/z_1 (f(z_1, z_2) - f(0, z_2))$ for every $f \in H^2(L_{--})$. A similar proof shows that $\Phi V_1^* \Phi^{-1}$ has the same form for $f \in H^2 L^2(L_{-+})$. It is easy to see that if $\Delta_1 L^2 H^2(L_{+-})$ or if $f \in L^2(L_{++})$ then $(\Phi V_1 \Phi^{-1})(f)(z_1, z_2) = 1/z_1 (f)(z_1, z_2)$. Analogically we prove that for $f \in H^2(L_{--})$ or $f \in \Delta_1 L^2 H^2(L_{+-}) (\Phi V_2^* \Phi^{-1}) f(z_1, z_2) = 1/z_2 (f(z_1, z_2) - -f(z_1, 0))$ and for $f \in \Delta_2 H^2 L(L_{-+})$ or $f \in \Delta L^2(L_{++})$, $(\Phi V_2 \Phi^{-1})(f)(z_1, z_2) = 1/z_2 f(z_1, z_2)$. Since at the beginning we could have put T_1^* and T_2^* in place of T_1 and T_2, we have proved the following

THEOREM 1. *Suppose that T_1 and T_2 are completely nonunitary doubly commuting contractions on the separable Hilbert space H. Let U_1, U_2 be the minimal unitary dilation of the pair T_1, T_2, as in Theorem 0. Then there are bounded analytic operator functions $\{Q_1^+, L_{++}, L_{-+}\}$, $\{Q_1^-, L_{+-}, L_{--}\}$, $\{Q_2^+, L_{++}, L_{+-}\}$ and $\{Q_2^-, L_{-+}, L_{--}\}$ such that the space H is unitarily equivalent to the space*

$$H = H^2(L_{--}) \oplus \Delta_1 L^2 H^2(L_{+-}) \oplus \Delta_2 H^2 L^2(L_{-+}) \oplus \Delta L^2(L_{++}) \ominus$$

$$\ominus \{(Q_1^- f_1 + Q_2^- f_2) \oplus (\Delta_1 f_1 + \Delta_1 Q_2^+ f_3) \oplus (\Delta_2 f_2 + \Delta_2 Q_1^+ f_4) \oplus \Delta (f_3 + f_4)$$

where $f_1 \in H^2(L_{+-})$, $f_2 \in H^3(L_{-+})$, $f_3 \in L^2 H^2(L_{++})$ and $f_4 \in H^2 L^2(L_{++})$\}

where

$$L_{--}=\overline{(I-U_1T_1^*-U_2T_2^*+U_1U_2T_1^*T_2^*)H},$$
$$L_{+-}=\overline{(U_1-T_1-U_1U_2T_2^*+U_2T_1T_2^*)H},$$
$$L_{-+}=\overline{(U_2-T_2-U_1U_2T_1^*+U_1T_1^*T_2)H},$$
$$L_{++}=\overline{(U_1U_2-U_2T_2-U_2T_1+T_1T_2)H},$$
$$(Q_i^af)(z_1,z_2)=Q_i^a(z_i)f(z_1,z_2)\quad(i=1,2,\ a=+,-),$$
$$\Delta_if(z_1,z_2)=\Delta_i(z_i)f(z_1,z_2),\quad \Delta_i(z)=\left(I-Q_i^+(z)^*Q_i^+(z)\right)^{1/2}\quad(i=1,2),$$
$$(\Delta f)(z_1,z_2)=\Delta(z_1,z_2)f(z_1,z_2),$$
$$\Delta(z_1,z_2)=\big(I-Q_1^+(z_1)^*Q_1^+(z_1)-Q_2^+(z_2)^*Q_2^+(z_2)+$$
$$+Q_2^+(z_2)^*Q_1^-(z_2)Q_1^-(z_1)Q_2^+(z_2)\big)^{1/2}.$$

Operators T_1 and T_2 are unitarily equivalent to the operators $\tilde{T}_1$ and $\tilde{T}_2$ on H such that for $f=f_1+f_2+f_2+f_4$ where

$$f_1\in H^2(L_{--}),\quad f_2\in\overline{\Delta_1L^2H^2(L_{+-})},\quad f_3\in\overline{\Delta_2H^2L^2(L_{-+})},\quad f_4\in\overline{\Delta L^2(L_{++})},$$
$$(\tilde{T}_1f_1)(z_1,z_2)=1/z_1\left(f_1(z_1,z_2)-f_1(0,z_2)\right),\quad(\tilde{T}_1f_2)(z_1,z_2)=1/z_1\left(f_2(z_1,z_2)\right),$$
$$(\tilde{T}_1f_3)(z_1,z_2)=1/z_1\left(f_3(z_1,z_2)-f_3(0,z_2)\right),\quad(\tilde{T}_1f_4)(z_1,z_2)=1/z_1\left(f_4(z_1,z_2)\right),$$
$$(\tilde{T}_2f_1)(z_1,z_2)=1/z_2\left(f_1(z_1,z_2)-f_1(z_1,0)\right),\quad(\tilde{T}_2f_3)(z_1,z_2)=1/z_2\left(f_3(z_1,z_2)\right),$$
$$(\tilde{T}_2f_2)(z_1,z_2)=1/z_2\left(f_2(z_1,z_2)-f_2(z_1,0)\right),\quad(\tilde{T}_2f_4)(z_1,z_2)=1/z_2\left(f_4(z_1,z_2)\right).$$

Now we consider the case where T_1 is completely nonunitary and T_2 is unitary. If U_1 is the minimal unitary dilation of T_1 then there is exactly one extension U_2 of T_2 such U_2 commutes with U_1 and U_2 is a extension of T_2. Moreover, U_2 is unitary and the pair U_1, U_2 is the minimal unitary dilation of the pair T_1, T_2. By Theorem VI.2.1 there is a bounded analytic function Q, $L_{T_1}^+$, U, $L_{T_1}^-$ such that H is unitarily equivalent to the space

$$H=H_1^2(L_{T_1}^-)\oplus\Delta L_1^2(L_{T_1}^+)\ominus\{Qf+\Delta f\text{ where }f\in H_1^2(L_{T_2}^+)\}$$

where $\Delta(z)=\left(I-(Q)(z)^*Q(z)\right)^{1/2}$ and T_1 is unitarily equivalent to the operator T_1 on such that for $f=f_1+f_2$, where $f_1\in H_1^2(U_1L_{T_1}^-)$ and $f_2\in L_1^2(L_{T_1}^+)$, $(T_1f_1)(z)=1/z\left(f_1(z)-f_1(0)\right)$ and $(T_1f_2)(z)=1/z\left(f_2(z)\right)$. Now if we apply Lemma V.3.2 to the operators $U|_{M_1^+(U_1L_{T_1}^-)}$ and $U|_{M_1^+(L_{T_1}^-)}$ we get the following:

THEOREM 2. *Suppose that T_1 is completely nonunitary contraction on H. Let U be unitary operator on H which commutes with T_1. If U_1 is the minimal unitary dilation of T_1, then there is the analytic bounded operator function*

$\{Q, L_+, L_+\}$ such that H is unitarily equivalent to the space

$$H = \left(H_1^2(L_-) \oplus L_1^2(L_+)\right) \ominus \{Qf + \Delta f \text{ where } f \in H_1^2(L_+)\}$$

where $L_- = \overline{(I - U_1 T_1^*) H}$, $L_+ = \overline{(U_1 - T_1) H}$, $\Delta(z) = \left(I - Q(z)^* Q(z)\right)^{1/2}$. *Operators T_1 and U are unitarily equivalent to the operators $\tilde{T}_1$ and $\tilde{U}$, such that for $f = f_1 + f_2$ where $f_1 \in H_1(L_-)$ and $f_2 \in \Delta L_1(L_+)$ we have*

$$(\tilde{T}_1 f_1)(z) = 1/z\left(f_1(z) - f_1(0)\right), \quad (\tilde{T}_1 f_2)(z) = 1/z\left(f_2(z)\right),$$

$$\tilde{U} f_1(z) = U_- f_1(z) \quad \text{and} \quad (\tilde{U} f_2)(z) = U_+ f_2(z)$$

where, for $x \in H$,

$$U_-(I - U_1 T_1)x = (I - U_1 T_1)Ux \quad \text{and} \quad U_+(U_1 - T_1)x = (U_1 - T_1)Ux.$$

References

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