Estimates for the coefficients of polynomials and trigonometric polynomials

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1. Let \( P_n(z) = \sum_{r=0}^{n} a_r z^r \) be a polynomial of degree \( n \) such that \(|P_n(z)| \leq M\) for \(|z| = 1\). By a very simple method Visser [13] proved that

\[
|a_0| + |a_n| \leq M.
\]

By the method of Visser [13], van der Corput and Visser [4] later proved the following more general

**THEOREM A.** Let \( P_n(z) = \sum_{r=0}^{n} a_r z^r \) be a polynomial of degree \( n \) such that \(|P_n(z)| \leq M\) for \(|z| \leq 1\). If \( a_u, a_v \) \((u < v)\) are two coefficients such that for no other coefficient \( a_w \neq 0 \) we have \( w = u \mod (v - u) \), then

\[
|a_u| + |a_v| \leq M.
\]

Also

\[
|a_u| + |a_v| \leq \frac{1}{\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})| \, d\theta.
\]

Besides, they proved that if \( F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta} \) is a real trigonometric polynomial, i.e. \( c_{-k} = c_k \), then for \( k > n/2 \)

\[
|c_0| + 2|c_k| \leq \max |F(\theta)|,
\]

\[
|c_0| + \frac{2}{3}|c_k| \leq \frac{1}{\pi} \int_{0}^{2\pi} |F(\theta)| \, d\theta.
\]

The constant \( \frac{1}{4} \) in (1.5) was improved by Boas [2] to \( \frac{1}{2}(1 + \sqrt{2}/\pi) = 0.234... \). He later obtained [3] the best possible result and proved that for \( k > n/2 \) and any positive \( \gamma \)

\[
|c_0| + 2\gamma|c_k| \leq A_\gamma \int_{0}^{2\pi} |F(\theta)| \, d\theta,
\]
where $A_\gamma$ is given by

\begin{equation}
A_\gamma = \frac{1}{2\pi - 4\varphi},
\end{equation}

and $\varphi$ is the smallest positive root of $\sin \varphi = \frac{1}{2} \gamma (\pi - 2\varphi)$.

The more general problem of determining the best possible estimate for

$$
\max \frac{\lambda_0 |c_0| + \lambda_k |c_k|}{\int_0^{2\pi} |F(\theta)| d\theta}, \quad 0 < k \leq n,
$$

where $\lambda_0, \lambda_k$ are given non-negative numbers was solved by Geronimus [5]. He proved in particular the following:

**THEOREM B.** If $F(\theta) = \sum_{k=-n}^{n} c_k e^{i k \theta}$ is a real trigonometric polynomial, then

$$
|c_k| \leq \frac{1}{8h} \int_0^{2\pi} |F(\theta)| d\theta,
$$

where $h$ is the smallest positive root of the algebraic equation

\begin{equation}
\begin{vmatrix}
m_0 & m_1 & \cdots & m_\beta \\
m_1 & m_0 & \cdots & m_{\beta-1} \\
\vdots & \vdots & \ddots & \vdots \\
m_{-\beta} & m_{-\beta-1} & \cdots & m_0
\end{vmatrix} = 0
\end{equation}

with $m_0 = 2$, $m_s = m_{-s} = (2h)^s/s!$ ($s = 1, 2, \ldots$, $\beta = [n/k]$). The result is the best possible.

In the spacial case, when $k > n/3$ this result gives the estimate

\begin{equation}
|c_k| \leq \frac{1}{8} \int_0^{2\pi} |F(\theta)| d\theta,
\end{equation}

whereas, for $n/3 \geq k > n/5$ we get

\begin{equation}
|c_k| \leq 0.142 \int_0^{2\pi} |F(\theta)| d\theta.
\end{equation}

Generalizing (1.3) and (1.6) respectively Rahman ([8], Theorem 2 and 3) has recently proved the following theorems:
THEOREM C. If \( P_n(x) = \sum_{r=0}^{n} a_r x^r \) is a polynomial of degree \( n \) and \( a_u, a_v \) (\( u < v \)) are two coefficients such that for no other coefficient \( a_w \neq 0 \) we have \( w = u \mod (v-u) \), then for every \( \delta \geq 1 \)

\[
|a_u| + |a_v| \leq 2 (C_\delta)^{1/\delta} \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta},
\]

where

\[
C_\delta = \frac{2\pi}{\int_0^{2\pi} |1+e^{i\varphi}|^\delta d\varphi} = \frac{2^{-\delta} \sqrt{\pi \Gamma(\frac{1}{2} \delta + 1)}}{\Gamma\left(\frac{\delta}{2} + \frac{1}{2}\right)}.
\]

The result is best possible.

THEOREM D. If \( F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta} \) is a real trigonometric polynomial and \( \delta \geq 1 \), then for \( k > n/2 \) and any positive \( \gamma \)

\[
|c_0| + 2\gamma |c_k| \leq (C_{\gamma,\delta})^{1/\delta} \left( \frac{1}{2\pi} \int_0^{2\pi} |F(\theta)|^\delta d\theta \right)^{1/\delta},
\]

where

\[
C_{\gamma,\delta} = \max_{0<\gamma<\infty} \frac{(1+2\gamma^\gamma)^\delta}{\frac{1}{2\pi} \int_0^{2\pi} |1+2r \cos \varphi|^\delta dr dr}.
\]

Real-valued trigonometric polynomials have been studied in detail by Rogosinski [9] and [10], and Mulholland [7]. The following theorem, equivalent to a result of Mulholland ([7], Theorem 1) stands in analogy with Theorem B.

THEOREM E. If \( F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta} \) is a real trigonometric polynomial such that \( |F(\theta)| \leq M \) for \( 0 \leq \theta < 2\pi \), then for \( k \geq 1 \)

\[
|c_k| \leq \frac{M}{l \cot \frac{\pi}{2l}}, \quad \text{where } l = \left\lfloor \frac{n}{2k} + \frac{3}{2} \right\rfloor.
\]

The maximum is attained, in particular, for the polynomial

\[
\frac{1}{l^2} \sum_{r=1}^{l-1} (-1)^{r-1} \left\{ e^{ik(2r-1)\theta} + e^{-ik(2r-1)\theta} \right\} \times \left\{ (2l-2r+1) \cot \frac{(2r-1)\pi}{2l} + \cot \frac{\pi}{2l} \right\}.
\]
THEOREM F. If \( P_n(x) = \sum_{r=0}^{n} A_r x^r \) is a polynomial of degree \( n \) such that \(|P_n(x)| \leq 1\) for \(-1 \leq x \leq 1\), then

\[
|A_n| \leq 2^{n-1} \quad \text{(Tchebycheff)},
\]

\[
|A_{n-1}| \leq 2^{n-2} \quad \text{(W. Markoff)}.
\]

Generalizing Theorems B and E we obtain an estimate for \(|c_k|\) in terms of

\[
\left( \frac{1}{2\pi} \int_{0}^{2\pi} |F'(\theta)|^\delta d\theta \right)^{1/\delta}
\]

for every \( \delta \geq 1 \).

THEOREM 1. If \( F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta} \) is a real trigonometric polynomial and \( \delta \geq 1 \), then for \( k \geq 1 \)

\[
|c_k| \leq (C_\delta) \left( \frac{2}{l} \cot \left( \frac{\pi}{2l} \right) \right) \left( \frac{1}{2\pi} \int_{0}^{2\pi} |F'(\theta)|^\delta d\theta \right)^{1/\delta},
\]

where

\[
l = \left[ \frac{n}{2k + \frac{3}{2}} \right] \quad \text{and} \quad C_\delta = \frac{2\pi}{\int_{0}^{2\pi} \left| 1 + e^{i\varphi} \right|^\delta d\varphi}.
\]

On letting \( \delta \to \infty \) we get Theorem E. However, we do not assert that the result is best possible. Its interest lies in the fact that for \( \delta \) other than 1 and 2 it is the first and only result of its kind. In order to have an idea as to how close it is to the correct estimate we compare it in the case \( \delta = 1 \) with the result of Geronimus (Theorem B) which is known to be precise.

In the special case \( \delta = 1 \) our theorem gives the estimate

\[
|c_k| \leq \frac{1}{4} \cot \left( \frac{\pi}{2k + \frac{3}{2}} \right) \int_{0}^{2\pi} |F'(\theta)| d\theta
\]

since \( C_1 = \pi/4 \). Thus for \( k > n/3 \) we get

\[
|c_k| \leq \frac{1}{4} \int_{0}^{2\pi} |F'(\theta)| d\theta
\]

which agrees with estimate (1.9) of Geronimus.
If \( k \leq n/3 \) but \( > n/5 \), then we obtain
\[
|c_k| \leq 0.144 \int_0^{2\pi} |F'(\theta)| \, d\theta
\]
which differs from the precise estimate (1.10)
\[
|c_k| \leq 0.142 \int_0^{2\pi} |F'(\theta)| \, d\theta
\]
by 1.4 per cent. Thus our estimate (1.13) though not best possible, appears to be fairly good.

Our next theorem gives estimate for linear combinations of any two coefficients of a polynomial \( P_n(z) \) bounded by \( M \) on the unit circle.

**Theorem 2.** If \( P_n(z) = \sum_{r=0}^{n} a_r z^r \) is a polynomial of degree \( n \) such that \( |P_n(z)| \leq M \) for \( |z| \leq 1 \) and \( 0 \leq u < v \leq n \), then for any real \( \lambda > 0 \)
\[
|a_u + \lambda a_v| \leq \frac{M}{2\pi} \int_0^{2\pi} |1 + \lambda e^{i\theta}| \, d\theta.
\]
(1.14)
The result is best possible.

If \( u \) and \( v \) satisfy a separation condition like \( 0 \leq 2u < v \leq n \), then the bound in (1.14) can be considerably improved.

**Theorem 3.** If \( P_n(z) = \sum_{r=0}^{n} a_r z^r \) is a polynomial of degree \( n \) such that \( |P_n(z)| \leq M \) for \( |z| \leq 1 \), then for any real \( \lambda > 0 \)
\[
|a_u + \lambda a_v| \leq M \left( \lambda + \frac{1}{4\lambda} \right) \text{ or } M \left( 1 + \frac{\lambda^2}{4} \right)
\]
(1.15)
according as \( 0 \leq 2u < v \leq n \) or \( 0 \leq u < 2v - n \leq n \) respectively.

The case \( \lambda = \frac{1}{2} \) is particularly interesting, for then the right-hand side in (1.15) reduces to \( M \) when \( 0 \leq 2u < v \leq n \). For \( u = 0 \), \( v = 1 \) we have the stronger

**Theorem 4.** If \( P_n(z) = \sum_{r=0}^{n} a_r z^r \) is a polynomial of degree \( n \) such that \( \text{Re } P_n(z) \leq A \) for \( |z| \leq 1 \), and \( a_0 \) is real, then
\[
a_0 + \frac{1}{2} |a_1| \leq A.
\]
(1.16)

In an attempt to generalize Theorem F by finding estimates for the coefficients in terms of
\[
\left( \frac{1}{2} \int_{-1}^{1} |P_n(x)|^4 \, dx \right)^{1/8}
\]
we have obtained the following.
THEOREM 5. If \( P_n(x) = \sum_{r=0}^{n} A_r x^r \) is a polynomial of degree \( n \), then for every \( \delta \geq 1, n = n, n-1 \)

\[
|A_r| \leq 2^{r+1} \left( \frac{2}{\pi} C_\delta \right)^{1/\delta} \left( \int_{-1}^{1} |P_n(x)|^{\delta} (1-x^2)^{(\delta-1)/2} dx \right)^{1/\delta},
\]

where \( C_\delta \) is given by (1.11).

The following corollary is immediate.

COROLLARY. If \( P_n(x) = \sum_{r=0}^{n} A_r x^r \) is a polynomial of degree \( n \), then for every \( \delta \geq 1, p > 1, q = p/(p-1) \) and \( n = n, n-1 \)

\[
|A_r| \leq 2^{r+1} \left( \frac{2}{\pi} C_\delta \right)^{1/\delta} \left( \frac{1}{2} \int_{-1}^{1} (1-x^2)^{(\delta-1)/2} dx \right)^{1/\delta} \left( \frac{1}{2} \int_{-1}^{1} |P_n(x)|^{\delta p} dx \right)^{1/\delta p}.
\]

2. For one of our results we shall need the following

**Lemma 1.** Let \( \pi_n \) denote the linear space of polynomials

\( P_n(z) = a_0 + a_1 z + \ldots + a_n z^n \)

of fixed degree \( n \) with complex coefficients, normed by \( \|P_n\| = \max_{0 < \theta < 2\pi} |P_n(e^{i\theta})| \).

Define the linear functional \( L \) on \( \pi_n \) as

\[
L: P_n \rightarrow l_0 a_0 + \ldots + l_n a_n,
\]

where the \( l_r \) are complex numbers. If the norm of the functional is \( N \), then for every polynomial \( P_n(z) = \sum_{r=0}^{n} a_r z^r \)

\[
\int_{0}^{2\pi} \varphi \left( \frac{\sum_{r=0}^{n} l_r a_r e^{i\theta}}{N} \right) \, d\theta \leq \int_{0}^{2\pi} \varphi \left( \sum_{r=0}^{n} \left| a_r e^{i\theta} \right| \right) \, d\theta,
\]

where \( \varphi(t) \) is a non-decreasing convex function of \( t \).

This was proved by Shapiro ([11], Theorem 8) for the case \( \varphi(t) = t \).

**Proof of Lemma 1.** According to a theorem of Shapiro ([11], Theorem 3) \( L \) can be represented in the form

\[
L[P_n(z)] = \sum_{k=1}^{r} u_k P_n(z_k),
\]

where
for all $P_n \in \pi_n$, where $z_1, \ldots, z_r$ are distinct numbers of modulus 1 and $\sum_{k=1}^{r} |u_k| = N$. Let $\xi$ be any number of modulus one, and apply this to the polynomial $P_n(\xi z)$. We get

$$\left| \sum_{r=0}^{n} l, a_r, \xi^r \right| = \left| \sum_{r=1}^{r} u_r P_n(z, \xi) \right| \leq \sum_{r=1}^{r} |u_r| |P_n(z, \xi)| .$$

Hence if $\varphi(t)$ is a non-decreasing convex function of $t$, then by Jensen's inequality ([6], pp. 150-151)

$$\varphi \left( \frac{\sum_{r=1}^{r} |u_r| |P_n(z, \xi)|}{\sum_{r=1}^{r} |u_r|} \right) \leq \frac{\sum_{r=1}^{r} |u_r| \varphi \{|P_n(z, \xi)|\}}{\sum_{r=1}^{r} |u_r|} .$$

Setting $\xi = e^{i\theta}$ and integrating both sides with respect to $\theta$ from 0 to $2\pi$ we get the result.

3. Proof of Theorem 1. According to a theorem of Rahman ([8], Theorem 7) if $P_n(z) = \sum_{r=0}^{n} a_r z^r$ is a polynomial of degree $n$ such that $|P_n(z)| \leq M$ for $|z| \leq 1$, then for $\nu < n/2$

$$|a_r| + |a_{n-r}| \leq \frac{2M}{l} \cot \frac{\pi}{2l} ,$$

where $l = \left[ \frac{n - 2\nu + 3/2}{2(n-2\nu)} \right]$.

Since $|a_r + a_{n-r}| \leq |a_r| + |a_{n-r}|$ the norm of the functional $L$: $P_n \rightarrow 0 \cdot a_0 + \ldots + 0 \cdot a_{r-1} + 1 \cdot a_r + 0 \cdot a_{r+1} + \ldots + 0 \cdot a_{n-r-1} + 1 \cdot a_{n-r} + 0 \cdot a_{n-r+1} + \ldots + 0 \cdot a_n$

does not exceed $\frac{2}{l} \cot \frac{\pi}{2l}$. Applying the lemma with $\varphi(t) = t^{\delta}$ we get for every $\delta > 1$

$$\left( \frac{1}{2\pi} \int_{0}^{2\pi} |a_r e^{i\theta} + a_{n-r} e^{i(n-r)\theta}| d\theta \right)^{1/\delta} \leq \frac{2}{l} \cot \frac{\pi}{2l} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |P_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta} .$$

But by Theorem C

$$|a_r| + |a_{n-r}| \leq 2(C\delta)^{1/\delta} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |a_r e^{i\theta} + a_{n-r} e^{i(n-r)\theta}| d\theta \right)^{1/\delta} .$$
Therefore

\[ |a_r| + |a_{n-r}| \leq 2(C\delta)^{1/\alpha} \frac{\pi}{2\theta} \left( \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^{1/\alpha} d\theta \right)^{1/\alpha}. \]

If \( F(\theta) = \sum_{k=n}^{n} c_k e^{i\theta} \) is a real trigonometric polynomial of degree \( n \), then \( e^{i\alpha\theta}F(\theta) \) is a polynomial \( \sum_{r=0}^{2n} b_r e^{i\theta} \) of degree \( 2n \). On applying (3.2) to the polynomial \( e^{i\alpha\theta}F(\theta) = \sum_{r=0}^{2n} b_r e^{i\theta} \) and noting that \( |b_r| = |b_{2n-r}| \) for \( 0 \leq r \leq n \) we shall get the desired result.

**Proof of Theorem 2.** If \( 0 \leq u < v \leq n, \ 0 < \alpha < 2\pi \) and \( \lambda \) any positive real number, then

\[
|a_u e^{iu\theta} + \lambda a_v e^{iv\theta}| = \left| \frac{1}{2\pi i} \int_{|z|=1} P_n(z e^{iu\theta}) \left[ \frac{1}{z^{u+1}} + \frac{\lambda}{z^{v+1}} \right] dz \right|
\leq \frac{M}{2\pi} \int_0^{2\pi} |1 + \lambda e^{iu\theta}| d\theta.
\]

and by choosing \( \alpha \) such that \( |a_u e^{iu\theta} + \lambda a_v e^{iv\theta}| = |a_u| + \lambda |a_v| \) we shall get the result.

In order to prove the estimate is best possible let \( f(\theta) = e^{i\tan^{-1} \left( \frac{\lambda \sin \theta}{1 + \lambda \cos \theta} \right)} \) for \( 0 \leq \theta < 2\pi \) (we choose any fixed branch of \( \tan^{-1} \left( \frac{\lambda \sin \theta}{1 + \lambda \cos \theta} \right) \)) have the Fourier series representation

\[ f(\theta) = \sum_{-\infty}^{\infty} b_r e^{ir\theta}. \]

Then

\[
b_0 + \lambda b_1 = \frac{1}{2\pi} \int_0^{2\pi} \exp \left[ \tan^{-1} \left( \frac{\lambda \sin \theta}{1 + \lambda \cos \theta} \right) \right] (1 + \lambda e^{-i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} |1 + \lambda e^{i\theta}| d\theta.
\]

For every \( n > 0 \), we have by a classical result of Fejér ([12], p. 440, Ex. 9)

\[
\left| \sum_{-n}^{n} \frac{(n+1) - |v|}{n+1} b_v e^{iv\theta} \right| \leq 1
\]
and so the Laurent polynomial

\[ L(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \frac{(n+1)-|\nu|}{n+1} b_n e^{in\theta} = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \]

is such that \(|L(e^{i\theta})| \leq 1\) for \(0 \leq \theta < 2\pi\) and

\[ |a_0| + \lambda |a_1| = |b_0| + \frac{\lambda n}{n+1} |b_1| > \frac{1}{2\pi} \int_{0}^{2\pi} |1 + \lambda e^{i\theta}| d\theta - \varepsilon, \]

where \(\varepsilon\) can be made arbitrary small by choosing \(n\) sufficiently large. Thus \(z^n L(z)\) is the polynomial which we wanted to construct.

Proof of Theorem 3. It is known ([1], pp. 140) that if \(f(z) = \sum_{r=0}^{\infty} a_r z^r\) is regular and \(|f(z)| \leq M\) in \(|z| \leq 1\), then

\begin{equation}
|a_1| \leq M - \frac{|a_0|^2}{M}.
\end{equation}

Now for \(0 \leq 2u < v \leq n\), putting \(P(z) = z^{v-2u} P_n(z)\), \(w = e^{2\pi i/v-u}\) and \(\xi = z^{v-u}\), the function

\[ \frac{P(z) + P(wz) + \ldots + P(w^{v-u-1}z)}{(v-u)z^{v-u}} = a_u + a_v z^{v-u} + a_{2v-u} z^{2(v-u)} + \ldots \]

of \(\xi\) is regular and in absolute value \(\leq M\) in \(|\xi| \leq 1\). Hence by (3.3)

\[ |a_v| \leq M - \frac{|a_u|^2}{M} \]

provided \(0 \leq 2u < v \leq n\). Thus for \(\lambda > 0\) and \(0 \leq 2u < v \leq n\),

\[ |a_u| + \lambda |a_v| \leq \lambda M + |a_u| - \frac{\lambda |a_u|^2}{M}. \]

Whatever be the value of \(|a_u|\) the right-hand side of the last inequality is \(\leq M(\lambda + 1/4\lambda)\) and so the first part of the theorem follows. For the second part we may consider \(z^n P_n(1/z)\) instead of \(P_n(z)\).

In order to prove (1.16) we note that the function \(A - P_n(z)\) is regular for \(|z| < 1\), where its real part is positive (we assume that \(P_n(z)\) is not a constant; clearly (1.16) is true even if \(P_n(z)\) is a constant). It is well known (see for example [12], pp. 194-195) that if \(f(z)\) is regular for \(|z| < 1\),
Re \{f(z)\} > 0, and \( f(0) = a > 0 \), then \(|f'(0)| \leq 2a\). Applying this result to the function \( A - P_n(x) \) we get

\[ |a_1| \leq 2(A - a_0) \quad \text{or} \quad a_0 + \frac{1}{2}|a_1| \leq A. \]

Proof of Theorem 5. We denote by

\[ V_0(x), V_1(x), \ldots \]

the polynomials defined by

\[ V_r(t) = \frac{\sin((r+1)t)}{\sin t}, \]

where \( x = \text{cost.} \)

The polynomial \( P_n(x) \) has a unique expansion

\[ P_n(x) = \sum_{r=0}^{n} d_r V_r(x). \]

We put

\[ G(t) = P_n(\text{cost}) \sin t = \sum_{r=0}^{n} d_r \sin((r+1)t) \]

\[ = \sum_{r=0}^{n} d_r \frac{e^{i(r+1)t} - e^{-i(r+1)t}}{2i} \]

\[ = \sum_{r=-n-1}^{n+1} c_r e^{i rt}, \]

where \( c_0 = 0, -c_{-r} = c_r = \frac{1}{2i} d_{r-1} \) for \( r \geq 1 \).

If \( s \) and \( m \) are two integers of which the first is positive, then ([4], pp. 383)

\[ \sum_{p=0}^{s-1} e^{-\frac{2\pi ipm}{s}} G(t + \frac{2\pi p}{s}) = s \sum_{r=m \pmod{s}} c_r e^{i rt}. \]

Putting \( s = 2N \) and \( m = -N \) for all \( N \) for which \( N > \frac{n+1}{3} \) the left-hand side reduces to \( 2N c_N(e^{iNt} - e^{-iNt}) \).

Clearly, for any complex \( \lambda \),

\[ \int_{0}^{2\pi} |1 + e^{i(\lambda - \arg \lambda)}| \delta t = \int_{0}^{2\pi} |1 + e^{i\lambda}| \delta t. \]
so that

\[ |\lambda| \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{it}|^3 \, dt \right)^{1/3} = \left( \frac{1}{2\pi} \int_0^{2\pi} |\lambda + \bar{\lambda}e^{it}|^3 \, dt \right)^{1/3}. \]

Thus for every \( \delta \geq 1 \) and \( 0 \leq \gamma < 2\pi \),

\[ |ae^{-i\gamma} + \bar{b}e^{i\gamma}| \left( \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{it}|^3 \, dt \right)^{1/3} \]

\[ = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| (ae^{-i\gamma} + \bar{b}e^{i\gamma}) + (be^{-i\gamma} + \bar{a}e^{i\gamma})e^{it} \right|^3 \, dt \right)^{1/3} \]

\[ = \left( \frac{1}{2\pi} \int_0^{2\pi} \left| e^{-i\gamma}(a + be^{it}) + e^{i\gamma}(\bar{b} + \bar{a}e^{it}) \right|^3 \, dt \right)^{1/3} \]

\[ \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |a + be^{it}|^3 \, dt \right)^{1/3} + \left( \frac{1}{2\pi} \int_0^{2\pi} |b + ae^{-it}|^3 \, dt \right)^{1/3} \]

\[ = 2 \left( \frac{1}{2\pi} \int_0^{2\pi} |a + be^{it}|^3 \, dt \right)^{1/3}. \]

Choosing \( \gamma \) such that \( |ae^{-i\gamma} + \bar{b}e^{i\gamma}| = |a| + |b| \) we get

\[ (3.5) \quad |a| + |b| \leq 2 \left( C_3 \right)^{1/3} \left( \frac{1}{2\pi} \int_0^{2\pi} |a + be^{it}|^3 \, dt \right)^{1/3} \]

\[ = 2 \left( C_3 \right)^{1/3} \left( \frac{1}{2\pi} \int_0^{2\pi} |ae^{itN} + be^{-itN}|^3 \, dt \right)^{1/3}. \]

Putting \( a = 2Nc_N \) and \( b = -2Nc_N \) in (3.5) and making use of (3.4) we get

\[ 4N |c_N| \leq 2 \left( C_3 \right)^{1/3} \left( \frac{1}{2\pi} \int_0^{2\pi} |2Nc_N(e^{itN} - e^{-itN})|^3 \, dt \right)^{1/3}. \]

\[ = 2 \left( C_3 \right)^{1/3} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{p=0}^{2N-1} e^{i\pi p} G \left( t + \frac{\pi p}{N} \right) \right|^3 \, dt \right)^{1/3} \]

\[ \leq 2 \left( C_3 \right)^{1/3} 2N \left( \frac{1}{2\pi} \int_0^{2\pi} |G(t)|^3 \, dt \right)^{1/3}. \]
by Minkowski inequality. Hence

\[ |d_{N-1}| = 2|c_N| \leq 2(C_\delta)^{1/\delta} \left( \frac{1}{2\pi} \int_0^{2\pi} |G(t)|^{1/\delta} dt \right)^{1/\delta} \]

\[ = 2 \left( \frac{2}{\pi} C_\delta \right)^{1/\delta} \left( \frac{1}{2} \int_{-1}^{1} |P_n(x)|^\delta (1-x^2)^{(\delta-1)/2} dx \right)^{1/\delta}. \]

In particular

\[ |A_n| = 2^n|d_n| \leq 2^{n+1} \left( \frac{2}{\pi} C_\delta \right)^{1/\delta} \left( \frac{1}{2} \int_{-1}^{1} |P_n(x)|^\delta (1-x^2)^{(\delta-1)/2} dx \right)^{1/\delta}, \]

\[ |A_{n-1}| = 2^{n-1}|d_{n-1}| \leq 2^n \left( \frac{2}{\pi} C_\delta \right)^{1/\delta} \left( \frac{1}{2} \int_{-1}^{1} |P_n(x)|^\delta (1-x^2)^{(\delta-1)/2} dx \right)^{1/\delta}. \]

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References


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