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On infinite systems of differential equations with deviated argument I

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Abstract. The aim of this paper is to indicate some effect caused by the deviated argument in infinite systems of differential equations. We give theorems on existence and uniqueness of the solution of the system

$$\begin{aligned} x_n'(t) = f_n[t;\ & x_1(t), x_1(\varphi_1^{(1)}(t)), x_1(\varphi_2^{(1)}(t)), \ldots; \\ & x_2(t), x_2(\varphi_1^{(2)}(t)), x_2(\varphi_2^{(2)}(t)), \ldots; \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & x_1'(t), x_1'(\psi_1^{(1)}(t)), x_1'(\psi_2^{(1)}(t)), \ldots; \\ & x_2'(t), x_2'(\psi_1^{(2)}(t)), x_2'(\psi_2^{(2)}(t)), \ldots; \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots] \\ & \qquad\qquad (n = 1, 2, \ldots) \end{aligned}$$

with initial conditions $x_n(0) = 0$, $n = 1, 2, \ldots$

In [2] a discussion is made on the existence and uniqueness of the solution of the differential equation

$$x'(t) = f[t, x(t), x(\varphi_1(t)), \ldots, x(\varphi_n(t)), x'(t), x'(\psi_1(t)), \ldots, x'(\psi_m(t))]$$

with the initial condition $x(0) = 0$. The purpose of this paper is to point out some effects which the deviated argument causes on infinite systems of differential equations.

Infinite systems of differential equations were considered by A. N. Tichonov [7], [8], and K. P. Persidski [5], [6]. Results of these authors are generalized in [9] for the systems of equations whose right-hand sides contain also variable parameters. Infinite systems of integral-differential equations were studied in [3].

The purpose of this paper is to discuss the existence and uniqueness problems for infinite system of differential equations

$$\text{(I)} \qquad \begin{aligned} x_n'(t) = f_n[t;\ & x_1(t), x_1(\varphi_{11}(t)), x_1(\varphi_{12}(t)), \ldots; \\ & x_2(t), x_2(\varphi_{21}(t)), x_2(\varphi_{22}(t)), \ldots; \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & x_1'(t), x_1'(\psi_{11}(t)), x_1'(\psi_{12}(t)), \ldots; \\ & x_2'(t), x_2'(\psi_{21}(t)), x_2'(\psi_{22}(t)), \ldots; \ldots] \\ & \qquad\qquad (n = 1, 2, \ldots) \end{aligned}$$

with initial conditions $x_n(0) = 0$, $n = 1, 2, \ldots$

In the proofs of existence of at least one solution for the above stated problem we use the following form of Tichonov's fixed point theorem: *If in B_0-space a continuous operator maps a closed convex and compact set into itself, then it has a fixed point.*

1. Let s denote the space of all sequences of real numbers with the usual metrics ϱ. The sets

$$s_1 = s \times s \times \ldots, \quad s_2 = [0, a] \times s_1, \quad s_3 = [0, a] \times s_1 \times s_1$$

are considered with the "product" metrics ϱ_1, ϱ_2 and ϱ_3 respectively.

Let $\|\cdot\|$ be a norm in $C[0, a]$. In the vector space $E = C[0, a] \times \times C[0, a] \times \ldots$ define a sequence (p_n) of semi-norms $p_n(\Phi) = \|\varphi_n\|$, where $\Phi = (\varphi_1, \varphi_2, \ldots)$. Then (cf. [1]) the functional $|||\cdot|||$ defined by

$$|||\Phi||| = \sum_{1}^{\infty} 2^{-n} \frac{\|\varphi_n\|}{1+\|\varphi_n\|}$$

is a paranorm in E. It is known (see [4]) that the space E equipped with this paranorm is a B_0-space.

The convergence in the introduced spaces is equivalent to the coordinate-wise convergence. Given a non-negative constant R, let m_R be the set of real sequences bounded by R. Then evidently the product $Z = m_{R_1} \times \times m_{R_2} \times \ldots$ is a compact subset of s_1. Let $\bar{a} = (\bar{t}, \bar{\xi}) \in s_2$, where $\bar{\xi} = (\bar{u}_n) \in s_1$, $\bar{u}_i = (\bar{u}_{in}) \in s$. A function $f\colon s_2 \to (-\infty, \infty)$ is continuous at the point $\bar{a}$ if and only if, for every $\varepsilon > 0$ a number $\eta > 0$ and natural numbers $N_1, N_2, \ldots, N_M$ exist such that

$$|f(t; u_{11}, u_{12}, \ldots; \ldots) - f(\bar{t}; \bar{u}_{11}, \bar{u}_{12}, \ldots; \ldots)| < \varepsilon,$$

whenever $|t-\bar{t}| < \eta$, $|u_{ki} - \bar{u}_{ki}| < \eta$ for $i = 1, 2, \ldots, N_k$, $k = 1, 2, \ldots, M$. If $\bar{A} = (\bar{t}, \bar{\xi}, \bar{\eta}) \in s_3$, $\bar{\xi} = (\bar{u}_n)$, $\bar{\eta} = (\bar{v}_n) \in s_1$, $\bar{u}_i = (\bar{u}_{in})$, $\bar{v}_i = (\bar{v}_{in}) \in s$, then a function $F\colon s_3 \to (-\infty, \infty)$ is continuous at the point $\bar{A}$ if and only if for every $\varepsilon > 0$ there exist a number $\eta > 0$ and natural numbers $N_1, N_2, \ldots \ldots, N_M$ and $N'_1, N'_2, \ldots, N'_K$ such that

$$|F(t; u_{11}, u_{12}, \ldots; v_{11}, v_{12}, \ldots) - F(\bar{t}; \bar{u}_{11}, \bar{u}_{12}, \ldots; \bar{v}_{11}, \bar{v}_{12}, \ldots)| < \varepsilon,$$

when $|t-\bar{t}| < \eta$, $|u_{ki} - \bar{u}_{ki}| < \eta$ for $i = 1, 2, \ldots, N_k$, $k = 1, 2, \ldots, M$ and $|v_{ki} - \bar{v}_{ki}| < \eta$ for $i = 1, 2, \ldots, N'_k$, $k = 1, 2, \ldots, K$.

2. We prove here the existence theorem for problem (I) in the following special case:

$$\text{(II)} \quad \begin{aligned} x'_n(t) = f_n\big[t; x_1(t), x_1(\varphi_{11}(t)), x_1(\varphi_{12}(t)), \ldots; \\ x_2(t), x_2(\varphi_{21}(t)), x_2(\varphi_{22}(t)), \ldots; \ldots\big] \quad (n = 1, 2, \ldots) \end{aligned}$$

with $x_n(0) = 0$, $n = 1, 2, \ldots$ We introduce

Assumption (A). *Suppose that*

1° *the function* f_n: $s_2 \to (-\infty, \infty)$ $(n = 1, 2, \ldots)$ *is continuous,*

2° *there exists an* $r > 0$ *such that*

$$\sup |f_n(t; u_{11}, u_{12}, \ldots; u_{21}, u_{22}, \ldots; \ldots)| \leqslant ra^{-1}$$

for $n = 1, 2, \ldots, t \in [0, a]$, $|u_{ij}| \leqslant r$, $i, j = 1, 2, \ldots$,

3° *the function* φ_{ij}: $[0, a] \to [0, a]$ $(i, j = 1, 2, \ldots)$ *is continuous.*

Theorem 1. *Under Assumption* (A) *there exists at least one solution of the initial-value problem* (II); *this solution consists of uniformly bounded continuous functions on the interval* $[0, a]$.

Proof. It is obvious that problem (II) is equivalent to the following infinite system of integral equations

$$x_n(t) = \int_0^t f_n\big[s; x_1(s), x_1(\varphi_{11}(s)), x_1(\varphi_{12}(s)), \ldots; \ldots\big] ds \qquad (1)$$

$$(n = 1, 2, \ldots)$$

considered in E. We shall first prove that if $x_i \in C[0, a]$ $(i = 1, 2, \ldots)$, then the composition

$$h(t) = f_n\big(t; x_1(t), x_1(\varphi_{11}(t)), \ldots; x_2(t), x_2(\varphi_{21}(t)), \ldots; \ldots\big)$$

is continuous in the interval $[0, a]$. Let

$$a_k = \big(t_k; x_1(t_k), x_1(\varphi_{11}(t_k)), x_1(\varphi_{12}(t_k)), \ldots; \ldots\big),$$

$$a_0 = \big(t_0; x_1(t_0), x_1(\varphi_{11}(t_0)), x_1(\varphi_{12}(t_0)), \ldots; \ldots\big).$$

Then for $(t_k) \to t_0$, $t_k \in [0, a]$ we get $\lim x_i(t_k) = x_i(t_0)$ and $\lim x_i(\varphi_{ij}(t_k)) = x_i(\varphi_{ij}(t_0))$ $(i, j = 1, 2, \ldots)$. This means that $\lim \varrho_2(a_k, a_0) = 0$. Hence $(h(t_k)) \to h(t_0)$ by the continuity of f_n.

If T is an operator whose coordinates are defined by right-hand sides of system (1), and K_0 denotes the set consisting of $U = (x_1, x_2, \ldots) \in E$ such that

$$\|x_n\| \leqslant r, \qquad |x_n(t_1) - x_n(t_2)| \leqslant ra^{-1}|t_1 - t_2|$$

$$\text{for } t_1, t_2 \in [0, a] \text{ and } n = 1, 2, \ldots,$$

then T: $K_0 \to K_0$. We shall prove now that the operator T is continuous on K_0. Let $(U_k) \to U$, where $U_k = (x_{k1}, x_{k2}, \ldots)$, $U = (x_1, x_2, \ldots) \in K_0$. We must prove that

$$\|y_{kn} - y_n\| \to 0 \quad \text{with } k \to \infty \ (n = 1, 2, \ldots),$$

where

$$y_{kn}(t) = \int_0^t f_n[s;\, x_{k1}(s),\, x_{k1}(\varphi_{11}(s)),\, x_{k1}(\varphi_{12}(s)),\, \ldots;\, \ldots]\,ds,$$

$$y_n(t) = \int_0^t f_n[s;\, x_1(s),\, x_1(\varphi_{11}(s)),\, x_1(\varphi_{12}(s)),\, \ldots;\, \ldots]\,ds.$$

Because $\lim |x_n(s) - x_{kn}(s)| = 0$, for $\delta > 0$ the number N exists such that

$$\sum_{n=1}^{\infty} 2^{-n}\Bigg[2^{-1}\frac{|x_{kn}(s) - x_n(s)|}{1 + |x_{kn}(s) - x_n(s)|} + \\ + \sum_{i=2}^{\infty} 2^{-i}\frac{|x_{kn}(\varphi_{n,i-1}(s)) - x_n(\varphi_{n,i-1}(s))|}{1 + |x_{kn}(\varphi_{n,i-1}(s)) - x_n(\varphi_{n,i-1}(s))|}\Bigg] < \delta$$

for $k > N$ and $s \in [0, a]$. By uniform continuity of the function f_n on the set $Z_1 = [0, a] \times m_r \times m_r \times \ldots$ we can find, for $\varepsilon > 0$, such a number N that

$$|f_n(s;\, x_{k1}(s),\, x_{k1}(\varphi_{11}(s)),\, x_{k1}(\varphi_{12}(s)),\, \ldots;\, \ldots) - \\ - f_n(s;\, x_1(s),\, x_1(\varphi_{11}(s)),\, x_1(\varphi_{12}(s)),\, \ldots;\, \ldots)| < \varepsilon,$$

whenever $k > N$, $s \in [0, a]$. Hence $|y_{kn}(t) - y_n(t)| \leqslant \varepsilon a$ for $k > N$, $t \in [0, a]$, $n = 1, 2, \ldots$ Thus T is continuous in K_0. Since for $\|x_{in}\| \leqslant r$, $|x_{in}(t_1) - x_{in}(t_2)| \leqslant ra^{-1}|t_1 - t_2|$, where $t_1, t_2 \in [0, a]$, $i = 1, 2$, $n = 1, 2, \ldots$ and for $0 \leqslant \alpha \leqslant 1$ we have

$$\|(1-\alpha)x_{1n} + \alpha x_{2n}\| \leqslant r,$$

$$|(1-\alpha)x_{1n}(t_1) + \alpha x_{2n}(t_1) - (1-\alpha)x_{1n}(t_2) - \alpha x_{2n}(t_2)| \leqslant ra^{-1}|t_1 - t_2|,$$

K_0 is a convex set. It can be easily proved that K_0 is compact. Indeed, let (U_i) be an infinite sequence such that $U_i = (x_1^{(i)}, x_2^{(i)}, \ldots) \in K_0$. Let k be a fixed number. Let us consider the sequence $(x_k^{(i)})$. The functions of the above sequence satisfy the inequality

$$|x_k^{(i)}(t_1) - x_k^{(i)}(t_2)| \leqslant ra^{-1}|t_1 - t_2| \qquad (i = 1, 2, \ldots)$$

for $t_1, t_2 \in [0, a]$. Hence these functions are equally continuous and we know that they are uniformly bounded, thus we can apply the Arzelà theorem. Hence we can substract from the sequence $(x_1^{(i)})$ a subsequence $(x_1^{n_1^{(i)}})$ converging to some x_1. Similarly, from $(x_2^{n_1^{(i)}})$ a subsequence $(x_2^{n_2^{(i)}})$ converging to some x_2 can be substracted. Analogously we can verify that the sequence $(U_{n_k}(k))$ is convergent to $U = (x_1, x_2, \ldots) \in K_0$, since

$$\lim x_i^{n_k^{(k)}}(t) = x_i(t) \quad \text{uniformly in } [0, a]$$

for every $i = 1, 2, \ldots$

Now, Theorem 1 is implied by the Tichonov fixed-point theorem.

Remark 1. Theorem 1 is a generalization of the result of paper [2], where a theorem on the existence of solutions of the equation

$$x'(t) = f\big(t, x(t), x(\varphi_1(t)), \ldots, x(\varphi_n(t))\big)$$

was established.

3. In this section we shall consider the initial-value problem for (I), under the following

ASSUMPTION B. *Suppose that*

1° *the function* $f_n\colon s_3 \to (-\infty, \infty)$ $(n = 1, 2, \ldots)$ *is continuous,*

2° *there exists a positive number* r *such that*

$$\sup |f_n(t; u_{11}, u_{12}, \ldots, \ldots; v_{11}, v_{12}, \ldots, \ldots)| \leqslant r$$

for $n = 1, 2, \ldots, t \in [0, a]$; $|u_{ij}| \leqslant ra$, $|v_{ij}| \leqslant r$, $i, j = 1, 2, \ldots,$

3° *the functions* f_n, $n = 1, 2, \ldots$, *are equally continuous on the set* $Z_2 = [0, a] \times Z' \times Z''$, *where* $Z' = m_{ar} \times m_{ar} \times \ldots$, $Z'' = m_r \times m_r \times \ldots,$

4° *there exist non-negative constants* $\mu_{ij}^{(n)}$ $(n = 1, 2, \ldots)$, $i, j = 1, 2, \ldots,$ *such that the inequalities*

$$|f_n(t, \xi, \eta) - f_n(t, \xi, \bar{\eta})| \leqslant \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij}^{(n)} |v_{ij} - \bar{v}_{ij}|,$$

$$\mu = \sup \Big\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij}^{(n)} \colon n = 1, 2, \ldots \Big\} < 1,$$

are satisfied for any $t \in [0, a]$, $\xi, \eta = (v_n)$, $\bar{\eta} = (\bar{v}_n) \in s_1$, $v_i = (v_{in})$, $\bar{v}_i = (\bar{v}_{in}) \in s$,

5° *the functions* $\varphi_{ij}\colon [0, a] \to [0, a]$, $i, j = 1, 2, \ldots$, *are equi-continuous,*

6° *the function* $\psi_{ij}\colon [0, a] \to [0, a]$ $(i, j = 1, 2, \ldots)$ *is continuous,*

7° *for any* $t_1, t_2 \in [0, a]$ *the following inequality holds*

$$|\psi_{ij}(t_1) - \psi_{ij}(t_2)| \leqslant |t_1 - t_2| \qquad (i, j = 1, 2, \ldots).$$

THEOREM 2. *If Assumption* (B) *is satisfied, then there exists at least one solution of the initial-value problem* (I); *this solution consists of uniformly bounded continuous functions on the interval* $[0, a]$.

Proof. We reduce problem (I), by substitution $x_n'(t) = u_n(t)$ $(n = 1, 2, \ldots)$ to the following system of equations

$$(2) \qquad u_n(t) = f_n\Big[t; \int_0^t u_1(s)\,ds, \int_0^{\varphi_{11}(t)} u_1(s)\,ds, \int_0^{\varphi_{12}(t)} u_1(s)\,ds, \ldots; \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_1(t), u_1(\psi_{11}(t)), u_1(\psi_{12}(t)), \ldots; \ldots\Big]. \qquad (n = 1, 2, \ldots).$$

Let T be the operator whose coordinates are defined by the right-hand sides of system (2) and let

$$K_0 = \{U = (u_1, u_2, \ldots) \in E: \|u_n\| \leqslant r,\ n = 1, 2, \ldots\}.$$

From Assumption (B) it follows that the operator T is continuous in K_0, and T: $K_0 \to K_0$.

Let F_n: $K_0 \times [0, a] \times [0, a] \to (-\infty, \infty)$ $(n = 1, 2, \ldots)$ be the function of variables $U = (u_1, u_2, \ldots) \in K_0$, t, $\tau \in [0, a]$. This function arises from the right-hand side of equation (2), if we replace in the last group of the terms

$$u_i(t), \quad u_i(\psi_{i1}(t)), \quad u_i(\psi_{i2}(t)), \quad \ldots \quad (i = 1, 2, \ldots)$$

the variable t by τ. Consider the restrictions of f_n to the set Z_2; then for every $\varepsilon > 0$ there exist $\eta(\varepsilon) > 0$ and natural numbers $N_1^{(n)}, N_2^{(n)}, \ldots, N_{M_n}^{(n)}$, such that

$$|F_n(U, t_1, t_1) - F_n(U, t_2, t_1)| < \varepsilon$$

for every $U \in K_0$ and every $t_1, t_2 \in [0, a]$, for which

$$(3) \qquad |t_1 - t_2| < \eta, \quad r|\varphi(t_2) - \varphi(t_1)| < \eta \quad \text{for all } \varphi \in \mathscr{F},$$

where $\mathscr{F}$ denotes the set of functions:

$$\varphi_{1j}, \quad j = 1, 2, \ldots, N_1^{(n)} - 1 \quad (n = 1, 2, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\varphi_{M_n j}, \quad j = 1, 2, \ldots, N_{M_n}^{(n)} - 1 \quad (n = 1, 2, \ldots).$$

For our $\eta = \eta(\varepsilon)$ there exists $\nu = \nu(\eta) > 0$, such that $r|\varphi(t_1) - \varphi(t_2)| \leqslant \eta$ for $\varphi \in \mathscr{F}$ and $|t_1 - t_2| < \nu$. Therefore inequalities (3) hold for any $t_1, t_2 \in [0, a]$, $|t_1 - t_2| < \delta$, where $\delta = \min(\eta, \eta r^{-1}, \nu)$. Thus we see that: there exists a function δ: $(0, \infty) \to (0, \infty)$ such that iff $U \in K_0$, $t_1, t_2 \in [0, a]$, $|t_1 - t_2| < \delta(\varepsilon)$, then

$$|F_n(U, t_1, t_1) - F_n(U, t_2, t_1)| < \varepsilon \quad \text{for } n = 1, 2, \ldots$$

Consider the set K_0^ε of those $U = (u_1, u_2, \ldots) \in K_0$ which have the following property: for every $\varepsilon > 0$ and $t_1, t_2 \in [0, a]$, $|t_1 - t_2| < \delta(\varepsilon)$,

$$\sup\{|u_n(t_1) - u_n(t_2)|:\ n = 1, 2, \ldots\} \leqslant (1-\mu)^{-1}\cdot\varepsilon$$

holds.

Now we can prove the inclusion $T[K_0^\varepsilon] \subset K_0^\varepsilon$. In fact, if $U = (u_1, u_2, \ldots) \in K_0^\varepsilon$ and $TU = (Tu_1, Tu_2, \ldots)$, then

$$\begin{aligned} |Tu_n(t_1) - Tu_n(t_2)| &\leqslant \varepsilon + |F_n(U, t_2, t_1) - F_n(U, t_2, t_2)| \\ &\leqslant \varepsilon + \sum_{i=1}^{\infty}\Big[\mu_{i1}^{(n)}|u_i(t_1) - u_i(t_2)| + \sum_{j=2}^{\infty}\mu_{ij}^{(n)}\big|u_i(\psi_{i,j-1}(t_1)) - u_i(\psi_{i,j-1}(t_2))\big|\Big] \\ &\leqslant \varepsilon + (1-\mu)^{-1}\varepsilon\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu_{ij}^{(n)} \leqslant \varepsilon + (1-\mu)^{-1}\varepsilon\mu = (1-\mu)^{-1}\varepsilon, \end{aligned}$$

for $\varepsilon > 0$ and $|t_1 - t_2| < \delta(\varepsilon)$.

Thus we see that the continuous operator T maps the convex compact set K_0^ε into itself; therefore, according to Tichonov's fixed-point theorem there exists at least one fixed point of T. This fixed-point is evidently a solution of system (2).

Remark 2. Theorem 2 is a generalization of a result from [2], where the existence of solutions of the equation

$$x'(t) = f\big(t, x(t), x(\varphi_1(t)), \ldots, x(\varphi_n(t)), x'(t), x'(\psi_1(t)), \ldots, x'(\psi_m(t))\big)$$

was established.

We shall formulate another theorem on the existence of a solution of problem (I).

ASSUMPTION (C). *Suppose that*

1° *conditions* 1°, 3°, 4°, 5° *and* 6° *of Assumption* (B) *are satisfied,*

2° *there exists a constant* $q \geqslant 0$ *such that*

$$|f_n(t, \xi, \theta)| \leqslant q \qquad (n = 1, 2, \ldots)$$

for any $t \in [0, a]$, $\xi \in s_1$, *where* θ *denotes the zero element of the space* s_1.

THEOREM 3. *If Assumption* (C) *is satisfied, then there exists at least one solution of the initial-value problem* (I).

Proof. Let T be the operator defined in the proof of Theorem 2. Given $r > 0$, the inequality $|Tu_n(t)| \leqslant q + \mu r$ holds for all $U = (u_1, u_2, \ldots) \in E$, $|u_n(t)| \leqslant r$ $(n = 1, 2, \ldots)$. Choosing $r_0 > 0$ so that $q + \mu r_0 \leqslant r_0$, we see that $T\colon K_0 \to K_0$, where

$$K_0 = \{U = (u_1, u_2, \ldots):\ \|u_n\| \leqslant r_0,\ n = 1, 2, \ldots\}.$$

Then we proceed similarly as in the proof of Theorem 2.

4. In this section we shall establish theorems on the existence of a unique solution of the initial-value problem (II).

Assumption (D). *Suppose that*

1° *Assumption* (A) *is satisfied,*

2° *there exists a function* $g \in C[0, a]$ *such that*

$$|f_n(t, \xi) - f_n(t, \bar{\xi})| \leqslant g(t) \cdot \sup\{|u_{i1} - \bar{u}_{i1}|,\ |u_{i2} - \bar{u}_{i2}|, \ldots : i = 1, 2, \ldots\}$$

for every $n = 1, 2, \ldots, t \in [0, a]$, $\xi = (u_n)$, $\bar{\xi} = (\bar{u}_n) \in s_1$, *where* $u_i = (u_{in})$, $\bar{u}_i = (\bar{u}_{in}) \in s$,

3° *the function* φ_{ij} $(i, j = 1, 2, \ldots)$ *fulfils the inequality*

$$\varphi_{ij}(t) \leqslant t \quad \textit{for every } t \in [0, a].$$

Assumption (E). *Suppose that*

1° *conditions* 1° *and* 2° *of Assumption* (D) *are satisfied,*

2° $a \cdot \|g\| < 1$.

Theorem 4. *Under Assumption* (D) *there exists a unique solution of problem* (II). *This solution consists of uniformly bounded and equi-continuous functions defined on the interval* $[0, a]$; *it is the limit of the sequence of Picard's successive approximations*

$$\begin{aligned} &x_n^{(0)}(t) = 0, \\ (4) \qquad &x_n^{(k)}(t) = \int_0^t f_n\big[s; x_1^{(k-1)}(s), x_1^{(k-1)}(\varphi_{11}(s)), x_1^{(k-1)}(\varphi_{12}(s)), \ldots; \\ &\qquad\qquad x_2^{(k-1)}(s), x_2^{(k-1)}(\varphi_{21}(s)), x_2^{(k-1)}(\varphi_{22}(s)), \ldots; \\ &\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\big]\,ds \\ &\qquad\qquad\qquad \textit{for } k = 1, 2, \ldots \ (n = 1, 2, \ldots) \end{aligned}$$

Proof. For any fixed n, let us consider the sequence $(x_n^{(k)})$ of the functions defined by formulas (4). The functions are continuous and bounded by r on the interval $[0, a]$, and

$$|x_n^{(k+1)}(t) - x_n^{(k)}(t)| \leqslant \frac{r\|g\|^k |t|^{k+1}}{a(k+1)!} \leqslant \frac{r(\|g\|a)^k}{(k+1)!}.$$

Since the series

$$x_n^{(0)} + (x_n^{(1)} - x_n^{(0)}) + \ldots + (x_n^{(k+1)} - x_n^{(k)}) + \ldots$$

is uniformly convergent on $[0, a]$, we also have

$$(5) \qquad \lim x_n^{(k)}(t) = x_n(t) \quad \text{uniformly on } [0, a].$$

Now it is easily seen that the functions x_n, $n = 1, 2, \ldots$, satisfy system (1).

Since

$$|x_n'(t)| \leqslant |f_n(t;\, 0,\, 0,\, \ldots;\, 0,\, 0\, \ldots;\, \ldots)| +$$
$$+ g(t)\sup\{|x_i(t)|,\, |x_i(\varphi_{i1}(t))|,\, |x_i(\varphi_{i2}(t))|,\, \ldots :\, i = 1, 2, \ldots\}$$
$$\leqslant ra^{-1} + \|g\| r \qquad (n = 1, 2, \ldots),$$

our solution of (1) consists of equi-continuous functions.

Suppose that $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are two solutions of (1). By (5), for every n we have

$$|y_n(t) - x_n^{(k)}(t)| \leqslant \frac{r\|g\|^k |t|^{k+1}}{a(k+1)!} \leqslant \frac{r(\|g\| a)^k}{(k+1)!}$$

for $t \in [0, a]$, $k = 0, 1, 2, \ldots$ Hence $y_n(t) = \lim x_n^{(k)}(t) = x_n(t)$ for $n = 1, 2, \ldots$ Thus completes the proof.

Let us further state

THEOREM 5. *Under Assumption* (E) *the conclusion of Theorem 4 is valid and, moreover, if* $x_n^{(k)}$ $(n = 1, 2, \ldots)$ *are defined by formulas* (4), *then*

$$|x_n(t) - x_n^{(k)}(t)| \leqslant ra^k \|g\|^k$$

for $t \in [0, a]$, $k = 0, 1, 2, \ldots$

5. We now want to formulate a theorem on the existence and uniqueness of the solution of problem (I).

ASSUMPTION (F). *Suppose that*

1° *conditions* 1°, 2° *and* 6° *of Assumption* (B) *are satisfied,*

2° *the function* $\varphi_{ij}\colon [0, a] \to [0, a]$ $(i, j = 1, 2, \ldots)$ *is continuous,*

3° *there exist functions* $g_1 \in C[0, a]$ *and* $g_2 \in C[0, a]$ *such that*

$$|f_n(t,\, \xi,\, \eta) - f_n(t,\, \bar{\xi},\, \bar{\eta})|$$
$$\leqslant g_1(t)\sup\{|u_{i1} - \bar{u}_{i1}|,\, |u_{i2} - \bar{u}_{i2}|,\, \ldots :\, i = 1, 2, \ldots\} +$$
$$+ g_2(t)\sup\{|v_{i1} - \bar{v}_{i1}|,\, |v_{i2} - \bar{v}_{i2}|,\, \ldots :\, i = 1, 2, \ldots\}$$

for $n = 1, 2, \ldots, t \in [0, a]$, $\xi = (u_n)$, $\eta = (v_n)$, $\bar{\xi} = (\bar{u}_n)$, $\bar{\eta} = (\bar{v}_n) \in s_1$, *where* $u_i = (u_{in})$, $v_i = (v_{in})$, $\bar{u}_i = (\bar{u}_{in})$, $\bar{v}_i = (\bar{v}_{in}) \in s$,

4° $a\|g_1\| + \|g_2\| < 1$.

THEOREM 6. *Under Assumption* (F) *there exists a unique solution of problem* (I). *This solution consists of uniformly bounded and equi-continuous functions defined on the interval* $[0, a]$; *it is the limit of the sequence* *of Picard's successive approximations*

$$x_n^{(0)}(t) = 0,$$

(7)
$$\begin{aligned} x_n^{(k)}(t) = \int_0^t f_n\big[s;\, & x_1^{(k-1)}(s), x_1^{(k-1)}(\varphi_{11}(s)), x_1^{(k-1)}(\varphi_{12}(s)), \ldots; \\ & x_2^{(k-1)}(s), x_2^{(k-1)}(\varphi_{21}(s)), x_2^{(k-1)}(\varphi_{22}(s)), \ldots; \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \dot{x}_1^{(k-1)}(s), \dot{x}_1^{(k-1)}(\psi_{11}(s)), \dot{x}_1^{(k-1)}(\psi_{12}(s)), \ldots; \\ & \dot{x}_2^{(k-1)}(s), \dot{x}_2^{(k-1)}(\psi_{21}(s)), \dot{x}_2^{(k-1)}(\psi_{22}(s)), \ldots; \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big]\,ds, \end{aligned}$$

$$\text{for } k = 1, 2, \ldots \quad (n = 1, 2, \ldots)$$

Proof. For any fixed n, let us consider the sequence $(x_n^{(k)})$ of the functions defined by formulas (7), and the sequence $(\dot{x}_n^{(k)})$ of their derivatives. Since

$$x_n^{(k)}(t) - x_n^{(k-1)}(t) = \int_0^t [\dot{x}_n^{(k)}(s) - \dot{x}_n^{(k-1)}(s)]\,ds, \tag{8}$$

it easily follows that

$$\|\dot{x}_n^{(k+1)} - \dot{x}_n^{(k)}\| \leqslant (a\|g_1\| + \|g_2\|)\sup\{\|\dot{x}_i^{(k)} - \dot{x}_i^{(k-1)}\|:\ i = 1, 2, \ldots\}.$$

Hence, by induction on n,

$$\|\dot{x}_n^{(k+1)} - \dot{x}_n^{(k)}\| \leqslant 2r(a\|g_1\| + \|g_2\|)^{k-1}. \tag{9}$$

From (8) and (9) it follows that

$$\|x_n^{(k)} - x_n^{(k-1)}\| \leqslant 2ra(a\|g_1\| + \|g_2\|)^{k-2},$$

therefore the series

$$x_n^{(0)} + \sum_{k=1}^{\infty}(x_n^{(k)} - x_n^{(k-1)}), \qquad \dot{x}_n^{(0)} + \sum_{k=1}^{\infty}(\dot{x}_n^{(k)} - \dot{x}_n^{(k-1)})$$

are uniformly convergent on $[0, a]$, provided that $a\|g_1\| + \|g_2\| < 1$. Hence

$$\lim x_n^{(k)}(t) = x_n(t), \qquad \lim \dot{x}_n^{(k)}(t) = \dot{x}_n(t)$$

uniformly on $[0, a]$.

Continuing similarly as in the proof of Theorem 4, we verify that the limit functions x_n, $n = 1, 2, \ldots$ satisfy the following system of integral equations

$$\begin{aligned} x_n(t) = \int_0^t f_n\big[s;\, & x_1(s), x_1(\varphi_{11}(s)), x_1(\varphi_{12}(s)), \ldots, \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \dot{x}_1(s), \dot{x}_1(\psi_{11}(s)), \dot{x}_1(\psi_{12}(s)), \ldots; \ldots\big]\,ds \qquad (n = 1, 2, \ldots) \end{aligned}$$

and have all the properties asserted in the theorem.

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