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Note on a general dilation theorem

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Abstract. The paper deal with the extension of a positive definite function on an involution semigroup S to such a function on the unitization of S. A related dilation problem is discussed.

Suppose A is a star algebra without unit and f is a positive linear functional on A, that is, $f(a^*a) \geqslant 0$, $a \in A$. Then it is well known [4], p. 223, that f extends to a positive linear functional f_1 on A_1, the unitization of A, if and only if $f(a^*) = \overline{f(a)}$, $a \in A$, and there is a constant $c > 0$ such that $|f(a)|^2 \leqslant cf(a^*a)$, $a \in A$. R. J. Lindahl and P. H. Maserick have shown [2] that a bounded, positive definite complex-valued function φ on a $*$-semigroup without unit is extendable to a positive definite one on S_1, the unitization of S, if and only if $\varphi(s^*) = \overline{\varphi(s)}$, $s \in S$, and there is a $c > 0$ such that

$$\Big|\sum_k c_k \varphi(s_k)\Big|^2 \leqslant c \sum_{i,k} \bar{c}_i c_k \varphi(s_i^* s_k)$$

for all $s_1, \ldots, s_n \in S$ and any complex numbers $c_1, \ldots, c_n$.

They have employed the fact mentioned at the outset and have established an isomorphism between such functions defined on a $*$-semigroup S and extendable linear functionals on the $*$-semigroup algebra $l_1(S)$.

In the present note we work out similar conditions for (not necessarily bounded) positive definite operator-valued functions on $*$-semigroups. Then we place these conditions into a general dilation theorem with emphasis on the boundedness condition involved in there.

1. Let S be a semigroup with an involution $s \to s^*$, $s \in S$ (called a *$*$-semigroup*) and let H be a complex Hilbert space. Denote by $F(S, H)$ the set of all maps from S to H with finite supports. A function $\varphi\colon A \to L(H)$ ($L(H)$ stands for the algebra of all bounded linear operators in H) is said to be *positive definite* [5] if

$$\sum_{s,t}\big(\varphi(s^*t)f(t),f(s)\big)\geqslant 0,\qquad f\in F(S,H).$$

Let f and g belong to $F(S,H)$ and let λ,μ be complex numbers. The positive definiteness of φ implies that

$$\sum_{s,t}\Big(\varphi(s^*t)\big(\lambda f(t)+\mu g(t)\big),\ \lambda f(s)+\mu g(s)\Big)$$
$$=\sum_{s,t}\big(\varphi(s^*t)f(t),f(s)\big)\lambda\bar\lambda+\sum_{s,t}\big(\varphi(s^*t)f(t),g(s)\big)\lambda\bar\mu+$$
$$+\sum_{s,t}\big(\varphi(s^*t)g(t),f(s)\big)\bar\lambda\mu+\sum_{s,t}\big(\varphi(s^*t)g(t),g(s)\big)\mu\bar\mu\geqslant 0.$$

We get a positive definite quadratic form in two complex variables λ and μ. It is a rudimentary matter to check that

$$\varphi(s^*t)=\varphi(t^*s)^*,\qquad t,s\in S, \tag{1}$$

$$\Big|\sum_{s,t}\big(\varphi(s^*t)f(t),g(s)\big)\Big|^2 \leqslant\Big(\sum_{s,t}\big(\varphi(s^*t)f(t),f(s)\big)\Big)\Big(\sum_{s,t}\big(\varphi(s^*t)g(t),g(s)\big)\Big),\qquad f,g\in F(S,H). \tag{2}$$

Inequality (2) is just the *Schwarz inequality* for positive definite operator functions on $*$-semigroups.

2. If S has no unit, we adjoin 1, define $1^*=1$ and denote the resulting $*$-semigroup by S_1. If S already has a unit, we set $S_1=S$.

Suppose for the moment $S_1=S$, choose $f\in F(S,H)$ and set $g(s)=0$ if $s\neq 1$ and $g(1)=\sum\varphi(s)f(s)$. Then the Schwarz inequality (2) yields

$$\Big\|\sum\varphi(s)f(s)\Big\|^4\leqslant\Big(\sum_{s,t}\big(\varphi(s^*t)f(t),f(s)\big)\Big)\leqslant\Big(\varphi(1)\sum_s\varphi(s)f(s),\ \sum_s\varphi(s)f(s)\Big).$$

This implies

$$\Big\|\sum_s\varphi(s)f(s)\Big\|^2\leqslant c\sum_{s,t}\big(\varphi(s^*t)f(t),f(s)\big),\qquad f\in F(S,H). \tag{3}$$

Here $c=\|\varphi(1)\|$.

Moreover, (1) with $t=1$ takes the form

$$\varphi(s^*)=\varphi(s)^*,\qquad s\in S. \tag{4}$$

Conditions (3) and (4) bear a resemblance to those which guarantee the extendability of positive definite scalar functions. We will show that they suffice to extend the positive definiteness of φ from S to S_1.

Suppose that φ satisfies (3) (with c independent of $f\in F(S,H)$) and (4). Let A be a positive operator in $L(H)$ such that $A\geqslant cI$, where I denotes the identity operator. Define $\varphi_1\colon S_1 \to L(H)$ as follows:

$$\varphi_1(s) = \begin{cases} A & \text{if } s = 1, \\ \varphi(s) & \text{otherwise}. \end{cases} \tag{5}$$

Then, by (3) and (4), we get

$$\sum_{s,t} \big(\varphi_1(s^*t)f(t), f(s)\big) = \sum_{s,t\neq 1} \big(\varphi(s^*t)f(t), f(s)\big) +$$
$$+ \sum_{s\neq 1} \big(\varphi(s^*)f(1), f(s)\big) + \sum_{t\neq 1} \big(\varphi(t)f(t), f(1)\big) + \big(Af(1), f(1)\big)$$
$$\geqslant c^{-1}\Big(\Big\|\sum_{s\neq 1}\varphi(s)f(s)\Big\|^2 + \sum_{s\neq 1}\big(cf(1), \varphi(s)f(s)\big) +$$
$$+ \sum_{t\neq 1}\big(\varphi(t)f(t),\, cf(1)\big) + \big(cf(1), cf(1)\big)\Big) = c^{-1}\Big\|\sum_{s\neq 1}\varphi(s)f(s) + cf(1)\Big\|^2 \geqslant 0.$$

This shows that $\varphi_1\colon S_1 \to L(H)$, defined above by (5), is a positive definite function. Notice also that, by (3), φ is positive definite itself.

We summarize all what we have proved in the following

PROPOSITION 1. *An operator function $\varphi\colon S \to L(H)$ extends to a positive definite operator function on the $*$-semigroup S_1 if and only if φ satisfies* (3) *and* (4).

Choosing $A = cI$ we may require in addition that:

If $T \in L(H)$ commutes with φ, then so does it with φ_1.

3. Now we deal with dilation problems. A semigroup homomorphism $\Phi\colon S \to L(K)$ (K is a complex Hilbert space) is called a *representation* of a $*$-semigroup S on K (or a *$*$-representation* of S on K) if it preserves involution, that is, if $\Phi(s^*) = \Phi(s)^*$, $s \in S$. Given $\varphi\colon S \to L(H)$, a $*$-representation Φ of S on K is called a *dilation* of φ if there exists a bounded linear operator $V\colon H \to K$ such that

$$\varphi(s) = V^*\Phi(s)V, \qquad s \in S.$$

We call φ *dilatable* if it has a dilation.

We will say that $\varphi\colon S \to L(H)$ satisfies the *boundedness condition* [5] if

$$\sum_{s,t} \big(\varphi(s^*u^*ut)f(t), f(s)\big) \leqslant c(u) \sum_{s,t} \big(\varphi(s^*t)f(t), f(s)\big), \tag{6}$$

where $c(u) < +\infty$ does not depend on $f \in F(S, H)$.

The general dilation theorem(¹) in the case where S has a unit says that:

(D) *An operator function $\varphi\colon S \to L(H)$ is dilatable if and only if it is positive definite and satisfies the boundedness condition* (6).

(¹) This theorem has been proved by B. Sz.-Nagy [5] provided $\varphi(1) = I$. In the general case the proof is much the same as that in [5].

On the other hand, it is evident that if Φ is a $*$-representation of S and S has no unit, then Φ_1 defined as $\Phi(s)$ for $s \in S$ and I if $s = 1$, is a $*$-representation of S_1. This implies that *an operator function* φ: $S \to L(H)$ *is dilatable if and only if it extends to a dilatable operator function* φ_1: $S_1 \to L(H)$. Combining this with (D), we see that φ: $S \to L(H)$ is dilatable if and only if it has an extention φ_1: $S_1 \to L(H)$ which is positive definite and satisfies the boundedness condition (6). We would like to prove that (3) and (4) together guarantee that there is an extension φ_1 of φ which is positive definite and satisfies the boundedness condition. Proposition 1 gives a partial success in this direction. Unfortunately, we are not able to show that φ_1 as defined in the proof of Proposition 1 satisfies the boundedness condition (6).

5. The obstructions with the boundedness condition disappear in the case of operator functions on Banach star algebras (we refer here to [1]). Let A be a Banach star algebra and let A_1 denote its *unitization*, that is, the Banach star algebra $A + C$. Let φ: $A \to L(H)$ be a linear map. Suppose φ has the extension property (3) and satisfies the symmetry relation (4). Define φ_1: $A_1 \to L(H)$ by $\varphi_1(a+\lambda) = \varphi(a) + \lambda A$ with $A \geqslant cI$. Then

$$
\begin{aligned}
(7)\qquad & \sum_{a,b,\lambda,\mu} \big(\varphi_1((a+\lambda)^*(b+\mu))f(b+\mu), f(a+\lambda)\big) \\
&= \sum_{a,b,\lambda,\mu} \big((\varphi(a^*b) + \bar\lambda\varphi(b) + \mu\varphi(a^*) + \bar\lambda\mu A)f(b+\mu), f(a+\lambda)\big) \\
&= \sum_{a,b,\lambda,\mu} \Big(\big(\varphi(a^*b)f(b+\mu), f(a+\lambda)\big) + \big(\varphi(b)f(b+\mu), \lambda f(a+\lambda)\big) + \\
&\qquad + \big(\mu f(b+\mu), \varphi(a)f(a+\lambda)\big) + \big(A\mu f(b+\mu), \lambda f(a+\lambda)\big)\Big) \\
&\geqslant \frac{1}{c}\Big(\Big\|\sum_{a,\lambda}\varphi(a)f(a+\lambda)\Big\|^2 + \Big(\sum_{b,\mu}\varphi(b)f(b+\mu), \sum_{a,\lambda}c\lambda f(a+\lambda)\Big) + \\
&\qquad + \Big(\sum_{b,\mu}c\mu f(b+\mu), \sum_{a,\lambda}\varphi(a)f(a+\lambda)\Big) + \Big(\sum_{b,\mu}c\mu f(b+\mu), \sum_{a,\lambda}c\lambda f(a+\lambda)\Big)\Big) \\
&= \frac{1}{c}\Big\|\sum_{a,\lambda}(\varphi(a) + c\lambda I)f(a+\lambda)\Big\|^2 \geqslant 0.
\end{aligned}
$$

This proves that φ_1 is positive definite. Now fix $f \in F(A_1, H)$ and define p_f: $A_1 \to C$ by

$$
(8)\qquad p_f(c_1) = \sum_{a_1,b_1}\big(\varphi_1(a_1^* c_1 b_1)f(b_1), f(a_1)\big).
$$

p_f is a positive linear functional on A_1. It is known [1], p. 198, Remarks, that then it must be

$$|p_f(c_1)| \leqslant p_f(1)\big(r(c_1^* c_1)\big)^{1/2},$$

where r denotes the spectral radius of $c_1^* c_1$. With (7) in mind we verify that φ_1 satisfies the boundedness condition (6). The general dilation theorem (D) implies the following

THEOREM. *Let A be a Banach star algebra and let $\varphi\colon A \to L(H)$ be a linear map. Then φ is dilatable if and only if it has the extention property, i.e.,*

$$\Big\|\sum_a \varphi(a)f(a)\Big\|^2 \leqslant c \sum_{a,b} \big(\varphi(a^*b)f(b), f(a)\big)$$

with c independent of $f \in F(A, H)$, and satisfies the symmetry relation (4).

The "only if" part of the above theorem is trivial. The dilation Φ of φ may be chosen to be a linear map (cf. [5]). Thus Φ is a $*$-representation of A and, consequently [1], p. 196, Theorem 3, is continuous. We get therefore

COROLLARY. *The function φ in Theorem 2 must be necessarily a continuous linear map of A into $L(H)$.*

The corollary shows that our theorem contains Theorem 2 of [3].

Note added in proof. October 1978. In the meantime we have answered in the affirmative the question left open at the end of Section 3, see: Bull. Acad. Polon. Sci., Sér. sci. math., astr., phys. 25 (1977), p. 263–267, and also Springer's Lecture Notes in Math., vol. 656, p. 245–249. Our related papers are: Bull. Acad. Polon. Sci., Sér. Sci. math., astr., phys. 24 (1976), p. 877–881, and Proc. Amer. Math. Soc. 66 (1977), p. 30–32.

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