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On the coefficients of starlike functions of some classes

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1. Let m, M be arbitrary fixed numbers which satisfy the condition $(m, M) \in D$, where $D = D_1 \cup D_2$,

$$D_1 = \{(m, M)\colon \tfrac{1}{2} < m < 1,\ 1-m < M \leqslant m\},$$

$$D_2 = \{(m, M)\colon 1 \leqslant m,\ m-1 < M \leqslant m\}.$$

Denote by $\mathfrak{P}_{m,M}$, $(m, M) \in D$, the family of all regular functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

defined in the circle $K = \{z\colon |z| < 1\}$ which satisfy in this circle the condition

$$|p(z) - m| < M.$$

The following sharp estimations

$$|p_n| \leqslant a+b, \quad n = 1, 2, \ldots \tag{1.1}$$

with

$$a = (M^2 - m^2 + m)\, M^{-1}, \quad b = (m-1)\, M^{-1} \tag{1.2}$$

have been found with the aid of the Clunie method [2] in [4].

The aim of this paper is to prove a more general theorem and then to employ it for finding the estimations of the coefficients of starlike functions of some classes.

2. Denote by Q the family of regular functions of the form

$$q(z) = \sum_{k=1}^{\infty} q_k z^k \tag{2.1}$$

which are defined in the circle K and satisfy the condition

$$|q(z)| < 1 \quad \text{for } z \in K. \tag{2.2}$$

Then $|q_k| \leqslant 1$, $k = 1, 2, \ldots$, and if $|q_n| = 1$, then $q(z) = \varepsilon z^n$, $|\varepsilon| = 1$.

Suppose we are given regular functions $g(z)$ and $h(z)$ of the form

$$g(z) = \sum_{k=0}^{\infty} g_k z^k, \quad h(z) = \sum_{k=0}^{\infty} h_k z^k \tag{2.3}$$

defined in the circle K with $g_0 = h_0 \neq 0$ or $g_0 = h_0 = 0$ and $g_1 = h_1 \neq 0$. Assume, moreover, that $h(z) \neq 0$ for $z \in K$, $z \neq 0$ and that

$$|g(z)h^{-1}(z) - m| < M \quad \text{for every } z \in K. \tag{2.4}$$

Then there exists a function $q \in Q$ such that (comp. [4])

$$g(z)h^{-1}(z) = [1 + aq(z)][1 - bq(z)], \quad z \in K, \tag{2.5}$$

with a and b defined by formulas (1.2) and $(m, M) \in D$. Conversely, for an arbitrary function $q \in Q$ there exist a pair of functions satisfying all the conditions mentioned above.

We shall prove the following

THEOREM 1. *For every fixed pair* $(m, M) \in D$ *and every fixed pair of functions* $g(z)$ *and* $h(z)$ *satisfying conditions* (2.3), (2.4) *and* (2.5), *the following conditions hold:*

$$g_1 - h_1 = (a+b) g_0 q_1, \tag{2.6}$$

$$g_n - h_n = (a+b) g_0 q_n + \sum_{k=1}^{n-1} (a h_k + b g_k) q_{n-k}, \quad n = 2, 3, \ldots, \tag{2.7}$$

and

$$|g_1 - h_1|^2 \leqslant (a+b)^2 |g_0|^2, \tag{2.8}$$

$$|g_n - h_n|^2 \leqslant (a+b)^2 |g_0|^2 - \sum_{k=1}^{n-1} A_k, \quad n = 2, 3, \ldots, \tag{2.9}$$

and

$$\sum_{k=1}^{\infty} A_k \leqslant (a+b)^2 |g_0|^2 \tag{2.10}$$

with

$$A_k = |g_k - h_k|^2 - |a h_k + b g_k|^2, \quad k = 1, 2, \ldots \tag{2.11}$$

Proof. (2.1), (2.3) and (2.5) yield the identity

$$\sum_{k=1}^{\infty} (g_k - h_k) z^k = \Big[(a+b) g_0 + \sum_{k=1}^{\infty} (a h_k + b g_k) z^k\Big]\Big(\sum_{k=1}^{\infty} q_k z^k\Big), \quad z \in K. \tag{2.12}$$

Equating the corresponding coefficients, we get relationships (2.6) and (2.7).

Since $|q_1| \leqslant 1$, from relationship (2.6) we obtain inequality (2.8). For $n \geqslant 2$ identity (2.12) may be written in the form

$$\sum_{k=1}^{n} (g_k - h_k) z^k + \sum_{k=n+1}^{\infty} d_k z^k = [(a+b) g_0 + \sum_{k=1}^{n-1} (a h_k + b g_k) z^k] q(z),$$

where the coefficients d_k have been chosen suitably. Making use of condition (2.2), we find in the circle K the condition

$$\Big|\sum_{k=1}^{n}(g_k-h_k)z^k+\sum_{k=n+1}^{\infty}d_kz^k\Big|^2<\Big|(a+b)g_0+\sum_{k=1}^{n-1}(ah_k+bg_k)z^k\Big|.$$

Assume $z=re^{it}$, $0<r<1$, $0\leqslant t\leqslant 2\pi$ and integrate the resulting inequality in the interval $\langle 0,2\pi\rangle$. Then we find the inequality

$$\sum_{k=1}^{n}|g_k-h_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|d_k|^2r^{2k}<(a+b)^2|g_0|^2+\sum_{k=1}^{n-1}|ah_k+bg_k|^2r^{2k}.$$

Hence we get

$$\sum_{k=1}^{n}|g_k-h_k|^2r^{2k}<(a+b)^2|g_0|^2+\sum_{k=1}^{n-1}|ah_k+bg_k|^2r^{2k}.$$

Passing to the limit as $r\to 1$ and taking into consideration estimations (2.11), we hence obtain inequality (2.9).

Employing condition (2.2) immediately in identity (2.12), by a similar argument we deduce inequality (2.10).

Remark. Estimation (1.1) immediately follows from inequalities (2.8) and (2.9). Because assuming $g(z)=p(z)$, $p\in\mathfrak{P}_{m,M}$, $h(z)=1$ ($g_0=h_0=1$, $g_k=p_k$, $h_k=0$, $k=1,2,\ldots$) since for $(m,M)\in D$, we have $|b|<1$, therefore $A_k=(1-b^2)|p_k|^2\geqslant 0$ for $k=1,2,\ldots$ Thus $|p_n|^2\leqslant(a+b)^2$ for $n=1,2,\ldots$ But if $(m,M)\in D$, then $a+b>0$; therefore estimation (1.1) is true.

From Theorem 1 we get

COROLLARY 1. *If for given m, M, g, h all the terms A_k are non-negative, then*

$$|g_n-h_n|\leqslant(a+b)|g_0|,\qquad n=1,2,\ldots, \tag{2.13}$$

equality being the case only if $A_k=0$ for $k=1,2,\ldots$

COROLLARY 2. *If for given m, M, g, h there exists an index N ($N=3,4,\ldots$) such that $A_k\leqslant 0$ for $k=1,2,\ldots,N-2$ and $A_k\geqslant 0$ for $k=N-1,N,\ldots$, then the following estimations hold:*

$$|g_{n-1}-h_{n-1}|^2\leqslant\begin{cases}(a+b)^2|g_0|^2, & n=2,\\ (a+b)^2|g_0|^2-\sum\limits_{k=1}^{n-2}A_k, & n=3,4,\ldots,N,\\ (a+b)^2|g_0|^2-\sum\limits_{k=1}^{N-2}A_k, & n=N+1,N+2,\ldots\end{cases} \tag{2.14}$$

3. Consider the class $\tilde{S}$ of all regular functions

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{3.1}$$

defined in the circle K. Let m, M, a, β be arbitrary fixed numbers satisfying the conditions $(m, M)\in D$, $a\in \langle 0,1)$, $\beta\in(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Denote by $S^{*}_{m,M}(a,\beta)$ the subclass of the family $\tilde{S}$ of functions satisfying at every point of the circle K the inequality

$$(3.2)\qquad |[zf'(z)f^{-1}(z)e^{i\beta}-i\sin\beta-a\cos\beta](1-a)^{-1}\cos^{-1}\beta-m|<M.$$

We introduce the following notation:

$$(3.3)\qquad A=(a+b)(1-a)\cos\beta,$$

$$(3.4)\qquad k_0=A\left(b\cos\beta+(1-b^2\sin^2\beta)^{1/2}\right)(1-b^2)^{-1},$$

$$(3.5)\qquad N=[k_0]+2,$$

$$(3.6)\qquad B=(1-a)^{-1}\cos^{-1}\beta\left[\left((1-m)(1-a)+1\right)^2\cos^2\beta+\sin^2\beta\right]^{1/2}$$

with a and b defined by formulas (1.2). We will prove

THEOREM 2. *Let f be an arbitrary function of the family $S^{*}_{m,M}(a,\beta)$. If $(m,M)\in D$ and $M\leqslant B$, then*

$$(3.7)\qquad |a_n|\leqslant A(n-1)^{-1},\quad n=2,3,\ldots$$

If, on the other hand, $(m,M)\in D$ and $M>B$, then

$$(3.8)\qquad |a_n|\leqslant\frac{1}{(n-1)!}\prod_{k=0}^{n-2}|A+kbe^{i\beta}|,\quad n=2,\ldots,N,$$

and

$$(3.9)\qquad |a_n|\leqslant\frac{1}{(n-1)(N-2)!}\prod_{k=0}^{N-2}|A+kbe^{i\beta}|,\quad n=N+1,\ldots$$

Estimations (3.7) and (3.8) are sharp, the equality sign being realized for the functions

$$(3.10)\qquad f(z)=z\exp\left(\varepsilon Ae^{-i\beta}\frac{z^{n-1}}{n-1}\right),$$

$$|\varepsilon|=1\quad \textit{when } M\leqslant B \textit{ and } b=0,$$

$$(3.11)\qquad f(z)=z(1-b\varepsilon z^{n-1})^{-Ab^{-1}(n-1)^{-1}e^{-i\beta}},$$

$$|\varepsilon|=1\quad \textit{when } M\leqslant B \textit{ and } b\neq 0,$$

$$(3.12)\qquad f(z)=z(1-\varepsilon bz)^{-Ab^{-1}e^{-i\beta}},\quad |\varepsilon|=1\quad \textit{when } M>B$$

respectively. Moreover,

$$(3.13)\qquad \sum_{k=1}^{\infty}[k^2-|A+kbe^{i\beta}|^2]|a_{k+1}|^2\leqslant A^2.$$

Proof. Since $f \in S^*_{m,M}(a, \beta)$, it follows from (3.2) that the functions

$$g(z) = f'(z)e^{i\beta} - (i\sin\beta + a\cos\beta)f(z)/z, \tag{3.14}$$

$$h(z) = (1-a)\cos\beta \cdot f(z)/z \tag{3.15}$$

satisfy the assumptions of Theorem 1. From formulas (2.3), (3.1), (3.14) and (3.15) we obtain

$$g_0 = h_0 = (1-a)\cos\beta, \quad h_k = a_{k+1}(1-a)\cos\beta,$$
$$g_k = h_k + ke^{i\beta}a_{k+1}, \quad k = 1, 2, \ldots$$

Thus by (2.6)-(2.11) and (3.3) we find the relationships

$$a_2 e^{i\beta} = Aq_1, \tag{3.16}$$

$$(n-1)a_n e^{i\beta} = Aq_{n-1} + \sum_{k=1}^{n-2} (A + kbe^{i\beta})a_{k+1}q_{n-k-1}, \quad n = 3, 4, \ldots, \tag{3.17}$$

$$|a_2| \leqslant A, \tag{3.18}$$

$$(n-1)^2|a_n|^2 \leqslant A^2 - \sum_{k=1}^{n-2} |a_{k+1}|^2 B_k, \quad n = 3, 4, \ldots, \tag{3.19}$$

$$\sum_{k=1}^{\infty} B_k|a_{k+1}|^2 \leqslant A^2, \tag{3.20}$$

where

$$B_k = k^2 - |A + kbe^{i\beta}|^2 = (1-b^2)k^2 - 2Abk\cos\beta - A^2, \tag{3.21}$$
$$k = 1, 2, \ldots$$

If $n = 2$, estimations (3.7) and (3.8) result immediately from inequality (3.18). Because of (3.16) equality can be the case only if $|q_1| = 1$, i.e. if function (2.1) is of the form $q(z) = \varepsilon z$, $|\varepsilon| = 1$. Thus relationships (2.5), (3.14) and (3.15) imply formulas (3.10)-(3.12).

Let $n \geqslant 3$. The polynomial $(1-b^2)x^2 - 2Abx\cos\beta - A^2$ has two roots, x_1 and $x = k_0$ (comp. (3.4)) and $x_1 < 0$, $x_2 > 0$.

If $M \leqslant B$, then by (3.4) and (3.6) we have $k_0 \leqslant 1$; thus it follows rom (3.21) that $B_1 \geqslant 0$ and $B_k > 0$, $k = 2, 3, \ldots$ Thus estimation (3.7) for $n \geqslant 3$ is immediately obtained from inequalities (3.19) (comp. Corollary 1). The extremal functions are here of form (3.10) or (3.11) according as $b = 0$ or $b \neq 0$.

If $M > B$, then $k_0 > 1$; thus $B_1, \ldots, B_{N-2} \leqslant 0$ and $B_{N-1}, \ldots > 0$, where N is defined by formula (3.5). Thus from (3.19) we get the following inequality (comp. Corollary 2):

$$(n-1)^2|a_n|^2 \leqslant A^2 - \sum_{k=1}^{n-2} |a_{k+1}|^2 B_k \quad \text{for } n = 3, \ldots, N$$

and

$$(n-1)^2|a_n|^2 \leqslant A^2 - \sum_{k=1}^{N-2} |a_{k+1}|^2 B_k \quad \text{for } n = N+1, \ldots$$

Employing the estimation $|a_2| \leqslant A$ and notation (3.21) from the first of these inequalities we get estimation (3.8), while the second implies inequality (3.9).

Since functions (3.10)-(3.12) belong to the class of functions under consideration, estimations (3.7) and (3.8) are sharp.

The last part of the assertion, inequality (3.13), results immediately from relations (3.20) and (3.21).

4. Some results obtained earlier by other authors can be obtained immediately from Theorem 2. So, assuming in this theorem in succession $m = M \geqslant 1$, $a = 0$, $\beta = 0$ and $m = M \geqslant 1$, $a \in \langle 0, 1)$, $\beta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we have the results published in papers [5] and [10].

In particular, we also have the following corollaries.

COROLLARY 3 [7]. *If $f \in S^*_{\infty,\infty}(a, \beta)$ (β-spiral starlike functions of order a), then*

$$(4.1) \qquad |a_n| \leqslant \frac{1}{(n-1)!} \prod_{k=0}^{n-2} |2(1-a)\cos\beta + ke^{i\beta}|, \qquad n = 2, 3, \ldots$$

In particular, assuming $a = 0$, we find the estimation of the modulus of the coefficients [15] in the class of spiral starlike functions [13].

COROLLARY 4 [12]. *If $f \in S^*_{\infty,\infty}(a, 0)$ (starlike functions of order a) then*

$$(4.2) \qquad |a_n| \leqslant \frac{1}{(n-1)!} \prod_{k=0}^{n-2} \big(2(1-a)+k\big), \qquad n = 2, 3, \ldots$$

Assuming $a = 0$, we obtain the estimation of the coefficients [9] in the family S^* of all starlike functions [1]. Estimations (4.1) and (4.2) are sharp, the extremal functions being of the form

$$f(z) = z(1-\varepsilon z)^{-2(1-a)e^{-i\beta}\cos\beta}, \qquad |\varepsilon| = 1$$

and

$$f(z) = z(1-\varepsilon z)^{-2(1-a)}, \qquad |\varepsilon| = 1$$

respectively.

Assuming, in Theorem 2, $m = 1$ $(0 < M \leqslant 1)$, we obtain

COROLLARY 5. *If $f \in S^*_{1,M}(a, \beta)$, then*

$$|a_n| \leqslant (n-1)^{-1} M(1-a)\cos\beta, \qquad n = 2, 3, \ldots,$$

the extremal function being of the form

$$f(z) = z\exp\Big(\varepsilon M(1-a)\frac{z^{n-1}}{n-1}e^{-i\beta}\cos\beta\Big), \qquad |\varepsilon| = 1.$$

In particular, we get the well-known estimation $|a_n| \leqslant (n-1)^{-1}$, $n = 2, 3, \ldots$, [10] in the class $S^*_{1,1}(0, 0)$.

5. Denote by $\hat{S}$ the well-known family of convex functions [14] of the form

$$F(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in K. \tag{5.1}$$

It is known ([3], p. 204) that $F \in \hat{S}$ if and only if the function

$$f(z) = zF'(z) \tag{5.2}$$

belongs to the class S^*.

Consider the family $\hat{S}_{m,M}(a, \beta)$ of functions of form (5.1) which satisfy in the circle K the condition

$$\left|\left[\big(zF''(z) + F'(z)\big)F'^{-1}(z)e^{i\beta} - i\sin\beta - a\cos\beta\right](1-a)^{-1}\cos^{-1}\beta - m\right| < M.$$

Then function (5.2) belongs to the family $S^*_{m,M}(a, \beta)$. Since $a_k = kb_k$, $k = 2, 3, \ldots$, from Theorem 2 we obtain

COROLLARY 6. *Let F be an arbitrary function of the family $\hat{S}_{m,M}(a, \beta)$. If $(m, M) \in D$ and $M \leqslant B$, then*

$$|b_n| \leqslant \frac{A}{n(n-1)}, \quad n = 2, 3, \ldots \tag{5.3}$$

If, on the other hand, $(m, M) \in D$ and $M > B$, then

$$|b_n| \leqslant \frac{1}{n!} \prod_{k=0}^{n-2} |A + kbe^{i\beta}|, \quad n = 2, \ldots, N, \tag{5.4}$$

and

$$|b_n| \leqslant \frac{1}{n(n-1)(N-2)!} \prod_{k=0}^{N-2} |A + kbe^{i\beta}|, \quad n = N+1, N+2, \ldots$$

Estimations (5.3) and (5.4) are sharp, the extremal functions being of the form $F(z) = \int_0^z f(z) z^{-1} dz$ with $f(z)$ defined by formulas (3.10)-(3.12) respectively.

Moreover,

$$\sum_{k=1}^{\infty} [k^2 - |A + kbe^{i\beta}|^2](k+1)^2 |b_{k+1}|^2 \leqslant A^2.$$

COROLLARY 7. *If $F \in \hat{S} = \hat{S}_{\infty,\infty}(0, 0)$, then the sharp estimation* [8] $|b_n| \leqslant 1$, $n = 2, 3, \ldots$, *holds.*

6. Let F denote the family of all regular and univalent functions

$$F(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \tag{6.1}$$

defined in the ring $Q = \{z\colon 0 < |z| < 1\}$.

We know the subclasses of the family F: 1° the subclass $F^*(a)$, $0 \leqslant a < 1$ of starlike functions of order a, 2° the subclass $F^*(a, \beta)$, $0 \leqslant a < 1$, $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$ of β-spiral starlike functions of order a and others.

Denote by $F^*_{m,M}(a, \beta)$, $(m, M) \in D$ the family of regular functions of form (6.1) defined in the ring Q and such that

$$|[-zF'(z)F^{-1}(z)e^{i\beta} - i\sin\beta - a\cos\beta](1-a)^{-1}\cos^{-1}\beta - m| < M \qquad \text{for } z \in Q.$$

Applying Theorem 1 (and, in particular, Corollary 1) to the function

$$g(z) = -z[zF'(z)e^{i\beta} + (i\sin\beta + a\cos\beta)F(z)]$$

$$= (1-a)\cos\beta - \sum_{k=1}^{\infty}[ke^{i\beta} + i\sin\beta + a\cos\beta]a_k z^{k+1},$$

$$h(z) = z(1-a)\cos\beta \cdot F(z) = (1-a)\cos\beta + (1-a)\cos\beta \sum_{k=1}^{\infty} a_k z^{k+1},$$

by a similar argument as in the proof of Theorem 2 we obtain

THEOREM 3. *If* $F \in F^*_{m,M}(a, \beta)$, *then*

$$|a_n| \leqslant A(n+1)^{-1}, \qquad n = 1, 2, \ldots, \tag{6.2}$$

and

$$\sum_{k=2}^{\infty}[k^2 - |A - bke^{i\beta}|^2]|a_{k-1}|^2 \leqslant A^2,$$

where $A = (a+b)(1-a)\cos\beta$, a *and* b *are defined by formulas* (1.2).

Estimations (6.2) *are sharp. The extremal functions are of the form*

$$F(z) = z^{-1}(1-\varepsilon b z^{n+1})^{Ab^{-1}(n+1)^{-1}e^{-i\beta}}, \qquad |\varepsilon| = 1 \qquad \textit{when } b \neq 0$$

or

$$F(z) = z^{-1}\exp\left(-A\varepsilon e^{-i\beta}\frac{z^{n+1}}{n+1}\right), \qquad |\varepsilon| = 1 \qquad \textit{when } b = 0.$$

In particular, the known estimations in the classes $F^*_{M,M}(a, \beta)$ and $F^*_{M,M}(a, 0)$, $M \geqslant 1$, [6] can be obtained from Theorem 3. In the family $F^*_{\infty,\infty}(a, 0)$ we obtain the sharp estimations [6]

$$|a_n| \leqslant 2(1-a)(n+1)^{-1}, \qquad n = 1, 2, \ldots, \tag{6.3}$$

and the inequality [11]

$$\sum_{k=2}^{\infty}(k-1+a)|a_{k-1}|^2 \leqslant 1-a.$$

Result (6.3) for $a = 0$ has been obtained earlier by Clunie [2].

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