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A new theory of Operational Calculus

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I. Introduction

The aim of this paper is to suggest a new theory of Operational Calculus, as general as possible.

Existent theories of Operational Calculus are based mostly on the Laplace Transformation

$$(1) \qquad F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

or on the Laplace–Carson Transformation

$$(2) \qquad F(p) = p \int_0^\infty e^{-pt} f(t)\, dt$$

which is an unessential modification of (1). To such theories one objection can be made, that they exclude all such functions $f(t)$ for which the integrals (1) and (2) diverge. Thus, in problems which admit such functions, they do not give the possibility to prove that solutions obtained are unique.

J. G. -Mikusiński ([2]), [3]) constructed — without use of the transformation (1) or (2) — a theory of Operational Calculus, which enables the introduction of all functions $f(t)$ integrable in every finite interval $0 \leqslant t \leqslant T$, i. e. also functions, for which the integrals (1) and (2) diverge. Funtions $F(s)$, obtained as Laplace transforms (1), correspond to operators $\{f(t)\}$ in the theory of Mikusiński. Only in case of rational and certain exponential functions $F(s)$ operators of Mikusiński are denoted by symbols

$$\frac{a_m s^m + \ldots + a_1 s + a_0}{b_n s^n + \ldots + b_1 s + b_0}, \quad e^{-s\lambda}, \quad e^{-\sqrt{s\lambda}} \text{ etc.}$$

$(a_m, \ldots, a_0; b_n, \ldots, b_0; \lambda$ — numerical coefficients), where, however, s does not denote a variable but an operator. Hence many methods useful on the base of Laplace Transformation have no analogue in the theory of Mikusiński.

The theory presented in this paper enables — like the theory of Mikusiński — the introduction of all functions $f(t)$ integrable in every finite interval $0 \leqslant t \leqslant T$ and moreover, the use of all methods and theorems from the theory of Laplace–Carson Transformation (or Laplace Transformation) in the smaller class of functions, for which the integrals (1) and (2) converge.

The principal idea is the following. We shall divide a class of functions $F(p, t)$ into abstract classes in such a way that:

1° the functions

$$\int_0^t F(p, \tau)\, d\tau \quad \text{and} \quad \frac{1}{p} F(p, t)$$

will be equivalent, i. e. they will belong to the same abstract class,

2° a continuous function $f(t)$ will be equivalent to zero, if and only if it equals zero identically.

The first condition is an extension of the well-known fact that the functions

$$\int_0^t f(\tau)\, d\tau \quad \text{and} \quad \frac{1}{p} f(t)$$

must correspond in the Operational Calculus. The second condition is necessary to enable a pass from a solution given as an equivalence to a solution given as an equality.

It follows that every function

$$H(p, t) \equiv F(p, t) - p \int_0^t F(p, \tau)\, d\tau \tag{3}$$

must be equivalent to zero. But the identity (3) has an unique solution

$$F(p, t) \equiv H(p, t) + p e^{pt} \int_0^t e^{-p\tau} H(p, \tau)\, d\tau. \tag{4}$$

Therefore we must construct such a class of feasible functions $F(p, t)$, that the equality $H(p, t) = f(t)$ where $f(t)$ is a continuous function, will imply $f(t) \equiv 0$, according to the second condition.

Let us notice, that if we defined the class of feasible functions as the class of all such functions $F(p, t)$, that for sufficiently great fixed p

$$\lim_{t \to \infty} e^{-pt} F(p, t) = 0$$

and $f(t)$ was a function for which — p being sufficiently large and fixed — the integral

$$\int_0^\infty e^{-pt} f(t)\, dt$$

converges with $\lim_{t \to \infty} e^{-pt} f(t) = 0$, then putting $H(p, t) = g(p) - f(t)$ we should obtain from (4)

$$e^{-pt} F(p, t) = e^{-pt} g(p) - e^{-pt} f(t) + p \int_0^t e^{-p\tau} g(p)\, d\tau - p \int_0^t e^{-p\tau} f(\tau)\, d\tau$$

$$= g(p) - e^{-pt} f(t) - p \int_0^t e^{-p\tau} f(\tau)\, d\tau$$

and as $t \to \infty$

$$g(p) = p \int_0^\infty e^{-p\tau} f(\tau)\, d\tau$$

according to (2). Thus, the simple first condition would already imply the formula (2).

But in this paper we shall deal with a larger class of feasible functions. It will be also convenient to define this class by means of formula (4) instead of (3).

I want to express my personal gratitude to Professor C. Ryll-Nardzewski, for his useful suggestions during the final preparation of the manuscript.

II. Ring $\mathscr{R}$. Subring $\mathscr{F}$

Let us consider such a class of functions $F(p, t)$, real or complex, of a real or complex variable p and a real variable t, that for every $F(p, t)$ there exists such a real non-negative number a_F that for every fixed p_0 satisfying the condition $\operatorname{Re} p_0 > a_F$ the function $F(p_0, t)$ is defined for almost every $t > 0$ and integrable in the Lebesgue sense in every finite interval $0 \leqslant t \leqslant T$. Putting $F(p, t) = 0$ for $\operatorname{Re} p \leqslant a_F$, we obtain a class of functions $F(p, t)$, which for every fixed p satisfying the condition $\operatorname{Re} p \geqslant 0$ are defined for almost every $t > 0$ and integrable in the Lebesgue sense in every finite interval $0 \leqslant t \leqslant T$ (1).

In our class of functions $F(p, t)$ we introduce a multiplication defined as follows

$$(5) \qquad F_1(p, t) \times F_2(p, t) = p \int_0^t F_1(p, t-\tau) F_2(p, \tau)\, d\tau .$$

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(1) For the sake of this theory it would be sufficient to consider a fixed sequence $\lambda_1, \lambda_2, \ldots$ of real positive numbers, satisfying the following two conditions:

1° $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$,

2° $\lambda_{n+1} - \lambda_n > \delta > 0 \quad (n = 1, 2, \ldots)$

and to assume that for every function $F(p, t)$ there exists such a positive integer N_F that for every $n \geqslant N_F$ the functions $\varphi_n(t) = F(\lambda_n, t)$ are defined for almost every $t > 0$ and integrable in the Lebesgue sense in every finite interval $0 \leqslant t \leqslant T$. In such a way it would be sufficient to consider sequences of functions $\varphi_N(t), \varphi_{N+1}(t), \ldots$ instead of functions $F(p, t)$, but then we should not be able to include the whole theory of Laplace Transformation and all its methods.

In virtue of Convolution Theorem ([1], p. 110) for every fixed p ($\operatorname{Re} p \geqslant 0$) the integral (5) exists for almost all $t > 0$ and it is integrable in the Lebesgue sense in every finite interval $0 \leqslant t \leqslant T$.

It is easy to prove that the multiplication (5) is associative, commutative and distributive with respect to ordinary addition.

It follows that our class of functions $F(p, t)$ forms a ring with respect to ordinary addition and multiplication (5). We shall call it *ring* $\mathscr{R}$.

If $F(p, t) = f(t)$ and $F(p, t) \in \mathscr{R}$, we shall write simply $f(t) \in \mathscr{R}$. Similarly, if $F(p, t) = g(p)$ and $F(p, t) \in \mathscr{R}$, we shall write $g(p) \in \mathscr{R}$.

DEFINITION. A function $F(p, t)$ will be called *feasible*, if and only if:

1° $F(p, t) \in \mathscr{R}$,

2° there exists such a real non-negative number a, that for every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-ap} F(p, \tau)| \, d\tau = 0, \tag{6}$$

where we shall always assume, that $p \to \infty$ along the real axe.

Let $\mathscr{F}$ be the set of all feasible functions $F(p, t)$. We shall prove that $\mathscr{F}$ forms a subring of $\mathscr{R}$.

Indeed, let $F_1(p, t) \in \mathscr{F}$ and $F_2(p, t) \in \mathscr{F}$. It means that there exist such real non-negative numbers a_1 and a_2 that

$$\lim_{p\to\infty} \int_0^t |e^{-a_k p} F_k(p, \tau)| \, d\tau = 0 \qquad (k = 1, 2).$$

Putting $a = \max(a_1, a_2)$ we obtain for every $t > 0$

$$\begin{aligned}
&\int_0^t |e^{-ap} [F_1(p, \tau) \pm F_2(p, \tau)]| \, d\tau \\
&\leqslant \int_0^t |e^{-ap} F_1(p, \tau)| \, d\tau + \int_0^t |e^{-ap} F_2(p, \tau)| \, d\tau \\
&= e^{-(a-a_1)\operatorname{Re} p} \int_0^t |e^{-a_1 p} F_1(p, \tau)| \, d\tau + e^{-(a-a_2)\operatorname{Re} p} \int_0^t |e^{-a_2 p} F_2(p, \tau)| \, d\tau \\
&\leqslant \int_0^t |e^{-a_1 p} F_1(p, \tau)| \, d\tau + \int_0^t |e^{-a_2 p} F_2(p, \tau)| \, d\tau .
\end{aligned}$$

Hence

$$\lim_{p\to\infty} \int_0^t |e^{-ap} [F_1(p, \tau) \pm F_2(p, \tau)]| \, d\tau = 0$$

and

$$F_1(p, t) + F_2(p, t) \in \mathscr{F}, \qquad F_1(p, t) - F_2(p, t) \in \mathscr{F}.$$

Putting any $\beta > \alpha_1 + \alpha_2$ we have further

$$\int_0^t \left| e^{-\beta p} p \int_0^\sigma F_1(p, \sigma-\tau) F_2(p, \tau)\, d\tau \right| d\sigma$$

$$\leqslant \int_0^t \int_0^\sigma |e^{-\beta p} p F_1(p, \sigma-\tau) F_2(p, \tau)|\, d\tau d\sigma$$

$$= \int_0^t \int_\tau^t |e^{-\beta p} p F_1(p, \sigma-\tau) F_2(p, \tau)|\, d\sigma d\tau$$

$$= |pe^{-(\beta-\alpha_1-\alpha_2)p}| \int_0^t |e^{-\alpha_2 p} F_2(p, \tau)| \int_0^{t-\tau} |e^{-\alpha_1 p} F_1(p, \vartheta)|\, d\vartheta d\tau$$

$$\leqslant |pe^{-(\beta-\alpha_1-\alpha_2)p}| \int_0^t |e^{-\alpha_2 p} F_2(p, \tau)|\, d\tau \int_0^t |e^{-\alpha_1 p} F_1(p, \vartheta)|\, d\vartheta .$$

Hence

$$\lim_{p\to\infty} \int_0^t \left| e^{-\beta p} p \int_0^\sigma F_1(p, \sigma-\tau) F_2(p, \tau)\, d\tau \right| d\sigma = 0$$

and

$$F_1(p, t) \times F_2(p, t) \in \mathscr{F}.$$

Thus $\mathscr{F}$ forms a ring with respect to ordinary addition and multiplication (5). It is a subring of $\mathscr{R}$.

Let us notice, that in virtue of (6) if $F(p, t) \in \mathscr{F}$ then also $|F(p, t)| \in \mathscr{F}$. Moreover, if $F(p, t) = f(t)$, where $f(t)$ is a function integrable in the Lebesgue sense in every finite interval $0 \leqslant t \leqslant T$, then $f(t) \in \mathscr{F}$.

It is also easy to show, that if $F(p, t) = p^\alpha$, α being an arbitrary real number, then $F(p, t) \in \mathscr{F}$.

THEOREM 1. *If $F(p, t) \in \mathscr{F}$ and $G(p, t) = g(p) \in \mathscr{F}$, then*

$$g(p) F(p, t) \in \mathscr{F}.$$

Proof. By assumption, $F(p, t) \in \mathscr{R}$ and $g(p) \in \mathscr{R}$. It is obvious, that also $g(p) F(p, t) \in \mathscr{R}$.

By assumption, there exist such real non-negative numbers α_1 and α_2, that for every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-\alpha_1 p} F(p, \tau)|\, d\tau = 0, \quad \lim_{p\to\infty} \int_0^t |e^{-\alpha_2 p} g(p)|\, d\tau = 0.$$

But

$$\int_0^t |e^{-\alpha_2 p} g(p)|\, d\tau = t |e^{-\alpha_2 p} g(p)|$$

and thus for every $t > 0$

$$\lim_{p\to\infty} |e^{-a_2 p} g(p)| = 0.$$

Hence for every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-(a_1+a_2)p} g(p) F(p, \tau)| d\tau = \lim_{p\to\infty} |e^{-a_2 p} g(p)| \int_0^t |e^{-a_1 p} F(p, \tau)| d\tau = 0$$

and the function $g(p)F(p, t)$ is, really, feasible, which was to be proved.

THEOREM 2. *If* $F(p, t) \in \mathscr{F}$ *then also*

$$\int_0^t F(p, \tau) d\tau \in \mathscr{F} \quad \textit{and} \quad p \int_0^t F(p, \tau) d\tau \in \mathscr{F}.$$

Proof. Let $F_1(p, t) = 1$. Since $1 = p^0$, we have $F_1(p, t) \in \mathscr{F}$ and hence

$$1 \times F(p, t) \in \mathscr{F}.$$

But

$$1 \times F(p, t) = p \int_0^t F(p, \tau) d\tau.$$

It follows that

$$F_2(p, t) = p \int_0^t F(p, \tau) d\tau \in \mathscr{F}.$$

Since

$$\int_0^t F(p, \tau) d\tau = \frac{1}{p} F_2(p, t)$$

and $1/p = p^{-1} \in \mathscr{F}$, we obtain

$$\int_0^t F(p, \tau) d\tau \in \mathscr{F}$$

in virtue of Theorem 1. This completes the proof.

III. Class $\mathscr{C}$. Distributions. Congruent functions

The class of all functions $f(t)$ continuous in every finite interval $0 \leqslant t \leqslant T$ will be called *class* $\mathscr{C}$.

It is obvious that $f(t) \in \mathscr{C}$ implies $f(t) \in \mathscr{F}$. On the other hand, $f(t) \in \mathscr{F}$ implies

$$\int_0^t f(\tau) d\tau \in \mathscr{C}.$$

Let $\mathscr{Z}$ be the class of all such functions $Z(p, t) \in \mathscr{F}$, for which there exists such a function $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$, that

$$e^{pt} \times a(t) \times Z(p, t) \in \mathscr{F}. \tag{7}$$

THEOREM 3. *Class $\mathscr{Z}$ forms an ideal of ring $\mathscr{F}$.*

Proof. First, we shall show that $\mathscr{Z}$ is a group with respect to addition. Let $Z_1(p, t) \in \mathscr{Z}$ and $Z_2(p,t) \in \mathscr{Z}$. Thus, there exist such functions $a_1(t)$, $a_2(t) \in \mathscr{C}$, $a_1(t) \not\equiv 0$, $a_2(t) \not\equiv 0$, that

$$e^{pt} \times a_1(t) \times Z_1(p, t) \in \mathscr{F} \quad \text{and} \quad e^{pt} \times a_2(t) \times Z_2(p, t) \in \mathscr{F}.$$

Hence — in virtue of Theorem 1 —

$$e^{pt} \times \left[\frac{1}{p} a_1(t) \times a_2(t)\right] \times Z_1(p, t) \in \mathscr{F},$$

$$e^{pt} \times \left[\frac{1}{p} a_1(t) \times a_2(t)\right] \times Z_2(p, t) \in \mathscr{F}$$

and also

$$e^{pt} \times \left[\frac{1}{p} a_1(t) \times a_2(t)\right] \times [Z_1(p, t) + Z_2(p, t)] \in \mathscr{F}.$$

Since

$$\frac{1}{p} a_1(t) \times a_2(t) = p \int_0^t \frac{1}{p} a_1(t-\tau) a_2(\tau)\, d\tau = \int_0^t a_1(t-\tau) a_2(\tau)\, d\tau \in \mathscr{C}$$

and in virtue of Theorem of Titchmarsh ([3], p. 15)

$$\int_0^t a_1(t-\tau) a_2(\tau)\, d\tau \not\equiv 0,$$

it follows that

$$Z_1(p, t) + Z_2(p, t) \in \mathscr{Z}.$$

If $Z(p, t) \in \mathscr{Z}$, then also $-Z(p, t) \in \mathscr{Z}$, because

$$e^{pt} \times a(t) \times Z(p, t) = e^{pt} \times [-a(t)] \times [-Z(p, t)].$$

Thus, $\mathscr{Z}$ forms a group with respect to addition.

Now, it is sufficient to show, that if $Z(p, t) \in \mathscr{Z}$, then for every $F(p, t) \in \mathscr{F}$ we have $Z(p, t) \times F(p, t) \in \mathscr{Z}$. Indeed, by assumption, there exists such a function $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$, that

$$e^{pt} \times a(t) \times Z(p, t) \in \mathscr{F}.$$

Hence

$$[e^{pt} \times a(t) \times Z(p, t)] \times F(p, t) \in \mathscr{F},$$

i. e.

$$e^{pt} \times a(t) \times [Z(p, t) \times F(p, t)] \in \mathscr{F}.$$

But it means that

$$Z(p,t) \times F(p,t) \in \mathscr{Z}.$$

We see that, really, class $\mathscr{Z}$ forms an ideal of ring $\mathscr{F}$.

The ideal $\mathscr{Z}$ divides the ring $\mathscr{F}$ into classes, which form the residue class ring $\mathscr{F}/\mathscr{Z}$. We shall call these residue classes *distributions*(²) and each function $F(p,t)$ belonging to a distribution A will be called a *representative of this distribution*, which will be written in the form

$$A = \{F(p,t)\}.$$

The ring $\mathscr{F}/\mathscr{Z}$ will be called *Distribution Ring* or, simply, *ring* $\mathscr{D}$.

If any functions $F_1(p,t) \in \mathscr{F}$ and $F_2(p,t) \in \mathscr{F}$ are representatives of the same distribution, they are *congruent* and we shall write

$$F_1(p,t) \sim F_2(p,t). \tag{8}$$

Such a relation holds if and only if $F_1(p,t) - F_2(p,t) \in \mathscr{Z}$. The congruence $F(p,t) \sim 0$ means that $F(p,t) \in \mathscr{Z}$.

The relation (8) is a congruence modulo $\mathscr{Z}$, i. e. it has the following properties:

(i) $F(p,t) \sim F(p,t)$;

(ii) if $F_1(p,t) \sim F_2(p,t)$ then $F_2(p,t) \sim F_1(p,t)$;

(iii) if $F_1(p,t) \sim F_2(p,t)$ and $F_2(p,t) \sim F_3(p,t)$ then

$$F_1(p,t) \sim F_3(p,t);$$

(iv) if $F_k(p,t) \sim G_k(p,t)$ $(k = 1, \ldots, n)$ then

$$\sum_{k=1}^{n} F_k(p,t) \sim \sum_{k=1}^{n} G_k(p,t);$$

(v) if $F_k(p,t) \sim G_k(p,t)$ $(k = 1, \ldots, n)$ then

$$F_1(p,t) \times \ldots \times F_n(p,t) \sim G_1(p,t) \times \ldots \times G_n(p,t);$$

(vi) if $F_k(p,t) \sim G_k(p,t)$ and $H_k(p,t) \in \mathscr{F}$ $(k = 1, \ldots, n)$ then

$$\sum_{k=1}^{n} H_k(p,t) \times F_k(p,t) \sim \sum_{k=1}^{n} H_k(p,t) \times G_k(p,t).$$

THEOREM 4. *If $F(p,t) \in \mathscr{F}$ and $e^{pt} \times F(p,t) \in \mathscr{F}$ then $F(p,t) \sim 0$.*

Proof. If $e^{pt} \times F(p,t) \in \mathscr{F}$ then for any $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$, we have $a(t) \in \mathscr{F}$ and $e^{pt} \times F(p,t) \times a(t) \in \mathscr{F}$. It means that $F(p,t) \sim 0$.

(²) Here, the word "distribution" has not the meaning used in the theory of distributions.

THEOREM 5. *If $F(p,t)\in\mathscr{F}$ then*

$$F(p,t) \sim p\int_0^t F(p,\tau)\,d\tau. \tag{9}$$

Proof. Let us consider the function

$$H(p,t) = F(p,t) - p\int_0^t F(p,\tau)\,d\tau.$$

In virtue of Theorem 2 we have $H(p,t)\in\mathscr{F}$. Furthermore

$$\begin{aligned} e^{pt}\times H(p,t) &= e^{pt}\times F(p,t) - e^{pt}\times p\int_0^t F(p,\tau)\,d\tau \\ &= e^{pt}\times F(p,t) - e^{pt}\times 1\times F(p,t) \\ &= e^{pt}\times F(p,t) - (e^{pt}-1)\times F(p,t) = 1\times F(p,t)\in\mathscr{F}. \end{aligned}$$

By Theorem 4 we obtain $H(p,t)\sim 0$, i. e. (9), which was to be proved.

THEOREM 6. *If $g(p)\in\mathscr{F}$ and $F(p,t)\in\mathscr{F}$ then*

$$g(p)\times F(p,t) \sim g(p)F(p,t).$$

Proof. In virtue of Theorem 1 we have $g(p)F(p,t)\in\mathscr{F}$ and by Theorem 5

$$g(p)\times F(p,t) = p\int_0^t g(p)F(p,\tau)\,d\tau \sim g(p)F(p,t)$$

which was to be proved.

THEOREM 7. *If $F(p,t)\sim H(p,t)$ and $g(p)\in\mathscr{F}$, then*

$$g(p)F(p,t) \sim g(p)H(p,t).$$

Proof. According to property (vi) of congruences, we have

$$g(p)\times F(p,t) \sim g(p)\times H(p,t).$$

Hence by Theorem 6 and property (iii) we obtain the assertion.

THEOREM 8. *A necessary and sufficient condition that a function $f(t)\in\mathscr{F}$ safisfy the relation*

$$f(t)\sim 0 \tag{10}$$

is that the relation

$$f(t) = 0 \tag{11}$$

be true for almost all $t>0$.

Proof. It is obvious, that (11) implies (10), because by Theorem 5

$$f(t) \sim p\int_0^t f(\tau)\,d\tau \equiv 0.$$

We shall prove that, conversely, (10) implies (11). Multiplying (10) on both sides by $1/p$ we obtain

$$\frac{1}{p}\times f(t) \sim 0 \quad \text{i. e.} \quad h(t) = \int_0^t f(\tau)\,d\tau \sim 0.$$

Since $h(t) \equiv 0$ implies $f(t) = 0$ for almost all $t > 0$, it suffices to prove, that the congruence

(12) $$h(t) \sim 0$$

where $h(t) \in \mathscr{C}$, implies

(13) $$h(t) \equiv 0.$$

The congruence (12) means, that there exists such a function $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$, that

$$e^{pt}\times a(t)\times h(t) \in \mathscr{F} \quad \text{i. e.} \quad p^2\int_0^t e^{p(t-\tau)}b(\tau)\,d\tau \in \mathscr{F}$$

where

(14) $$b(t) = \int_0^t a(t-\tau)h(\tau)\,d\tau \in \mathscr{C}.$$

Since $1/p \in \mathscr{F}$, we have by Theorem 1 $p\int_0^t e^{p(t-\tau)}b(\tau)\,d\tau \in \mathscr{F}$. It means that there exists such a real non-negative number a that

(15) $$\lim_{p\to\infty}\int_0^t \left| e^{-ap}p\int_0^\sigma e^{p(\sigma-\tau)}b(\tau)\,d\tau \right| d\sigma = 0.$$

Hence

$$\lim_{p\to\infty}\left|\int_0^t e^{-ap}p\int_0^\sigma e^{p(\sigma-\tau)}b(\tau)\,d\tau d\sigma\right| = 0$$

and

(16) $$\lim_{p\to\infty}\int_0^t\int_0^\sigma pe^{p(\sigma-\tau-a)}b(\tau)\,d\tau d\sigma = 0.$$

But for every $t > 0$

(17) $$\lim_{p\to\infty}\int_0^t\int_0^\sigma pe^{p(\sigma-\tau-a)}b(\tau)\,d\tau d\sigma$$

$$= \lim_{p\to\infty}\int_0^t e^{-p(\tau+a)}b(\tau)\int_\tau^t pe^{p\sigma}\,d\sigma d\tau$$

$$= \lim_{p\to\infty}\int_0^t e^{p(t-\tau-a)}b(\tau)\,d\tau - \lim_{p\to\infty} e^{-ap}\int_0^t b(\tau)\,d\tau.$$

If (15) is true for $a = 0$, then it is true also for every $a > 0$. Therefore we can assume that $a > 0$. Then for every $t > 0$

$$\lim_{p\to\infty} e^{-ap}\int_0^t b(\tau)\,d\tau = 0$$

and, in virtue of (16) and (17), we obtain for every $t > 0$

$$\lim_{p\to\infty}\int_0^t e^{p(t-\tau-a)}b(\tau)\,d\tau = 0. \tag{18}$$

Now, let be $t > a$. Then

$$\lim_{p\to\infty}\int_0^t e^{p(t-\tau-a)}b(\tau)\,d\tau \tag{19}$$

$$= \lim_{p\to\infty}\int_0^{t-a} e^{p\tau}b(t-a-\tau)\,d\tau + \lim_{p\to\infty}\int_{-a}^{0} e^{p\tau}b(t-a-\tau)\,d\tau.$$

Since $b(t)\in\mathscr{C}$, the function $b(t-a-\tau)$ is bounded in the interval $-a \leqslant \tau \leqslant 0$ for every $t > a$. Therefore

$$\lim_{p\to\infty}\int_{-a}^{0} e^{p\tau}b(t-a-\tau)\,d\tau = 0$$

and by (18) and (19) we obtain for every $t > a$

$$\lim_{p\to\infty}\int_0^{t-a} e^{p\tau}b(t-a-\tau)\,d\tau = 0$$

i. e. for every $t > 0$

$$\lim_{p\to\infty}\int_0^{t} e^{p\tau}b(t-\tau)\,d\tau = 0.$$

In virtue of Theorem on bounded moments ([3], p. 395), we obtain $b(t-\tau) = 0$ for every τ from the interval $0 \leqslant \tau \leqslant t$, i. e. $b(\tau) = 0$ for every τ from that interval. Since t can be arbitrarily great, we obtain $b(t) \equiv 0$ and, by Theorem of Titchmarsh and (14) $h(t) \equiv 0$ because, by assumption, $a(t) \not\equiv 0$. This completes the proof.

Corollary. *The congruence $f_1(t) \sim f_2(t)$ holds if and only if for almost all $t > 0$ $f_1(t) = f_2(t)$.*

We obtain this assertion immediately, when we replace the function $f(t)$ in the previous theorem by $f_1(t)-f_2(t)$.

Theorem 9. *If $g(p)\in\mathscr{F}$ and*

$$g(p) \sim 0 \tag{20}$$

then for every $t \geqslant 0$

$$\liminf_{p\to\infty} e^{pt}g(p) = 0. \tag{21}$$

Proof. By assumption, there exists such a function $a(t)\in\mathscr{C}$, $a(t)\not\equiv 0$, that

$$e^{pt}\times a(t)\times g(p)\in\mathscr{F},$$

i. e. there exists such a real non-negative number α that for every $t>0$

$$\lim_{p\to\infty}\int_0^t\left|e^{-\alpha p}p^2g(p)\int_0^\tau e^{p(\tau-\sigma)}h(\sigma)\,d\sigma\right|d\tau=0 \tag{22}$$

where $h(t)\equiv\int_0^t a(\tau)\,d\tau\in\mathscr{C}$, $h(t)\not\equiv 0$.

On the other hand, by assumption, there exists such a real non-negative number β, that for every $t>0$

$$\lim_{p\to\infty}\int_0^t|e^{-\beta p}g(p)|\,d\tau=\lim_{p\to\infty}t|e^{-\beta p}g(p)|=0$$

which implies

$$\lim_{p\to\infty}e^{-\beta p}g(p)=0.$$

Thus, for any real positive $\delta>\max(\alpha,\beta)$ we have for every $t>0$

$$\lim_{p\to\infty}g(p)\int_0^t\int_0^\tau pe^{p(\tau-\sigma-\delta)}h(\sigma)\,d\sigma d\tau=0 \tag{23}$$

and

$$\lim_{p\to\infty}e^{-\delta p}g(p)=0. \tag{24}$$

But

$$\lim_{p\to\infty}g(p)\int_0^t\int_0^\tau pe^{p(\tau-\sigma-\delta)}h(\sigma)\,d\sigma d\tau$$

$$=\lim_{p\to\infty}g(p)\int_0^t h(\sigma)\int_\sigma^t pe^{p(\tau-\sigma-\delta)}\,d\tau d\sigma$$

$$=\lim_{p\to\infty}g(p)\int_0^t e^{p(t-\sigma-\delta)}h(\sigma)\,d\sigma-\lim_{p\to\infty}e^{-\delta p}g(p)\int_0^t h(\sigma)\,d\sigma.$$

In virtue of (23) and (24) we obtain

$$\lim_{p\to\infty}g(p)\int_0^t e^{p(t-\sigma-\delta)}h(\sigma)\,d\sigma=0. \tag{25}$$

Now, let us suppose that there exists such a real non-negative number γ that (21) is false for $t=\gamma$. Then, there would exist such two real positive numbers μ and ν that

$$|e^{\gamma p}g(p)|>\mu \tag{26}$$

for real $p > \nu$. By (25) and (26) it would follow

$$\lim_{p\to\infty} \int_0^t e^{p(t-\sigma-\gamma-\delta)} h(\sigma)\, d\sigma = 0$$

and, analogically to (18) we should obtain $h(t) \equiv 0$, which contradicts the assumption.

THEOREM 10. *If for every* $t \geqslant 0$

$$\lim_{p\to\infty} e^{pt} g(p) = 0 \tag{27}$$

then $g(p) \in \mathscr{F}$ *and*

$$g(p) \sim 0. \tag{28}$$

Proof. By assumption, we have for $t = 0$

$$\lim_{p\to\infty} g(p) = 0$$

and for any $\delta > 0$ and every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-\delta p} g(p)|\, d\tau = \lim_{p\to\infty} t |e^{-\delta p} g(p)| = 0.$$

Therefore $g(p) \in \mathscr{F}$. We have, further, for any $a > 0$ and every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-ap} e^{p\tau} g(p)|\, d\tau = \lim_{p\to\infty} \left|\frac{1}{p} e^{-ap} g(p)\right| \int_0^t p e^{p\tau}\, d\tau$$

$$= \lim_{p\to\infty} \left|\frac{1}{p} e^{-ap} e^{pt} g(p)\right| - \lim_{p\to\infty} \left|\frac{1}{p} e^{-ap} g(p)\right| = 0.$$

It means that

$$e^{pt} g(p) \in \mathscr{F}.$$

It follows that

$$e^{pt} \times g(p) = p \int_0^t g(p) e^{p\tau}\, d\tau = e^{pt} g(p) - g(p) \in \mathscr{F}.$$

By Theorem 4 we obtain (28). It completes the proof.

EXAMPLES. (1) For any $a > 1$ we have $e^{-p^a} \sim 0$, because for every $t > 0$

$$\lim_{p\to\infty} e^{pt} e^{-p^a} = 0.$$

(2) If any function $g(p)$ satisfies the condition $g(p) = 0$ for sufficiently great real p, then $g(p) \sim 0$.

From the second example we conclude, that, if any functions $g_1(p) \in \mathscr{F}$ and $g_2(p) \in \mathscr{F}$ satisfy the condition

$$g_1(p) = g_2(p)$$

for sufficiently great real p, then already

$$g_1(p) \sim g_2(p).$$

This conclusion can be generalized. Namely, the following theorem is true.

THEOREM 11. *If there exists such a real positive number μ, that for every real $p > \mu$ a function $F(p, t) \in \mathscr{R}$ satisfies for almost all $t > 0$ the condition*

$$F(p, t) = 0 \tag{29}$$

then

$$F(p, t) \in \mathscr{F} \quad \text{and} \quad F(p, t) \sim 0. \tag{30}$$

Proof. We have for any real positive α and every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-\alpha p} F(p, \tau)| \, d\tau = 0.$$

Therefore $F(p, t) \in \mathscr{F}$. Furthermore

$$e^{pt} \times F(p, t) \in \mathscr{F} \tag{31}$$

because for any real positive α and every $t > 0$

$$\lim_{p\to\infty} \int_0^t \left| e^{-\alpha p} p \int_0^\tau e^{p(\tau-\sigma)} F(p, \sigma) \, d\sigma \right| d\tau = 0.$$

By Theorem 4, (31) implies $F(p, t) \sim 0$, which was to be proved.

COROLLARY. *If there exists such a real positive number μ, that for every real $p > \mu$ functions $F_1(p, t)$, $F_2(p, t) \in \mathscr{F}$ satisfy for almost all $t > 0$ the condition $F_1(p, t) = F_2(p, t)$ then, already, $F_1(p, t) \sim F_2(p, t)$.*

We obtain this corollary immediately, when we replace the function $F(p, t)$ in Theorem 11 by $F_1(p, t) - F_2(p, t)$.

We see, that for any function $F(p, t)$ only its behaviour for real p as $p \to \infty$ decides, whether $F(p, t)$ belongs to a distribution A or not.

THEOREM 12. *If $F(p, t) \in \mathscr{F}$ and for every real non-negative number δ there exists such a real non-negative number μ that the integral*

$$\int_\delta^\infty e^{p(\delta-t)} F(p, t) \, dt = \int_0^\infty e^{-pt} F(p, t+\delta) \, dt \tag{32}$$

converges uniformly with respect to real $p \geqslant \mu$, then

$$g(p) \equiv p \int_0^\infty e^{-pt} F(p, t) \, dt \in \mathscr{F} \tag{33}$$

and

$$g(p) \sim F(p, t). \tag{34}$$

Proof. First, we shall show that

$$(35)\qquad P(p,t) \equiv p\int_t^\infty e^{p(t-\tau)}F(p,\tau)\,d\tau \in \mathscr{F}.$$

By assumption there exists such a real non-negative number a that for every $T > 0$

$$\lim_{p\to\infty}\int_0^T |e^{-ap}F(p,\tau)|\,d\tau = 0$$

i. e. for every $T > 0$, every $t > 0$ and every $\varepsilon > 0$ there exists such a real non-negative number ν that

$$e^{-ap}\int_0^T |F(p,\tau)|\,d\tau < \frac{\varepsilon}{2t} \quad \text{for real} \quad p > \nu.$$

It follows that for every u from the interval $0 \leqslant u \leqslant T$ and real $p > \nu$

$$(36)\qquad e^{-ap}\left|\int_u^T e^{-p(\tau-u)}F(p,\tau)\,d\tau\right| \leqslant e^{-ap}\int_u^T |F(p,\tau)|\,d\tau$$

$$\leqslant e^{-ap}\int_0^T |F(p,\tau)|\,d\tau < \frac{\varepsilon}{2t}.$$

On the other hand, by assumption, for every $t > 0$ the integral

$$\int_t^\infty e^{p(t-\tau)}F(p,\tau)\,d\tau$$

converges uniformly with respect to real $p > \mu$, where μ depends on t. It means, that for every $t > 0$ and every $\varepsilon > 0$ there exist such real non-negative numbers μ and $T > t$ that

$$\left|\int_T^\infty e^{p(t-\tau)}F(p,\tau)\,d\tau\right| < \frac{\varepsilon}{2t} \quad \text{for real} \quad p > \mu$$

Thus, for every u from the interval $0 \leqslant u \leqslant t \leqslant T$ and real $p > \mu$

$$(37)\qquad e^{-ap}\left|\int_T^\infty e^{-p(\tau-u)}F(p,\tau)\,d\tau\right| \leqslant \left|\int_T^\infty e^{-p(\tau-u)}F(p,\tau)\,d\tau\right|$$

$$= e^{-p(t-u)}\left|\int_T^\infty e^{p(t-\tau)}F(p,\tau)\,d\tau\right|$$

$$\leqslant \left|\int_T^\infty e^{p(t-\tau)}F(p,\tau)\,d\tau\right| < \frac{\varepsilon}{2t}.$$

It follows from (36) and (37) that for every $0 \leqslant u \leqslant t$ and real $p > \max(\mu, \nu)$

$$e^{-ap}\left|\int_u^\infty e^{p(u-\tau)}F(p, \tau)\,d\tau\right| < \frac{\varepsilon}{t}$$

i. e.

$$e^{-ap}\int_0^t\left|\int_u^\infty e^{p(u-\tau)}F(p, \tau)\,d\tau\right| du < \int_0^t \frac{\varepsilon}{t}\,du = \varepsilon.$$

But it means that for every $t > 0$

$$\lim_{p\to\infty}\int_0^t\left|e^{-ap}\int_u^\infty e^{p(u-\tau)}F(p, \tau)\,d\tau\right| du = 0$$

i. e. (35).

Now, we shall show that $g(p)\in\mathscr{F}$. Indeed, by assumption, the integral (32) with $\delta = 0$ converges for real $p > \mu$. Thus, the integral (33) converges for real $p > \mu$. Furthermore, by Theorem 2

$$H(p, t) \equiv P(p, t) - p\int_0^t P(p, \tau)\,d\tau \in \mathscr{F}. \tag{38}$$

But

$$H(p, t) = p\int_t^\infty e^{p(t-\tau)}F(p, \tau)\,d\tau - p^2\int_0^t\int_u^\infty e^{p(u-\tau)}F(p, \tau)\,d\tau du$$

$$= p\int_0^\infty e^{p(t-\tau)}F(p, \tau)\,d\tau - p\int_0^t e^{p(t-\tau)}F(p, \tau)\,d\tau -$$

$$-p^2\int_0^t\int_0^\infty e^{p(u-\tau)}F(p, \tau)\,d\tau du + p^2\int_0^t\int_0^u e^{p(u-\tau)}F(p, \tau)\,d\tau du$$

$$= e^{pt}g(p) - p\int_0^t e^{p(t-\tau)}F(p, \tau)\,d\tau -$$

$$-pg(p)\int_0^t e^{pu}\,du + p\int_0^t e^{-p\tau}F(p, \tau)\int_\tau^t pe^{pu}\,du d\tau$$

$$= e^{pt}g(p) - p\int_0^t e^{p(t-\tau)}F(p, \tau)\,d\tau -$$

$$-e^{pt}g(p) + g(p) + p\int_0^t e^{p(t-\tau)}F(p, \tau)\,d\tau - p\int_0^t F(p, \tau)\,d\tau$$

$$= g(p) - p\int_0^t F(p, \tau)\,d\tau.$$

Thus, $g(p)\in\mathscr{F}$ because by Theorem 2 $p\int_0^t F(p,\tau)\,d\tau\in\mathscr{F}$ and

$$g(p) = H(p,t)+p\int_0^t F(p,\tau)\,d\tau. \tag{39}$$

Furthermore, by Theorem 5 and (38) $H(p,t)\sim 0$ and by (39) $g(p) \sim p\int_0^t F(p,\tau)\,d\tau$. Thus, by Theorem 5 $g(p)\sim F(p,t)$. This completes the proof.

THEOREM 13. *If $f(t)\in\mathscr{F}$ and there exists such a real non-negative number μ that the integral*

$$g(p) = p\int_0^\infty e^{-pt}f(t)\,dt \tag{40}$$

converges for $p=\mu$, then

$$g(p)\in\mathscr{F} \quad \textit{and} \quad g(p)\sim f(t). \tag{41}$$

Proof. It follows from the equality

$$\int_0^\infty e^{-pt}f(t+\delta)\,dt = e^{\delta p}\int_\delta^\infty e^{-pt}f(t)\,dt \tag{42}$$

that the integral (42) is convergent for $p=\mu$ and any real $\delta\geqslant 0$. In virtue of Fundamental Theorem for Laplace-Transformation ([1], p. 35) the integral (42) is uniformly convergent with respect to real $p>\mu$. Thus, we obtain (41) by Theorem 12.

Theorem 13 enables us to use all methods and theorems known for Laplace-Transformation (strictly speaking Laplace–Carson Transformation) in the smaller class of functions $f(t)$, for which the integral (40) converges. But many of these methods and theorems can be generalized for all functions $F(p,t)\in\mathscr{F}$, e. g. Theorems 5 and 8 .We shall prove several further theorems of such kind.

THEOREM 14. *If $F(p,t)\in\mathscr{F}$ and we put $F(p,t)=0$ for $t<0$, then for any real δ*

$$H(p,t)\equiv F(p,t+\delta)\in\mathscr{F}.$$

Proof. By assumption, there exists such a real non-negative number α, that for every $t>0$

$$\lim_{p\to\infty}\int_0^t |e^{-\alpha p}F(p,\tau)|\,d\tau = 0. \tag{43}$$

Hence for every $t > 0$ and every real δ

$$\lim_{p\to\infty}\int_0^t |e^{-\alpha p} H(p,\tau)|\,d\tau = \lim_{p\to\infty}\int_0^t |e^{-\alpha p} F(p,\tau+\delta)|\,d\tau$$

$$= \lim_{p\to\infty}\int_\delta^{t+\delta} |e^{-\alpha p} F(p,\tau)|\,d\tau$$

$$= \lim_{p\to\infty}\int_0^{t+\delta} |e^{-\alpha p} F(p,\tau)|\,d\tau - \lim_{p\to\infty}\int_0^{\delta} |e^{-\alpha p} F(p,\tau)|\,d\tau = 0$$

in virtue of (43) and the assumption that $F(p,t) = 0$ for $t < 0$. But it means, that $H(p,t)\in\mathscr{F}$, which was to be proved.

THEOREM 15. *If $F(p,t) \sim g(p)$ where $F(p,t)\in\mathscr{F}$, $g(p)\in\mathscr{F}$ and we put $F(p,t) = 0$ for $t < 0$, then for any real δ*

$$F(p,t+\delta) \sim g(p)e^{\delta p} - pe^{\delta p}\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau \tag{44}$$

where the congruent functions belong to $\mathscr{F}$.

Proof. By Theorem 14,

$$H(p,t) \equiv F(p,t+\delta)\in\mathscr{F}.$$

We shall show, that also

$$k(p) \equiv g(p)e^{\delta p} - pe^{\delta p}\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau\in\mathscr{F}.$$

By assumption, there exist such real non-negative numbers α, β that for every $t > 0$

$$\lim_{p\to\infty}\int_0^t |e^{-\alpha p}g(p)|\,d\tau = 0 \quad\text{and}\quad \lim_{p\to\infty}\int_0^t |e^{-\beta p}F(p,\tau)|\,d\tau = 0.$$

Hence, for every $t > 0$ and any $\gamma > \max(\alpha,\beta,|\delta|)$

$$\lim_{p\to\infty}\int_0^t |e^{-(\gamma+\delta)p}k(p)|\,d\tau$$

$$= \lim_{p\to\infty}\int_0^t \left|e^{-\gamma p}g(p) - pe^{-\gamma p}\int_0^\delta e^{-p\sigma}F(p,\sigma)\,d\sigma\right|\,d\tau$$

$$\leqslant \lim_{p\to\infty}\int_0^t |e^{-\gamma p}g(p)|\,d\tau + \lim_{p\to\infty}|pte^{-\gamma p}|\left|\int_0^\delta e^{-p\sigma}F(p,\sigma)\,d\sigma\right| = 0$$

because

$$\int_0^t |e^{-\gamma p}g(p)|\,d\tau = |e^{-(\gamma-\alpha)p}|\int_0^t |e^{-\alpha p}g(p)|\,d\tau \underset{p\to\infty}{\to} 0$$

and for $\delta \geqslant 0$

$$|pte^{-\gamma p}|\left|\int_0^\delta e^{-p\sigma}F(p,\sigma)\,d\sigma\right| \leqslant |pte^{-\gamma p}|\int_0^\delta |F(p,\sigma)|\,d\sigma$$

$$= t|pe^{-(\gamma-\beta)p}|\int_0^\delta |e^{-\beta p}F(p,\tau)|\,d\tau \underset{p\to\infty}{\to} 0$$

and for $\delta < 0$, by assumption, we have

$$\int_0^\delta e^{-p\sigma}F(p,\sigma)\,d\sigma = 0.$$

It follows that

$$\lim_{p\to\infty}\int_0^t \left|e^{-(\gamma+\delta)p}k(p)\right|\,d\tau = 0 \qquad \text{i. e.} \qquad k(p)\in\mathscr{F}.$$

Now, we shall prove (44). Since for every real δ we have $e^{\delta p}\in\mathscr{F}$, because for any $\varrho > \delta$, $\varrho \geqslant 0$ and every $t > 0$

$$\lim_{p\to\infty}\int_0^t \left|e^{-\varrho p}e^{\delta p}\right|\,d\tau = \lim_{p\to\infty} t\left|e^{-(\varrho-\delta)p}\right| = 0$$

then the congruence (44) is equivalent to the congruence

$$F(p,t) - p\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau - e^{-\delta p}F(p,t+\delta) \sim 0 \tag{45}$$

because, by assumption, $F(p,t) \sim g(p)$.

Let us consider the function

$$P(p,t) \equiv e^{pt}\times\left(F(p,t) - p\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau - e^{-\delta p}F(p,t+\delta)\right).$$

We have

$$P(p,t) = p\int_0^t e^{p(t-\tau)}F(p,\tau)\,d\tau - p^2\int_0^t e^{p(t-\sigma)}\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau d\sigma -$$

$$-p\int_0^t e^{p(t-\tau-\delta)}F(p,\tau+\delta)\,d\tau$$

$$= p\int_0^t e^{p(t-\tau)}F(p,\tau)\,d\tau - p^2\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau\int_0^t e^{p(t-\sigma)}\,d\sigma - p\int_\delta^{t+\delta} e^{p(t-\tau)}F(p,\tau)\,d\tau$$

$$= p\int_0^t e^{p(t-\tau)}F(p,\tau)d\tau - p(e^{pt}-1)\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau - p\int_\delta^{t+\delta} e^{p(t-\tau)}F(p,\tau)\,d\tau$$

$$= p\int_\delta^t e^{p(t-\tau)}F(p,\tau)\,d\tau + p\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau - p\int_\delta^{t+\delta} e^{p(t-\tau)}F(p,\tau)\,d\tau$$

$$= p\int_0^\delta e^{-p\tau}F(p,\tau)\,d\tau - p\int_t^{t+\delta} e^{p(t-\tau)}F(p,\tau)\,d\tau.$$

Furthermore, for any real $\gamma > \beta$ and every $t > 0$

$$\int_0^t |e^{-\gamma p} P(p, \tau)| d\tau$$

$$\leqslant |pe^{-\gamma p}| \int_0^t \left| \int_0^\delta e^{-p\tau} F(p, \tau)\, d\tau \right| d\sigma + |p| \int_0^t \left| \int_\sigma^{\sigma+\delta} e^{p(\sigma-\tau-\gamma)} F(p, \tau)\, d\tau \right| d\sigma$$

$$\leqslant t |pe^{-(\gamma-\beta)p}| \int_0^{|\delta|} |e^{-\beta p} F(p, \tau)| d\tau + |pe^{-(\gamma-\beta)p}| \int_0^t \int_0^{t+|\delta|} |e^{-\beta p} F(p, \tau)| d\tau d\sigma$$

$$\leqslant t |pe^{-(\gamma-\beta)p}| \left(\int_0^{|\delta|} |e^{-\beta p} F(p, \tau)| d\tau + \int_0^{t+|\delta|} |e^{-\beta p} F(p, \tau)| d\tau \right) \underset{p\to\infty}{\to} 0$$

and, thus, for every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-\gamma p} P(p, \tau)| d\tau = 0.$$

It means that $P(p, t) \in \mathscr{F}$, which implies, by Theorem 4, the congruence (45) equivalent to (44). The proof is complete.

COROLLARY 1. *If* $F(p, t) \sim g(p)$ *where* $F(p, t)$, $g(p) \in \mathscr{F}$, *and we put* $F(p, t) = 0$ *for* $t < 0$, *then for any real positive* δ

$$F(p, t-\delta) \sim g(p) e^{-\delta p}.$$

We obtain this corollary by replacing δ in Theorem 15 by $-\delta$.

COROLLARY 2. *If* $F(p, t) \sim g(p)$ *where* $F(p, t)$, $g(p) \in \mathscr{F}$ *and* $F(p, t) = 0$ *for* $0 \leqslant t < \delta$ *then*

$$F(p, t+\delta) \sim g(p) e^{\delta p}.$$

This corollary follows immediately from Theorem 15.

THEOREM 16. *If* $F_1(p, t)$, $F_2(p, t)$, $g_1(p)$, $g_2(p) \in \mathscr{F}$ *and* $F_1(p, t) \sim g_1(p)$, $F_2(p, t) \sim g_2(p)$ *then*

$$F_1(p, t) \times F_2(p, t) \sim g_1(p) g_2(p). \tag{46}$$

Proof. In virtue of property (v) of congruences we have

$$F_1(p, t) \times F_2(p, t) \sim g_1(p) \times g_2(p) \tag{47}$$

and, by Theorem 6,

$$g_1(p) \times g_2(p) \sim g_1(p) g_2(p). \tag{48}$$

The congruences (47) and (48) imply (46).

THEOREM 17. *Let $F(p, t)$ be absolutely continuous with respect to t for every real $p > \mu$ and $t > 0$. If $\frac{\partial}{\partial t} F(p, t) \in \mathscr{F}$ and there exists*

$$\lim_{t \to +0} F(p, t) = F_0(p) \in \mathscr{F}$$

then

(49) $$F(p, t) \in \mathscr{F}$$

and

$$\frac{\partial}{\partial t} F(p, t) \sim pF(p, t) - pF_0(p).$$

Proof. By Theorem 2, it follows from the assumption that

$$\int_0^t \frac{\partial}{\partial \tau} F(p, \tau)\, d\tau = F(p, t) - F_0(p) \in \mathscr{F}.$$

Since, by assumption, $F_0(p) \in \mathscr{F}$, we obtain (49). In virtue of Theorem 5

$$\frac{\partial}{\partial t} F(p, t) \sim p \int_0^t \frac{\partial}{\partial \tau} F(p, \tau)\, d\tau = p[F(p, t) - F_0(p)]$$

which completes the proof.

THEOREM 18. *Let $F(p, t)$ have derivatives with respect to t up to and including the k-th for every real $p > \mu$ and $t > 0$. If*

$$\frac{\partial^k}{\partial t^k} F(p, t) \in \mathscr{F}$$

and there exist

$$\lim_{t \to +0} F(p, t) = F_0(p) \in \mathscr{F}, \qquad \lim_{t \to +0} \frac{\partial^j}{\partial t^j} F(p, t) = F_j(p) \in \mathscr{F}, \quad j = 1, 2, \ldots, k-1,$$

then

$$F(p, t) \in \mathscr{F}, \qquad \frac{\partial^j}{\partial t^j} F(p, t) \in \mathscr{F}, \qquad j = 1, 2, \ldots, k-1,$$

and

$$\frac{\partial^k}{\partial t^k} F(p, t) \sim p^k F(p, t) - p^k F_0(p) - p^{k-1} F_1(p) - \ldots - pF_{k-1}(p).$$

We obtain Theorem 18 when we apply Theorem 17 k times.

IV. Distribution ring

If $A = \{F(p,t)\}$ we write

$$-A = \{-F(p,t)\}.$$

If $A_1 = \{F_1(p,t)\}$, $A_2 = \{F_2(p,t)\}$ we write

$$A_1 + A_2 = \{F_1(p,t) + F_2(p,t)\},$$

$$A_1 - A_2 = \{F_1(p,t) - F_2(p,t)\},$$

$$A_1 A_2 = \{F_1(p,t) \times F_2(p,t)\}.$$

If $F(p,t) \sim 0$ we write

$$\{F(p,t)\} = 0.$$

DEFINITION. The distribution B will be called the *k-th derivative of the distribution* A and we shall write

$$B = A^{(k)}$$

if there exists such a function $F(p,t)$, satisfying all the assumptions of Theorem 18, that

$$A = \{F(p,t)\}, \qquad B = \{p^k F(p,t)\}$$

which implies also

$$B = \left\{\frac{\partial^k}{\partial t^k} F(p,t) + p^k F_0(p) + p^{k-1} F_1(p) + \ldots + p F_{k-1}(p)\right\}.$$

The ring $\mathscr{D}$ of distributions includes zero divisors. For example, let $g_1(p)$ and $g_2(p)$ be defined as follows:

$$g_1(p) = \begin{cases} 0 & \text{for} \quad 2n \leqslant \operatorname{Re} p < 2n+1, \\ 1 & \text{for} \quad 2n+1 \leqslant \operatorname{Re} p < 2n+2; \end{cases}$$

$$g_2(p) = \begin{cases} 1 & \text{for} \quad 2n \leqslant \operatorname{Re} p < 2n+1, \\ 0 & \text{for} \quad 2n+1 \leqslant \operatorname{Re} p < 2n+2, \end{cases}$$

where $n = 1, 2, \ldots$

Since for every $t > 0$

$$\lim_{p\to\infty} \int_0^t |e^{-p} g_k(p)|\, d\tau = \lim_{p\to\infty} t\,|e^{-p} g_k(p)| = 0 \qquad (k = 1, 2)$$

then $g_1(p) \in \mathscr{F}$ and $g_2(p) \in \mathscr{F}$. Furthermore, by Theorem 6, the function

$$F(p,t) = g_1(p) \times g_2(p) \sim g_1(p)\, g_2(p) = 0$$

but, nevertheless, we shall show that none of the congruences

$$g_1(p) \sim 0, \quad g_2(p) \sim 0$$

is true. In order to do it, let us suppose that

(50) $$g_k(p) \sim 0 \quad (k = 1 \text{ or } k = 2).$$

It means that there exists such a function $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$, that

$$H(p, t) = e^{pt} \times a(t) \times g_k(p) \in \mathscr{F}.$$

We introduce the function

$$G(p, t) = a(t) \times g_k(p)$$

i. e. if $k = 1$ then

$$G(p, t) = \begin{cases} 0 & \text{for} \quad 2n \leqslant \operatorname{Re} p < 2n+1, \\ pb(t) & \text{for} \quad 2n+1 \leqslant \operatorname{Re} p < 2n+2 \end{cases}$$

and if $k = 2$ then

$$G(p, t) = \begin{cases} pb(t) & \text{for} \quad 2n \leqslant \operatorname{Re} p < 2n+1, \\ 0 & \text{for} \quad 2n+1 \leqslant \operatorname{Re} p < 2n+2 \end{cases} \quad n = 1, 2, \ldots$$

where

(51) $$b(t) = \int_0^t a(\tau)\, d\tau \in \mathscr{C}, \quad b(t) \not\equiv 0.$$

Thus, if $k = 1$ then

$$H(p, t) = \begin{cases} 0 & \text{for} \quad 2n \leqslant \operatorname{Re} p < 2n+1, \\ p^2 \int_0^t e^{p(t-\tau)}\, b(\tau)\, d\tau & \text{for} \quad 2n+1 \leqslant \operatorname{Re} p < 2n+2 \end{cases}$$

and if k = 2 then

$$H(p, t) = \begin{cases} p^2 \int_0^t e^{p(t-\tau)} b(\tau)\, d\tau & \text{for} \quad 2n \leqslant \operatorname{Re} p < 2n+1, \\ 0 & \text{for} \quad 2n+1 \leqslant \operatorname{Re} p < 2n+2 \end{cases} \quad n = 1, 2, \ldots$$

By assumption (50) there exists such a real non-negative number α that for every $t > 0$

$$\lim_{p \to \infty} \int_0^t |e^{-\alpha p} H(p, \tau)|\, d\tau = 0.$$

It follows that

$$\lim_{n \to \infty} \int_0^t |e^{-\alpha \beta_n} H(\beta_n, \tau)|\, d\tau = 0$$

for any such sequence $\beta_1, \beta_2, \ldots$ that

$$\lim_{n \to \infty} \beta_n = \infty.$$

Let

$$\beta_n = \begin{cases} 2n+1 & \text{for} \quad k = 1, \\ 2n & \text{for} \quad k = 2. \end{cases}$$

Thus

$$\lim_{n\to\infty} \beta_n^2 e^{-\alpha\beta_n} \int_0^t \left| \int_0^\tau e^{\beta_n(\tau-\sigma)} b(\sigma)\, d\sigma \right| d\tau = 0$$

and hence

$$\lim_{n\to\infty} \beta_n^2 e^{-\alpha\beta_n} \int_0^t \int_0^\tau e^{\beta_n(\tau-\sigma)} b(\sigma)\, d\sigma d\tau$$

$$= \lim_{n\to\infty} \beta_n e^{-\alpha\beta_n} \int_0^t b(\sigma) \int_\sigma^t \beta_n e^{\beta_n(\tau-\sigma)}\, d\tau d\sigma$$

$$= \lim_{n\to\infty} \beta_n e^{-\alpha\beta_n} \int_0^t e^{\beta_n(t-\sigma)} b(\sigma)\, d\sigma - \lim_{n\to\infty} \beta_n e^{-\alpha\beta_n} \int_0^t b(\sigma)\, d\sigma = 0.$$

Since we can assume that $\alpha > 0$ and therefore for every $t > 0$

$$\lim_{n\to\infty} \beta_n e^{-\alpha\beta_n} \int_0^t b(\sigma)\, d\sigma = 0$$

we obtain for every $t > 0$

$$\lim_{n\to\infty} \int_0^t e^{\beta_n(t-\alpha-\sigma)} b(\sigma)\, d\sigma = 0.$$

Analogically as it has been done in the case of (18) in proving Theorem 8, we obtain, in virtue of Theorem on bounded moments,

$$b(t) \equiv 0$$

and, further, in virtue of (51)

$$a(t) \equiv 0$$

which contradicts the assumption. It follows that (50) is not true. Thus, if $A_1 = \{g_1(p)\}$, $A_2 = \{g_2(p)\}$, $A = \{g_1(p) \times g_2(p)\}$ then $A = A_1 A_2 = 0$, but neither $A_1 = 0$ nor $A_2 = 0$.

Since the ring $\mathscr{D}$ of distributions has zero divisors, we can not construct a quotient field of distributions. Nevertheless, we can prove two following theorems.

THEOREM 19. *If $f(t) \in \mathscr{F}$ and $G(p, t) \in \mathscr{F}$ satisfy the congruence*

$$f(t) \times G(p, t) \sim 0 \tag{52}$$

then either $f(t) \sim 0$ or $G(p, t) \sim 0$.

Proof. By Theorem 2, we have

$$b(t) \equiv \int_0^t f(\tau)\,d\tau \in \mathscr{F}, \quad b(t) \in \mathscr{C} \tag{53}$$

and by Theorem 5

$$f(t) \sim pb(t).$$

Multiplying this congruence on both sides by $G(p, t)$ we obtain in virtue of (52)

$$pb(t) \times G(p, t) \sim 0.$$

Thus, there exists such a function $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$, that

$$e^{pt} \times a(t) \times pb(t) \times G(p, t) \in \mathscr{F} \quad \text{i. e.} \quad p^2 e^{pt} \times c(t) \times G(p, t) \in \mathscr{F}$$

where

$$c(t) = \int_0^t a(t-\tau)\,b(\tau)\,d\tau; \quad c(t) \in \mathscr{C}. \tag{54}$$

In virtue of Theorem 1 also

$$e^{pt} \times c(t) \times G(p, t) \in \mathscr{F}. \tag{55}$$

If $c(t) \equiv 0$, then — by (54) and already mentioned Theorem of Titchmarsh — $b(t) \equiv 0$, because, by assumption, $a(t) \not\equiv 0$. Then, by (53), $f(t) = 0$ for almost all $t > 0$ and, by Theorem 8, $f(t) \sim 0$.

If $c(t) \not\equiv 0$, then (55) means that $G(p, t) \sim 0$. The proof is complete.

THEOREM 20. *If $F(p, t)$, $G(p, t) \in \mathscr{F}$ satisfy the congruence*

$$F(p, t) \times G(p, t) \sim 0 \tag{56}$$

and there exist such functions $H(p, t) \in \mathscr{F}$, $a(t) \in \mathscr{C}$, $a(t) \not\equiv 0$ that

$$F(p, t) \times H(p, t) \sim a(t) \tag{57}$$

then

$$G(p, t) \sim 0.$$

Proof. Multiplying (56) on both sides by $H(p, t)$, we obtain by (57) $a(t) \times G(p, t) \sim 0$ and then, by Theorem 19, $G(p, t) \sim 0$, which completes the proof.

V. Quotient functions and quotient distributions

DEFINITION. A function $F(p, t)$ will be called a *quotient function*, if and only if

1° $F(p, t) \in \mathscr{F}$,

2° there exist such functions $a(t)$, $b(t) \in \mathscr{C}$, $b(t) \not\equiv 0$ that

$$F(p, t) \times b(t) \sim a(t).$$

THEOREM 21. *Every $f(t) \in \mathcal{F}$ is a quotient function.*

Proof. In virtue of Theorem 5 we have

$$f(t) \sim p \int_0^t f(\tau)\, d\tau$$

then, by Theorem 7,

$$\frac{1}{p} f(t) \sim \int_0^t f(\tau)\, d\tau \tag{58}$$

and by Theorem 6

$$f(t) \times \frac{1}{p} \sim \int_0^t f(\tau)\, d\tau . \tag{59}$$

But for $f(t) \equiv 1$ we obtain from (58)

$$\frac{1}{p} \sim t . \tag{60}$$

Thus, by (59), for every $f(t) \in \mathcal{F}$

$$f(t) \times t \sim \int_0^t f(\tau)\, d\tau .$$

It means that $f(t)$ is a quotient function.

THEOREM 22. *The set of all quotient functions forms a ring with respect to ordinary addition and multiplication* (5).

Proof. Let $F_1(p, t)$ and $F_2(p, t)$ be quotient functions. Then there exist such functions $a_k(t)$, $b_k(t) \in \mathcal{C}$, $b_k(t) \not\equiv 0$ $(k = 1, 2)$ that

$$F_1(p, t) \times b_1(t) \sim a_1(t), \qquad F_2(p, t) \times b_2(t) \sim a_2(t) . \tag{61}$$

Multiplying the first congruence by $\frac{1}{p} b_2(t)$ and the second by $\frac{1}{p} b_1(t)$ and adding the congruences thus obtained, we have

$$[F_1(p, t) + F_2(p, t)] \times \left[\frac{1}{p} b_1(t) \times b_2(t)\right] \sim \frac{1}{p} a_1(t) \times b_2(t) + \frac{1}{p} a_2(t) \times b_1(t)$$

i. e.

$$[F_1(p, t) + F_2(p, t)] \times \int_0^t b_1(t-\tau)\, b_2(\tau)\, d\tau \sim \int_0^t a_1(t-\tau)\, b_2(\tau)\, d\tau + \int_0^t a_2(t-\tau)\, b_1(\tau)\, d\tau .$$

In virtue of Convolution Theorem and Theorem of Titchmarsh

$$h(t) \equiv \int_0^t b_1(t-\tau)\, b_2(\tau)\, d\tau \in \mathscr{C}, \qquad h(t) \not\equiv 0. \tag{62}$$

Thus, $F_1(p,t)+F_2(p,t)$ is a quotient function.

If $F(p,t)$ is a quotient function, so is $-F(p,t)$.

Now, multiplying the congruences (61) we obtain

$$F_1(p,t)\times F_2(p,t)\times b_1(t)\times b_2(t) \sim a_1(t)\times a_2(t)$$

and also

$$[F_1(p,t)\times F_2(p,t)]\times\left[\frac{1}{p}\, b_1(t)\times b_2(t)\right] \sim \left[\frac{1}{p}\, a_1(t)\times a_2(t)\right]$$

which means, by (62), that $F_1(p,t)\times F_2(p,t)$ is a quotient function. Thus, the proof is complete.

Definition. The ring of all quotient functions will be called *ring* $\mathscr{Q}$. Obviously, $\mathscr{Q}$ is a subring of $\mathscr{F}$.

Theorem 23. *If* $F(p,t)\in\mathscr{Q}$, $G(p,t)\in\mathscr{F}$, *and* $F(p,t)\times G(p,t)\sim 0$ *then*

$$F(p,t)\sim 0 \quad \text{or} \quad G(p,t)\sim 0.$$

Proof. It is sufficient to prove that, if the congruence $F(p,t)\sim 0$ is not true, then $G(p,t)\sim 0$. By assumption, there exist such functions $a(t)$, $b(t)\in\mathscr{C}$, $b(t)\not\equiv 0$, that

$$F(p,t)\times b(t)\sim a(t).$$

If $a(t)\equiv 0$, then, by Theorem 19, $F(p,t)\sim 0$. It follows that, if the congruence $F(p,t)\sim 0$ is not true, then $a(t)\not\equiv 0$, and, by Theorem 20, $G(p,t)\sim 0$, which was to be proved.

Definition. If $F(p,t)\in\mathscr{Q}$ and $A=\{F(p,t)\}$ then A will be called a *quotient distribution.*

All representatives of a quotient distribution are, obviously, quotient functions.

Since all quotient functions form a ring, then all quotient distributions form also a ring. The ring of all quotient distributions will be called *ring* $\mathscr{S}$. Obviously, $\mathscr{S}$ is a subring of $\mathscr{D}$.

It follows from Theorem 21 that every $\{f(t)\}$ where $f(t)\in\mathscr{F}$, is a quotient distribution.

Theorem 24. *If m is an arbitrary integer and* $f(t)\in\mathscr{F}$, *then*

$$A=\{p^m f(t)\}$$

is a quotient distribution.

Proof. By Theorem 5 we have for any non-negative integer k

$$\frac{t^k}{p\cdot k!} \sim \int_0^t \frac{\tau^k}{k!}\,d\tau = \frac{t^{k+1}}{(k+1)!}$$

and, by (60), we obtain for any positive integer m

$$\frac{1}{p^m} \sim \frac{t^m}{m!}. \tag{63}$$

Thus, if m is a non-negative integer, we obtain, by Theorem 6, from the identity

$$p^m f(t)\cdot\frac{1}{p^{m+1}} = \frac{1}{p}f(t)$$

the congruence

$$p^m f(t)\times\frac{t^{m+1}}{(m+1)!} \sim \int_0^t f(\tau)\,d\tau \in \mathscr{C}$$

which means that $p^m f(t)$ is a quotient function.

If m is negative, we can write

$$A = \left\{\frac{1}{p^n}f(t)\right\}$$

where $n = -m$ is positive. By Theorem 6 we have also

$$A = \left\{\frac{1}{p}f(t)\times\frac{1}{p^{n-1}}\right\}.$$

Since (63) is true also for $m = 0$, if we put $0! = 1$, then

$$A = \left\{\frac{1}{p}f(t)\times\frac{t^{n-1}}{(n-1)!}\right\} = \left\{\int_0^t f(t-\tau)\,\frac{\tau^{n-1}}{(n-1)!}\,d\tau\right\}$$

which means that A is a quotient distribution. This completes the proof.

VI. Sequences of quotient distributions Distributional convergence

Definition. A sequence of quotient distributions $A_1, A_2, \ldots$ will be called *convergent to a quotient distribution* A and it will be written

$$\lim_{n\to\infty} A_n = A$$

if, and only if, there are such functions $a(t), f_1(t), f_2(t), \ldots \in \mathscr{C}$, $a(t) \not\equiv 0$, that

1° $A_n\{a(t)\} = \{f_n(t)\}$, $n = 1, 2, \ldots$,

2° the sequence $f_1(t), f_2(t), \ldots$ converges almost uniformly to a function $f(t) \in \mathscr{C}$ (we write $f_n(t) \underset{n\to\infty}{\Rightarrow} f(t)$).

3° $A\{a(t)\} = \{f(t)\}$ (3).

THEOREM 25. *If* $\lim\limits_{n\to\infty} A_n = A$ *and* $\lim\limits_{n\to\infty} A_n = A^*$ *then*

$$A = A^*.$$

Proof. By assumption, there exist in the class $\mathscr{C}$ such functions $a(t) \not\equiv 0$, $b(t) \not\equiv 0$, $f_n(t)$, $g_n(t)$ $(n = 1, 2, \ldots)$ that

$$A_n\{a(t)\} = \{f_n(t)\} \quad \text{and} \quad A_n\{b(t)\} = \{g_n(t)\}, \tag{64}$$

$$f_n(t) \underset{n\to\infty}{\Rightarrow} f(t) \quad \text{and} \quad g_n(t) \underset{n\to\infty}{\Rightarrow} g(t), \tag{65}$$

$$A\{a(t)\} = \{f(t)\} \quad \text{and} \quad A^*\{b(t)\} = \{g(t)\}. \tag{66}$$

From (64) we obtain

$$A_n\{a(t)\}\{b(t)\} = \{b(t)\}\{f_n(t)\} = \{a(t)\}\{g_n(t)\}$$

and then

$$\{b(t) \times f_n(t)\} = \{a(t) \times g_n(t)\}$$

which means

$$b(t) \times f_n(t) \sim a(t) \times g_n(t).$$

Hence

$$\frac{1}{p} b(t) \times f_n(t) \sim \frac{1}{p} a(t) \times g_n(t) \tag{67}$$

i. e.

$$\int_0^t b(t-\tau) f_n(\tau)\, d\tau \sim \int_0^t a(t-\tau) g_n(\tau)\, d\tau.$$

In virtue of Theorem 8 and its corollary the congruence (67) implies the identity

$$\frac{1}{p} b(t) \times f_n(t) = \frac{1}{p} a(t) \times g_n(t).$$

By (65) we obtain

$$\frac{1}{p} b(t) \times f(t) = \frac{1}{p} a(t) \times g(t). \tag{68}$$

If there are such functions $F(p, t)$, $G(p, t) \in \mathscr{F}$ that

$$A = \{F(p, t)\} \quad \text{and} \quad A^* = \{G(p, t)\}$$

(3) The idea of a convergent sequence of quotient distributions is analogous to that used by J. Mikusiński for convergent sequences of operators ([2], p. 41).

then, by (66)

$$F(p,t)\times a(t)\sim f(t),\qquad G(p,t)\times b(t)\sim g(t)$$

and, further,

$$F(p,t)\times\frac{1}{p}a(t)\times b(t)\sim\frac{1}{p}b(t)\times f(t),\quad G(p,t)\times\frac{1}{p}a(t)\times b(t)\sim\frac{1}{p}a(t)\times g(t).$$

In virtue of (68) we obtain

$$[F(p,t)-G(p,t)]\times\left[\frac{1}{p}a(t)\times b(t)\right]\sim 0$$

and by Theorem 19

$$F(p,t)-G(p,t)\sim 0.$$

It means, that $A=A^*$, which was to be proved.

THEOREM 26. *If* $\lim\limits_{n\to\infty}A_n=A$ *and* $\lim\limits_{n\to\infty}B_n=B$ *then*

(69) $$\lim_{n\to\infty}(A_n+B_n)=A+B,$$

(70) $$\lim_{n\to\infty}(A_n-B_n)=A-B,$$

(71) $$\lim_{n\to\infty}A_nB_n=AB.$$

Proof. By assumption, there are in the class $\mathscr{C}$ such functions $a(t)\not\equiv 0$, $b(t)\not\equiv 0$, $f_n(t)$, $g_n(t)$ $(n=1,2,\ldots)$ that

(72) $$A_n\{a(t)\}=\{f_n(t)\},\qquad B_n\{b(t)\}=\{g_n(t)\},$$

(73) $$f_n(t)\underset{n\to\infty}{\Rightarrow}f(t),\qquad g_n(t)\underset{n\to\infty}{\Rightarrow}g(t),$$

(74) $$A\{a(t)\}=\{f(t)\},\qquad B\{b(t)\}=\{g(t)\}.$$

From (72) we obtain

$$A_n\{a(t)\}\{b(t)\}=\{b(t)\}\{f_n(t)\},\qquad B_n\{a(t)\}\{b(t)\}=\{a(t)\}\{g_n(t)\}$$

and

$$(A_n+B_n)\left\{\frac{1}{p}a(t)\times b(t)\right\}=\left\{\frac{1}{p}b(t)\times f_n(t)+\frac{1}{p}a(t)\times g_n(t)\right\}.$$

On the other hand, from (74) we obtain analogously

$$(A+B)\left\{\frac{1}{p}a(t)\times b(t)\right\}=\left\{\frac{1}{p}b(t)\times f(t)+\frac{1}{p}a(t)\times g(t)\right\}.$$

Since by (73)

$$\frac{1}{p} b(t) \times f_n(t) + \frac{1}{p} a(t) \times g_n(t) = \int_0^t b(t-\tau) f_n(\tau)\, d\tau + \int_0^t a(t-\tau) g_n(\tau)\, d\tau$$

$$\underset{n\to\infty}{\Rightarrow} \int_0^t b(t-\tau) f(\tau)\, d\tau + \int_0^t a(t-\tau) g(\tau)\, d\tau$$

$$= \frac{1}{p} b(t) \times f(t) + \frac{1}{p} a(t) \times g(t)$$

then, by definition, we obtain (69).

We prove (70) analogously.

In order to prove (71), we multiply (72) and obtain

$$A_n B_n \{a(t)\}\{b(t)\} = \{f_n(t)\}\{g_n(t)\}$$

and hence

$$A_n B_n \left\{ \int_0^t a(t-\tau) b(\tau)\, d\tau \right\} = \left\{ \int_0^t f_n(t-\tau) g_n(\tau)\, d\tau \right\}.$$

Analogously, we have by (74)

$$AB \left\{ \int_0^t a(t-\tau) b(\tau)\, d\tau \right\} = \left\{ \int_0^t f(t-\tau) g(\tau)\, d\tau \right\}.$$

Since

$$\int_0^t f_n(t-\tau) g_n(\tau)\, d\tau \underset{n\to\infty}{\Rightarrow} \int_0^t f(t-\tau) g(\tau)\, d\tau$$

then we obtain (71). This completes the proof.

DEFINITION. A sequence of quotient functions $F_1(p,t), F_2(p,t), \ldots$ will be called *distributionally convergent to a quotient function* $F(p,t)$ and it will be written

$$\lim_{n\to\infty} \operatorname{dis} F_n(p,t) \sim F(p,t)$$

if, and only if,

$$\lim_{n\to\infty} \{F_n(p,t)\} = \{F(p,t)\}.$$

DEFINITION. A series of quotient distributions $A_1 + A_2 + \ldots$ will be called *convergent to a quotient distribution* A and it will be written

$$A = A_1 + A_2 + \ldots$$

if, and only if,

$$A = \lim_{n\to\infty} (A_1 + A_2 + \ldots + A_n).$$

DEFINITION. A series of quotient functions $F_1(p,t) + F_2(p,t) + \ldots$ will be called *distributionally convergent to a quotient function* $F(p,t)$ and it will be written

$$F(p,t) \sim F_1(p,t)+F_2(p,t)+\ldots$$

if, and only if,

$$F(p,t) \sim \lim_{n\to\infty} \operatorname{dis}[F_1(p,t)+F_2(p,t)+\ldots+F_n(p,t)].$$

EXAMPLE. Although the sequence

$$\frac{pt}{1!}, \frac{p^2t^2}{2!}, \frac{p^3t^3}{3!}, \ldots$$

is for every p almost uniformly convergent to zero with respect to t, we have

(75) $$\lim_{n\to\infty}\left\{\frac{p^n t^n}{n!}\right\} = \{1\}, \quad \text{i. e.} \quad \lim_{n\to\infty} \operatorname{dis}\frac{p^n t^n}{n!} \sim 1.$$

Indeed, since

$$\left\{\frac{p^n t^n}{n!}\right\}\{1\} = \left\{\frac{p^{n+1}t^{n+1}}{(n+1)!}\right\}$$

and by (63)

$$\frac{p^{n+1}t^{n+1}}{(n+1)!} \sim 1$$

then

$$\left\{\frac{p^n t^n}{n!}\right\}\{1\} = \{1\}$$

and for $n = 0$ $\{1\}\{1\} = \{1\}$. It means (75), by definition.

THEOREM 27. *If* $f(t), f_1(t), f_2(t), \ldots \in \mathscr{C}$ and

(76) $$f_n(t) \underset{n\to\infty}{\Rightarrow} f(t)$$

then

(77) $$\lim_{n\to\infty} \operatorname{dis} f_n(t) \sim f(t)$$

i. e.

(78) $$\lim_{n\to\infty}\{f_n(t)\} = \{f(t)\} \quad ([2], \text{ p. } 41).$$

Proof. Since

$$\{f_n(t)\}\{1\} = \left\{p\int_0^t f_n(\tau)\,d\tau\right\}$$

and, by Theorem 5,

(79) $$\{f_n(t)\}\{1\} = \{f_n(t)\}, \quad \{f(t)\}\{1\} = \{f(t)\}$$

then, by definition, (78) results from (76). Since (78) is equivalent to (77), the proof is complete.

Remark 1. Accordingly to (79) we can write simply $\{1\} = 1$. Thus, for every distribution A we have $1 \cdot A = A \cdot 1 = A$. Generally, for every number a we can write $\{a\} = a$.

Remark 2. There exist sequences $f_1(t), f_2(t), \ldots$, where $f_n(t) \in \mathscr{C}$ ($n = 1, 2, \ldots$), which are divergent in common sense, but they are distributionally convergent.

Example ([2], p. 41). The sequence $\sin t, 2\sin 2t, 3\sin 3t, \ldots$ is divergent in common sense, but

$$\lim_{n\to\infty} \operatorname{dis} n \sin nt \sim p. \tag{80}$$

Indeed, we have

$$n\sin nt \sim p \int_0^t n \sin n\tau d\tau = p(1-\cos nt)$$

$$\sim p\int_0^t p(1-\cos n\tau)\,d\tau = p^2\left(t-\frac{1}{n}\sin nt\right)$$

then, by (63)

$$\frac{1}{p^2} \sim \frac{t^2}{2}$$

and, thus,

$$\{n\sin nt\}\left\{\frac{t^2}{2}\right\} = \left\{p^2\left(t-\frac{1}{n}\sin nt\right)\right\}\left\{\frac{1}{p^2}\right\} = \left\{t-\frac{1}{n}\sin nt\right\}.$$

Since

$$t-\frac{1}{n}\sin nt \underset{n\to\infty}{\Rightarrow} t$$

and, by (63)

$$\{p\}\left\{\frac{t^2}{2}\right\} = \{p\}\left\{\frac{1}{p^2}\right\} = \left\{\frac{1}{p}\right\} = \{t\}$$

then

$$\lim_{n\to\infty}\{n\sin nt\} = \{p\}$$

i. e. (80).

VII. Examples of applications

All problems solvable in Operational Calculus or in the theory of Laplace Transformation can be solved also on the base of theory presented here. We shall show some examples.

First, we introduce one definition more.

DEFINITION. The function $g(p) \in \mathscr{F}$ will be called a *transform of a function* $F(p, t) \in \mathscr{F}$, if

(81) $$g(p) \sim F(p, t).$$

If there exists a transform (81) of a function $F(p, t)$, it is not unique, because we have also

$$g(p) + h(p) \sim F(p, t)$$

where $h(p) \in \mathscr{F}$ is any function satisfying the congruence $h(p) \sim 0$.

If $F(p, t)$ satisfies the assumptions of Theorem 12, its transform can be found by means of formula (33), which is a generalization of Laplace–Carson Transformation for functions of two variables p and t. But in many cases it is also possible to find this transform in another simplier way, supposing, of course, that such a transform exist and be a feasible function.

EXAMPLE. We shall find transforms of the following functions: $t^k/k!$ (k positive integer), e^{at} (a real and $a \neq 0$), $\sin \omega t$ and $\cos \omega t$ (ω real and $\omega \neq 0$), e^{-apt} (a positive), $e^{-ap^k t}$ (a, k positive), $\sin pt$ and $\cos pt$.

In virtue of (63) we have

(82) $$\frac{t^k}{k!} \sim \frac{1}{p^k}.$$

In virtue of Theorem 5

$$e^{at} \sim p\int_0^t e^{a\tau}\, d\tau = \frac{p}{a} e^{at} - \frac{p}{a}$$

and hence $ae^{at} \sim pe^{at} - p$ i. e.

(83) $$e^{at} \sim \frac{p}{p-a}.$$

In a similar way we obtain

$$\sin \omega t \sim p\int_0^t \sin \omega\tau\, d\tau = -\frac{p}{\omega}\cos \omega t + \frac{p}{\omega}$$

i. e.

(84) $$\omega \sin \omega t + p \cos \omega t \sim p$$

and

$$\cos \omega t \sim p\int_0^t \cos \omega\tau\, d\tau = \frac{p}{\omega}\sin \omega t$$

i. e.

(85) $$p \sin \omega t \sim \omega \cos \omega t.$$

From (84) and (85) we obtain

(86) $$\sin \omega t \sim \frac{p\omega}{p^2 + \omega^2}$$

and

(87) $$\cos\omega t \sim \frac{p^2}{p^2+\omega^2}.$$

Now, we have

$$e^{-apt} \sim p\int_0^t e^{-ap\tau}d\tau = -\frac{1}{a}e^{-apt}+\frac{1}{a}$$

and hence

$$e^{-apt} \sim \frac{1}{a+1}.$$

Similarly

$$e^{-ap^kt} \sim p\int_0^t e^{-ap^k\tau}d\tau = -\frac{1}{p^{k-1}a}e^{-ap^kt}+\frac{1}{p^{k-1}a}$$

and hence

$$e^{-ap^kt} \sim \frac{1}{ap^{k-1}+1}.$$

We find the transforms of $\sin pt$ and $\cos pt$ analogously to (84) and (85) and we obtain

$$\sin pt \sim \cos pt \sim \tfrac{1}{2} \sim e^{-pt}.$$

EXAMPLE 2. Let us solve the following differential equation

(88) $$x'''-x''+4x'-4x = 12t-32$$

with initial conditions

$$x(0) = 5, \quad x'(0) = -2, \quad x''(0) = 5.$$

In virtue of Theorem 18 we have

$$x' \sim px-5p,$$

$$x'' \sim p^2x-5p^2+2p,$$

$$x''' \sim p^3x-5p^3+2p^2-5p$$

and by (82)

$$t \sim \frac{1}{p}.$$

Thus, by (88), we obtain

$$(p^3x-5p^3+2p^2-5p)-(p^2x-5p^2+2p)+4(px-5p)-4x \sim \frac{12}{p}-32$$

i. e. after arrangement

$$p(p^3-p^2+4p-4)x \sim 5p^4-7p^3+27p^2-32p+12$$

and hence

$$x \sim \frac{5p^4-7p^3+27p^2-32p+12}{p(p^3-p^2+4p-4)} = 5-\frac{3}{p}+\frac{p}{p-1}-\frac{p^2}{p^2+4}.$$

In virtue of (82), (83), (87) we obtain

$$x \sim 5-3t+e^t-\cos 2t$$

and by Theorem 8 and its Corollary

$$x = 5-3t+e^t-\cos 2t.$$

EXAMPLE 3. Let us find the general solution of the following differential equation

(89) $$(t^3-2t+1)x'''+(-t^3+9t^2+2t-7)x''+(-6t^2+18t+4)x'+ \\ +(-6t+6)x = e^{-t}.$$

Introducing

(90) $$y = (t^3-2t+1)x$$

we obtain

$$y' = (t^3-2t+1)x'+(3t^2-2)x,$$
$$y'' = (t^3-2t+1)x''+(6t^2-4)x'+6tx,$$
$$y''' = (t^3-2t+1)x'''+(9t^2-6)x''+18tx'+6x$$

and we can write (89) in the form

$$y'''-y'' = e^{-t}.$$

In a similar way as in Example 2 we obtain

$$y \sim \frac{p^4y_0+p^3y_0'+p^2(y_0''-y_0)+p(y_0''-y_0'+1)}{(p+1)(p^3-p^2)}$$

where $y_0 = y(0)$, $y_0' = y_0'(0)$, $y_0'' = y''(0)$.

Hence

$$y \sim A+\frac{B}{p}+\frac{Cp}{p-1}-\frac{1}{2}\frac{p}{p+1}$$

where $A = -y_0''+y_0$, $B = -y_0''+y_0'-1$, $C = y_0''+\frac{1}{2}$ and by (82) and (83)

$$y \sim A+Bt+Ce^t-\tfrac{1}{2}e^{-t}$$

i. e. in virtue of Theorem 8

$$y = A+Bt+Ce^{t}-\tfrac{1}{2}e^{-t}.$$

Hence, by (90)

$$x = \frac{A+Bt+Ce^{t}-\frac{1}{2}e^{-t}}{t^{3}-2t+1}.$$

VIII. Quotient field $\mathscr{K}$

In virtue of Theorem 23 the ring $\mathscr{Q}$ has no zero divisors and by Remark 1 it has an unit element 1. Then we can construct a quotient field $\mathscr{K}$ with elements written in the form

$$\frac{A}{B} \tag{91}$$

where $A, B \in \mathscr{Q}$, $B \neq 0$, i. e. by definition, there exists such a quotient function $F(p, t)$ that

$$B = \{F(p, t)\}$$

and $F(p, t)$ does not satisfy the congruence $F(p, t) \sim 0$.

The arithmetical rules are as follows

$$\frac{A}{B} = \frac{C}{D} \quad \text{if and only if} \quad AD = BC,$$

$$\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD},$$

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD},$$

$$\frac{A}{B} : \frac{C}{D} = \frac{AD}{BC} \quad \text{if} \quad C \neq 0,$$

$$\frac{A}{1} = A.$$

Thus, by isomorphism, all quotient distributions belong to $\mathscr{K}$.

THEOREM 28. *Every element* (91) *can be written in the form*

$$\frac{\{f(t)\}}{\{g(t)\}}$$

where $f(t)$, $g(t) \in \mathscr{C}$, $g(t) \not\equiv 0$.

Proof. Let $A = \{F(p, t)\}$, $B = \{G(p, t)\}$ where, by definition of quotient distribution, $F(p, t)$, $G(p, t) \in \mathscr{Q}$. Moreover, $G(p, t)$ does not satisfy the congruence $G(p, t) \sim 0$. Hence there exist such functions $a(t), b(t), c(t), d(t) \in \mathscr{C}$, $b(t) \not\equiv 0$, $c(t) \not\equiv 0$, $d(t) \not\equiv 0$, that

$$F(p, t) \times b(t) \sim a(t), \quad G(p, t) \times d(t) \sim c(t).$$

Multiplying numerator and denominator of (91) by

$$\{b(t) \times d(t)\} = \{b(t)\}\{d(t)\}$$

we obtain

$$\frac{A}{B} = \frac{\{F(p, t)\}}{\{G(p, t)\}} = \frac{\{F(p, t)\}\{b(t)\}\{d(t)\}}{\{G(p, t)\}\{b(t)\}\{d(t)\}}$$

$$= \frac{\{F(p, t) \times b(t)\}\{d(t)\}}{\{G(p, t) \times d(t)\}\{b(t)\}} = \frac{\{a(t)\}\{d(t)\}}{\{c(t)\}\{b(t)\}}$$

$$= \frac{\left\{\frac{1}{p}\right\}\{a(t)\}\{d(t)\}}{\left\{\frac{1}{p}\right\}\{c(t)\}\{b(t)\}} = \frac{\{f(t)\}}{\{g(t)\}}$$

where $f(t) = \int_0^t a(t-\tau) d(\tau) d\tau$, $g(t) = \int_0^t c(t-\tau) b(\tau) d\tau$ which completes the proof.

Theorem 28 shows the isomorphism of $\mathscr{K}$ with the field of operators in the sense of Mikusiński ([2]), p. 41).

There is an open problem, whether such elements (91) exist which are not distributions. That such a case would be rare, we can see from the following theorem.

THEOREM 29. *If* $\frac{\{f(t)\}}{\{g(t)\}} \in \mathscr{K}$ *and the function* $g(t)$ *has such a transform* $h(p)$, *that also* $\frac{1}{h(p)} \in \mathscr{F}$, *then there exists such a function* $F(p, t) \in \mathscr{Q}$ *that*

$$\{F(p, t)\} = \frac{\{f(t)\}}{\{g(t)\}}.$$

Proof. By definition of a transform, we have

$$\{g(t)\} = \{h(p)\}$$

and thus

$$\frac{\{f(t)\}}{\{g(t)\}} = \frac{\{f(t)\}}{\{h(p)\}}.$$

Putting

$$F(p, t) \equiv \frac{1}{h(p)} f(t) \in \mathscr{F}$$

we obtain

$$F(p,t)\times g(t) \sim F(p,t)\times h(p) \sim h(p)F(p,t) = h(p)\frac{1}{h(p)}f(t) = f(t).$$

Thus, $F(p,t)\in\mathscr{Q}$ and $\{F(p,t)\}\{g(t)\} = \{f(t)\}$ and hence

$$\frac{\{f(t)\}}{\{g(t)\}} = \frac{\{F(p,t)\}\{g(t)\}}{\{g(t)\}} = \{F(p,t)\}$$

which was to be proved.

It is also an open problem, whether such a function $g(t)\in\mathscr{C}$ exists, which has transforms, but no such transform $h(p)$ that

$$\frac{1}{h(p)}\in\mathscr{F}.$$

IX. A theory corresponding to Laplace Transformation

Let us consider, how to change the theory presented here, in order to obtain a correspondence to Laplace Transformation and not to Laplace–Carson Transformation. First, let us notice, that the function $f(t)\equiv 1$ corresponds by Laplace Transformation to $g(s) = 1/s$ and not to 1, as it was by Laplace–Carson Transformation. Thus, we must construct the new theory in such a way, as to distinguish $f(t)\equiv 1$ and $g(s)\equiv 1$. (According to custom, we replace p used in the theory of Laplace–Carson Transformation by s used for Laplace Transformation).

We shall deal with pairs of functions from the ring $\mathscr{F}$, namely,

(92) $$\big(F(s,t),\, g(s)\big)$$

i. e. we consider only such functions $F(s,t), g(s)$ that $F(p,t), g(p)\in\mathscr{F}$ in the sense of previous theory.

The arithmetical rules, however, are now as follows:

$$\big(F_1(s,t),\, g_1(s)\big)+\big(F_2(s,t),\, g_2(s)\big) = \big(F_1(s,t)+F_2(s,t),\, g_1(s)+g_2(s)\big),$$

$$\big(F_1(s,t),\, g_1(s)\big)\times\big(F_2(s,t),\, g_2(s)\big)$$

$$= \Big(\int_0^t F_1(s,t-\tau)F_2(s,\tau)\,d\tau + g_1(s)F_2(s,t)+g_2(s)F_1(s,t),\, g_1(s)g_2(s)\Big).$$

The above arithmetical operations are associative and commutative, multiplication is distributive with respect to addition.

In virtue of isomorphism, we introduce the following notations:

$$\big(F(s,t),0\big) = \big[F(s,t)\big], \qquad \big(0,g(s)\big) = \langle g(s)\rangle.$$

Thus, by definition of addition,

$$\big(F(s,t),g(s)\big) = \big[F(s,t)\big]+\langle g(s)\rangle$$

and by definition of multiplication

$$\big[F_1(s,t)\big]\times\big[F_2(s,t)\big] = \big(F_1(s,t),0\big)\times\big(F_2(s,t),0\big)$$

$$= \Big(\int_0^t F_1(s,t-\tau)F_2(s,\tau)\,d\tau,\ 0\Big)$$

i. e.

(93) $$\big[F_1(s,t)\big]\times\big[F_2(s,t)\big] = \Big[\int_0^t F_1(s,\ t-\tau)F_2(s,\tau)\,d\tau\Big].$$

We have also

$$\langle 1\rangle\times\big(F(s,t),g(s)\big) = (0,1)\times\big(F(s,t),g(s)\big) = \big(F(s,t),g(s)\big)$$

or, more generally,

$$\langle h(s)\rangle\times\big(F(s,t),g(s)\big) = \big(h(s)F(s,t),h(s)g(s)\big).$$

In particular, we have

$$\langle h(s)\rangle\times\big[F(s,t)\big] = \big[h(s)F(s,t)\big], \qquad \langle h(s)\rangle\ \langle g(s)\rangle = \langle h(s)g(s)\rangle.$$

The set of all pairs (92) forms a commutative ring, which will be called *ring* $\mathscr{L}$. The functions $[F(s,t)]$ form a ring $\mathscr{T}$ with multiplication (93). Theorems 1–5 hold also for the ring $\mathscr{T}$. Theorem 6 changes as follows

If $\big[g(s)\big]$, $\big[F(s,t)\big]\in\mathscr{T}$ then

$$\big[g(s)\big]\times\big[F(s,t)\big]\sim\Big[\frac{1}{s}\,g(s)F(s,t)\Big].$$

Theorems 7–16 hold also for the ring $\mathscr{T}$.

In the ring $\mathscr{L}$ we introduce moreover the congruences $\big[sg(s)\big]\sim\langle g(s)\rangle$.

The propositions of Theorems 17 and 18 change as follows

$$\Big[\frac{\partial}{\partial t}F(s,t)\Big]\sim\big[sF(s,t)\big]-\langle F_0(s)\rangle,$$

$$\Big[\frac{\partial^k}{\partial t^k}F(s,t)\Big]\sim\big[s^kF(s,t)\big]-\langle s^{k-1}F_0(s)\rangle-\ \dots\ -\langle F_0^{(k-1)}(s)\rangle.$$

Theorems 19-29 hold for the ring $\mathscr{L}$.

The propositions of Theorems 12 and 13 can be written in the form

$$[F(s,t)] \sim \left[s\int_0^\infty e^{-st} F(s,t)\,dt \right] \sim \left\langle \int_0^\infty e^{-st} F(s,t)\,dt \right\rangle,$$

$$[f(t)] \sim \left[s\int_0^\infty e^{-st} f(t)\,dt \right] \sim \left\langle \int_0^\infty e^{-st} f(t)\,dt \right\rangle.$$

There are many problems connected with the theory presented here, which were not considered. The aim of this paper is to give only an outline of the whole theory.

Warszawa, February 1969

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[3] — *Operational Calculus*, 1959.