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Scalar differential concomitants of the second order of the metrical tensor in three-dimensional Riemann spaces

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Introduction. Let V^n be an n-dimensional Riemann space. The metrical tensor on V^n will be denoted by g_{ij} (the signature of this tensor is arbitrary).

The problem of the classification of Riemann spaces has been considered in many papers (cf. [4], [5] and [6]). The solution of this problem is equivalent to the determination of scalar differential concomitants of the second order of the tensor g_{ij}. It is known (cf. [3], also [8], p. 138) that such a concomitant is an algebraic concomitant of the tensor g_{ij} and of the curvature tensor. For $n = 2$ such concomitants have been found in paper [3].

In Section 1 we shall determine all scalar differential concomitants of the second order of the tensor g_{ij} for $n = 3$.

In Section 2 we characterize three-dimensional local Euclidean spaces and three-dimensional Einstein spaces by means of the scalars found in Section 1.

1. Let us consider a Riemann space V^3. The problem of finding the scalar differential concomitants of the second order of the tensor g_{ij} is brought to the solving of the equation

$$\omega(g_{ij}, R_{ijkl}) = \omega(g_{i'j'}, R_{i'j'k'l'}), \qquad i, j, k, l, i', j', k', l' = 1, 2, 3. \tag{1.1}$$

In the space V^3 the tensor R_{ijkl} satisfies the following relation:

$$\begin{aligned} R_{ijkl} &= R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} + \tfrac{1}{2}R(g_{jl}g_{ik} - g_{jk}g_{il}) \\ &= \varphi_{ijkl}(g_{rs}, R_{rs}), \end{aligned} \tag{1.2}$$

where $R_{jk} = g^{rs}R_{rjks}$ and R denotes the scalar curvature of the space V^3 (cf. [8], p. 520).

As follows from relation (1.2) our problem leads to the determination of the scalar concomitants of the pair of tensors (g_{ij}, R_{ij}) or (g_{ij}, R^i_j).

We put

$$R = R^i_i,$$

$$P = \begin{vmatrix} R^1_1 & R^1_2 \\ R^2_1 & R^2_2 \end{vmatrix} + \begin{vmatrix} R^1_1 & R^1_3 \\ R^3_1 & R^3_3 \end{vmatrix} + \begin{vmatrix} R^2_2 & R^2_3 \\ R^3_2 & R^3_3 \end{vmatrix},$$

$$Q = \mathrm{Det}\|R^i_k\|,$$

s is the signature of the tensor g_{ij}.

We prove the following theorem:

THEOREM 1. *If the signature of the tensor g_{ij} is $s = 3$ or $s = -3$, then every scalar differential concomitant of the second order of the tensor g_{ij} is an arbitrary function of the signature s and scalars R, P, Q.*

Proof. We shall prove the theorem in the case $s = 3$. In the case $s = -3$ the proof is analogous.

It is known that we may always find a non-singular matrix $\|A^i_{i'}\|$ such that

$$\|g_{i'j'}\| = \|A^i_{i'}, A^j_{j'}, g_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{1.3}$$

Hence we obtain the relations

$$R^{i'}_{j'} = R_{i'j'}, \tag{1.4}$$

i.e. the matrix $\|R^{i'}_{j'}\|$ is symmetric. Therefore there exists an orthogonal matrix $\|A^{i'}_{i''}\|$ such that

$$\|R^{i''}_{j''}\| = \|A^{i''}_{i'} A^{j'}_{j''} R^{i'}_{j'}\| = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix}, \tag{1.5}$$

where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of $\|R^i_k\|$. It follows from (1.3) and (1.4) that the following conditions

$$\|g_{i''j''}\| = \|A^i_{i''} A^j_{j''} g_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{1.6}$$

$$\|R^{i''}_{j''}\| = \|A^{i''}_i A^j_{j''} R^i_j\| = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix} \tag{1.7}$$

are fulfilled, where $\|A^i_{i''}\| = \|A^i_{i'} A^{i'}_{i''}\|$.

Inserting (1.6) and (1.7) into equation

$$\varrho(g_{ij}, R^i_j) = \varrho(g_{i'j'}, R^{i'}_{j'})$$

we get

$$(1.8)\qquad \varrho(g_{ij}, R^i_j) = \varrho\left(\begin{Vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{Vmatrix}, \begin{Vmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{Vmatrix}\right).$$

Values $\lambda_1, \lambda_2, \lambda_3$ are determined by R, P, Q. Hence

$$(1.9)\qquad \varrho(g_{ij}, R^i_j) = h(3, R, P, Q).$$

This completes the proof.

Now we shall determine the scalar differential concomitants of the second order of the tensor g_{ij} with an arbitrary signature s. Our considerations are based on the results of [9] (p. 10-18, 21-23).

We consider the tensor bundle

$$(1.10)\qquad g_{ij}\lambda + R_{ij}$$

and the matrix bundle

$$(1.11)\qquad G\lambda + K,$$

where

$$G = \|g_{ij}\|, \quad K = \|R_{ij}\|.$$

We denote by σ the partial signatures (Teilsignaturen) of the canonical form of bundle (1.11) (cf. [9], p. 18).

We shall prove the following theorem:

THEOREM 2. *Every scalar differential concomitant of the second order of the symmetric and regular tensor g_{ij} for $n = 3$, is an arbitrary function of the partial signatures σ of the canonical form of bundle* (1.11), *of the scalars R, P, Q and of the Weierstrass characteristic of the matrix $\|R^i_k\|$.*

Proof. The bundle $G\lambda + K$ is regular. In fact, we have

$$\operatorname{Det}(G\lambda+K) = \operatorname{Det}G(E\lambda+G^{-1}K) = \operatorname{Det}G\operatorname{Det}(E\lambda+G^{-1}K) \neq 0,$$

where E is the unit element of the group $Gl(3)$. Every scalar concomitant of the tensor bundle $g_{ij}\lambda + R_{ij}$ is a function of elementary divisors and of partial signatures of bundle (1.11) (cf. [9], p. 22). The bundles $G\lambda+K$ and $E\lambda+G^{-1}K$ are stricly equivalent. Hence these bundles have identical elementary divisors (cf. [1], p. 332). It is known that the scalars R, P, Q and the Weierstrass characteristic of the matrix $\|R^i_k\|$ determine elementary divisors of the bundle $E\lambda+G^{-1}K$. Thus any scalar differential concomitant of the second order of the tensor g_{ij} has the form

$$(1.12)\qquad f(\sigma, R, P, Q, [e_1 e_2 e_3]).$$

This completes the proof.

For $n = 3$ the Weierstrass characteristic $[e_1 e_2 e_3]$ can have the following values:

1° [1 1 1],

2° [2 1],

3° [3],

4° $[\bar{1}\ \bar{1}\ 1]$ (1).

In cases 1°-4° we have the following canonical forms of bundle (1.11) and of the tensor g_{ij}:

1° $$\begin{Vmatrix} \varepsilon_1(\lambda+\lambda_1) & 0 & 0 \\ 0 & \varepsilon_2(\lambda+\lambda_2) & 0 \\ 0 & 0 & \varepsilon_3(\lambda+\lambda_3) \end{Vmatrix}, \quad \begin{Vmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{Vmatrix},$$

2° $$\begin{Vmatrix} 0 & \varepsilon_1(\lambda+\lambda_1) & 0 \\ \varepsilon_1(\lambda+\lambda_1) & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2(\lambda+\lambda_2) \end{Vmatrix}, \quad \begin{Vmatrix} 0 & \varepsilon_1 & 0 \\ \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \end{Vmatrix},$$

3° $$\begin{Vmatrix} 0 & 0 & \varepsilon_1(\lambda+\lambda_1) \\ 0 & \varepsilon_1(\lambda+\lambda_1) & 0 \\ \varepsilon_1(\lambda+\lambda_1) & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & \varepsilon_1 \\ 0 & \varepsilon_1 & 0 \\ \varepsilon_1 & 0 & 0 \end{Vmatrix},$$

4° $$\begin{Vmatrix} \lambda+a_1 & \beta_1 & 0 \\ \beta_1 & -\lambda-a_1 & 0 \\ 0 & 0 & \varepsilon_1(\lambda+\lambda_2) \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \varepsilon_1 \end{Vmatrix},$$

where $(\varepsilon_i)^2 = 1, i = 1, 2, 3$.

We put

$$\underset{i}{d}_{jk} \overset{\text{df}}{=} -\lambda_i g_{jk} + R_{jk}, \qquad i = 1, 2, 3.$$

The signature of the tensor $\underset{i}{d}_{jk}$ will be denoted by s_i.

From Theorem 1 we obtain (on account of the above forms of bundle (1.11) and of the tensor g_{ij}) the following corollaries:

COROLLARY 1. *If* $[e_1 e_2 e_3] \neq [2\ 1]$ *and* $[e_1 e_2 e_3] \neq [1\ 1\ 1]$, *then every scalar differential concomitant of the second order of the tensor* g_{ij} *is a function of the signature s of the tensor* g_{ij}, *of the Weierstrass characteristic* $[e_1 e_2 e_3]$ *and of the scalars* R, P, Q.

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(1) The dash denotes that the matrix has complex eigenvalues.

COROLLARY 2. *If* $[e_1e_2e_3] = [2\ 1]$ *or* $[e_1e_2e_3] = [1\ 1\ 1]$, *then every scalar differential concomitant of the second order of the tensor* g_{ij} *is a function of the signature* s *of the tensor* g_{ij}, *of signatures* s_i, *of the Weierstrass characteristic* $[e_1e_2e_3]$ *and of the scalars* R, P, Q.

2. We use the results of Section 1 to obtain a certain classification of three-dimensional Riemann spaces.

We call the scalars s, $[e_1e_2e_3]$ arithmetical invariants. Now we have the following theorem:

THEOREM 3. *Arithmetical invariants give the following types of three-dimensional Riemann spaces:*

$$\begin{aligned} s &= 3\ [1\ 1\ 1], \\ s &= -3\ [1\ 1\ 1], \\ s &= 1\ [1\ 1\ 1], [\bar{1}\ \bar{1}\ 1], [3], [2\ 1], \\ s &= -1\ [1\ 1\ 1], [\bar{1}\ \bar{1}\ 1], [3], [2\ 1]. \end{aligned}$$

Proof. The proof follows from Corollaries 1 and 2.

Now we shall prove two theorems which characterize three-dimensional local Euclidean spaces and three-dimensional Einstein spaces by means of the scalars determined above.

THEOREM 4. *The space* V^3 *is a local Euclidean space if and only if*

$$[e_1e_2e_2] = [1\ 1\ 1], \quad R = P = 0. \tag{2.1}$$

Proof. We assume that the space V^3 is a local Euclidean space, i.e. $R_{ijkl} = 0$. Thus we obtain $R^i_k = 0$. This implies that $R = P = Q = 0$ and $[e_1e_2e_3] = 1\ 1\ 1$.

Now assume that (2.1) holds. Hence we get $Q = 0$. In fact, in the opposite case the characteristic polynomial of the matrix $\|R^i_k\|$ has complex roots which is a contradiction to the assumption $[e_1e_2e_3] = [1\ 1\ 1]$. Therefore the matrix $\|R^i_k\|$ is the zero-matrix. It follows from (1.2) that

$$R_{ijkl} = 0.$$

THEOREM 5. *The space* V^3 *is an Einstein space if and only if*

$$[e_1e_2e_3] = [1\ 1\ 1], \quad R = \text{const}, \quad P = \frac{R^2}{3}, \quad Q = \frac{R^3}{27}. \tag{2.2}$$

Proof. We assume that the space V^3 is an Einstein space, i.e.

$$R_{ij} = \tau g_{ij}, \quad i, j = 1, 2, 3. \tag{2.3}$$

It is known (cf. [2], p. 238) that $\tau = \text{const}$. We have, by (2.3),

$$R^i_j = \tau\delta^i_j, \quad \delta^i_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{2.4}$$

From (2.4) it follows that $[e_1e_2e_3] = [1\ 1\ 1]$ and $R = 3\tau = \text{const}$.

Now assume that (2.2) holds. Then the characteristic polynomial of the matrix $\|R^i_k\|$ has the following form:

$$w(\lambda) = (\lambda - \tfrac{1}{3}R)^3. \tag{2.5}$$

The Weierstrass characteristic of the matrix $\|R^i_k\|$ satisfies the condition $[e_1 e_2 e_3] = [1\ 1\ 1]$; so the matrix $\|R^i_k\|$ has the following canonical form:

$$\left\| \begin{matrix} \tfrac{1}{3}R & 0 & 0 \\ 0 & \tfrac{1}{3}R & 0 \\ 0 & 0 & \tfrac{1}{3}R \end{matrix} \right\|. \tag{2.6}$$

It follows from (2.6) that

$$R_{ij} = \tfrac{1}{3}Rg_{ij}$$

and thus the space V^3 is an Einstein space.

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