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The spectrum of the Laplace operator on conformally flat manifolds

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1. Introduction. We consider a compact orientable Riemannian manifold (M, g) of dimension n. We denote by $\Lambda^q(M)$, $q = 0, 1, \ldots, n$, the vector space of exterior q-forms on M. Let Δ be the Laplace operator acting on q-forms. We denote by $\mathrm{Sp}^q(M, g)$ the spectrum of Δ on q-forms.

The purpose of this article is to prove that $\mathrm{Sp}^0(M, g)$ and $\mathrm{Sp}^1(M, g)$ determine the conformal flatness with constant scalar curvature on (M, g).

The basic properties of the Laplace operator are given in Section 2.

The influence of $\mathrm{Sp}^{q_j}(M, g)$, $j = 1, 2$, on the conformal flatness with constant scalar curvature on (M, g) is studied in Section 3.

2. Let (M, g) be a compact, orientable Riemannian manifold of dimension n. Let $\Lambda^q(M)$ be the vector space of the exterior q-forms on M, where $q = 0, 1, \ldots, n$. For $q = 0$, we have the set $\Lambda^0(M)$ of all differentiable functions on M.

Let $\Delta = d\delta + \delta d$ be the Laplace operator which acts on the exterior algebras of M

$$\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \ldots \oplus \Lambda^n(M) = \bigoplus_{q=0}^{n} \Lambda^q(M)$$

as follows:

$$\Delta\colon \Lambda(M) \to \Lambda(M),$$

$$\Delta\colon \Lambda^q(M) \to \Lambda^q(M), \qquad \Delta\colon \alpha \to \Delta(\alpha) = d\delta(\alpha) + \delta d(\alpha), \qquad \forall \alpha \in \Lambda^q(M).$$

If the exterior q-form α is such that $\Delta\alpha = \lambda\alpha$, where $\lambda \in \boldsymbol{R}$, then α is called a *q-eigenform* (or simply a *q-form*) and λ the eigenvalue associated with α. The set of eigenvalues associated with the exterior q-forms is called the *spectrum of Δ* on $\Lambda^q(M)$. The spectrum of Δ on $\Lambda^q(M)$ is denoted by $\mathrm{Sp}^q(M, g)$. Hence we obtain

$$\mathrm{Sp}^q(M, g) = \{0 \leqslant \lambda_{1,q} = \ldots = \lambda_{1,q} < \lambda_{2,q} = \ldots = \lambda_{2,q} < \ldots < \infty\}$$

where each eigenvalue is repeated as many times as its multiplicity, which is finite and the spectrum $\mathrm{Sp}^q(M, g)$ is discrete since Δ is an elliptic operator.

The spectra $\mathrm{Sp}^{q_1}(M, g)$ and $\mathrm{Sp}^{q_2}(M, g)$ have an influece on the geometry of (M, g).

The purpose of the present paper is to prove that $\mathrm{Sp}^{q_1}(M, g)$ and $\mathrm{Sp}^{q_2}(M, g)$ determine the conformal flatness with constant scalar curvature.

In order to study the influence of $\mathrm{Sp}^{q_j}(M, g)$, $j = 1, 2$, on the geometry of (M, g) we need the Minakshisundarum–Pleijel–Gaffney asymptotic expansion which is given by

$$\sum_{i=1}^{\infty} e^{\lambda_{i,q_j}^t} \sim \sum_{\substack{t>0\\ t\to 0}} (4\pi t)^{-n/2}(\alpha_{0,q_j}+\alpha_{1,q_j}t+\ldots+\alpha_{m,q_j}t^m)+O(t^{m-n/2}), \qquad j=1, 2,$$

where α_{0,q_j}, α_{1,q_j}, α_{2,q_j}, ... are numbers which can be expressed by

$$\alpha_{i,q_j} = \int_M u_{i,q_j} dM, \qquad i = 0, 1, 2, \ldots,$$

where dM is the volume element of M and

$$u_{i,q_j}\colon M \to R, \qquad i = 0, 1, 2, \ldots,$$

are functions which are local Riemannian invariants. These can by expressed by the curvature tensor, its associated tensors, and their covariant derivatives.

Some of them have been computed

$$\alpha_{0,q_j} = \binom{n}{q_j}\mathrm{Vol}(M), \tag{2.1}$$

$$\alpha_{1,q_j} = \int_M C(n, q_j) S dM, \tag{2.2}$$

$$\alpha_{2,q_j} = \int_M [C_1(n, q_j)S^2 + C_2(n, q_j)|E|^2 + C_3(n, q_j)|R|^2]\, dM, \tag{2.3}$$

where

$$C(n, q_j) = \tfrac{1}{6}\binom{n}{q_j} - \binom{n-2}{q_j-1}, \tag{2.4}$$

$$C_1(n, q_j) = \tfrac{1}{72}\binom{n}{q_j} - \tfrac{1}{6}\binom{n-2}{q_j-1} + \tfrac{1}{2}\binom{n-4}{q_j-2}, \tag{2.5}$$

$$C_2(n, q_j) = -\tfrac{1}{180}\binom{n}{q_j} + \tfrac{1}{2}\binom{n-2}{q_j-1} - 2\binom{n-4}{q_j-2}, \tag{2.6}$$

$$C_3(n, q_j) = \tfrac{1}{180}\binom{n}{q_j} - \tfrac{1}{12}\binom{n-2}{q_j-1} + \tfrac{1}{2}\binom{n-4}{q_j-2}, \tag{2.7}$$

and R, E and S are the curvature tensor field, the Ricci curvature, and the scalar curvature of (M, g), respectively, and $|R|$, $|E|$ are the norms of R, E with respect to g.

3. We consider two compact, orientable Riemannian manifolds (M, g), (M', g') for which we further assume that

$$\text{(3.1)} \qquad \mathrm{Sp}^{q_1}(M, g) = \mathrm{Sp}^{q_1}(M', g'), \qquad \mathrm{Sp}^{q_2}(M, g) = \mathrm{Sp}^{q_2}(M', g').$$

THEOREM 3.1. *Let (M, g), (M', g') be two compact orientable Riemannian manifolds. If n is given, then we can find at least two integers q_1, q_2 (two of them are* 0 *and* 1) *such that $\mathrm{Sp}^{q_j}(M, g) = \mathrm{Sp}^{q_j}(M', g')$, $j = 1, 2$, imply that (M, g) is conformally flat with constant scalar curvature S if and only if (M', g') is conformally flat with constant scalar curvature S' and $S = S'$.*

Proof. Let C, G be the Weyl conformal curvature tensor field and the Einstein tensor field respectively on (M, g). The components (C_{ijkl}) and (G_{ij}) of C and G respectively, with respect to a local coordinate system $(x^1, \ldots, x^n)$ on the manifold (M, g) are given by

$$\text{(3.2)} \qquad C_{ijkl} = R_{ijkl} - \alpha(E_{jk}g_{il} - E_{jl}g_{ik} - E_{il}g_{jk} + E_{ik}g_{jl}),$$

where

$$\alpha = \frac{1}{n-1}, \qquad \beta = \frac{1}{(n-1)(n-2)},$$

and

$$\text{(3.3)} \qquad G_{ij} = E_{ij} - \gamma g_{ij} S,$$

where $\gamma = 1/n$.

From (3.2) and (3.3) we obtain

$$\text{(3.4)} \qquad |C|^2 = |R|^2 - \frac{4|E|^2}{n-2} + \frac{2S^2}{(n-1)(n-2)},$$

$$\text{(3.5)} \qquad |G|^2 = |E|^2 - \frac{S^2}{n}.$$

Relation (2.3), by means of (3.4) and (3.5), takes the form

$$\text{(3.6)} \qquad \alpha_{2,q_j} = \int_M [A_1(n, q_j)|C|^2 + A_2(n, q_j)|G|^2 + A_3(n, q_j)S^2]\, dM, \qquad j = 1, 2,$$

where

$$\text{(3.7)} \qquad A_1(n, q_j) = \frac{1}{360n(n-1)(n-2)(n-3)}\binom{n}{q_j} L_1(n, q_j),$$

(3.8) $$A_2(n, q_j) = \frac{2}{360(n-1)(n-2)^2}\binom{n}{q_j} L_2(n, q_j),$$

(3.9) $$A_3(n, q_j) = \frac{2}{360n^2(n-1)^2}\binom{n}{q_j} L_3(n, q_j), \quad j = 1, 2.$$

The expressions $L_1(n, q_j)$, $L_2(n, q_j)$ and $L_3(n, q_j)$ in relations (3.7), (3.8) and (3.9) are given by

(3.10) $$L_1(n, q_j) = 180q_j(q_j-1)(n-q_j)(n-q_j-1) - \\ -30q_j(n-q_j)(n-2)(n-3)+2n(n-1)(n-2)(n-3),$$

(3.11) $$L_2(n, q_j) = -360q_j(q_j-1)(n-q_j)(n-q_j-1)+ \\ +30q_j(n-q_j)(n-2)(3n-8)-n(n-1)(n-2)(n-6),$$

(3.12) $$L_3(n, q_j) = 180q_j(q_j-1)(n-q_j)(n-q_j-1)-60q_j(n-q_j)(n-2)^2+ \\ +n(n-1)(5n^2-7n+6).$$

From relations (3.1) we obtain

(3.13) $$\alpha_{0,q_1} = \alpha'_{0,q_1}, \quad \alpha_{1,q_1} = \alpha'_{1,q_1}, \quad \alpha_{2,q_1} = \alpha'_{2,q_1},$$

(3.14) $$\alpha_{0,q_2} = \alpha'_{0,q_2}, \quad \alpha_{1,q_2} = \alpha'_{1,q_2}, \quad \alpha_{2,q_2} = \alpha'_{2,q_2}.$$

The third of equalities in (3.13) can be written by means of (3.6) as

(3.15) $$\int_M [A_1(n, q_1)|C|^2 + A_2(n, q_1)|G|^2 + A_3(n, q_1)S^2]\,dM \\ = \int_{M'} [A_1(n, q_1)|C'|^2 + A_2(n, q_1)|G'|^2 + A_3(n, q_1)S'^2]\,dM'.$$

The third of (3.14) by virtue of (3.6) takes the form

(3.16) $$\int_M [A_1(n, q_2)|C|^2 + A_2(n, q_2)|G|^2 + A_3(n, q_2)S^2]\,dM \\ = \int_M [A_1(n, q_2)|C'|^2 + A_2(n, q_2)|G'|^2 + A_3(n, q_2)S'^2]\,dM'.$$

From relations (3.15) and (3.16) we have

(3.17) $$\int_M [T_1|C|^2 + T_2S^2]\,dM = \int_{M'} [T_1|C'|^2 + T_2S'^2]\,dM',$$

where

(3.18) $$T_1 = \lambda[\lambda_1(A_1B_2 - A_2B_1) + \lambda_2(A_1 - A_2) + \lambda_3(B_1 - B_2)],$$

(3.19) $$T_2 = \mu[\mu_1(A_1B_2 - A_2B_1) + \mu_2(A_1 - A_2) + \mu_3(B_1 - B_2)],$$

(3.20) $$\lambda = \frac{2\binom{n}{q_1}\binom{n}{q_2}}{360^2 n^2 (n-1)^2 (n-2)^3 (n-3)},$$

(3.21) $$\lambda_1 = 30(n-2)^2,$$

(3.22) $$\lambda_2 = 3n(n-1)(n-2)^2,$$

(3.23) $$\lambda_3 = -150n(n-1)(n-2)^3(n-3),$$

(3.24) $$\mu = \frac{4\binom{n}{q_1}\binom{n}{q_2}}{360^2 n^3 (n-1)^3 (n-2)^2},$$

(3.25) $$\mu_1 = -30n(n-2),$$

(3.26) $$\mu_2 = 3n^2(n-1)(3n-2),$$

(3.27) $$\mu_3 = -30n^2(n-1)(n-2)(13n^2-41n+18),$$

(3.28) $$A_j = 180q_j(q_j-1)(n-q_j)(n-q_j-1), \quad j = 1, 2,$$

(3.29) $$B_j = q_j(n-q_j), \quad j = 1, 2.$$

The second relation of (3.13) or (3.14) implies

(3.30) $$\int_M S\,dM = \int_{M'} S'\,dM'.$$

We assume that the manifold (M', q') is conformally flat, therefore we obtain $C' = 0$ and has constant scalar curvature S'.

Relation (3.17) becomes

(3.31) $$\int_M [T_1|C|^2 + T_2 S'^2]\,dM \int_{M'} T_2 S'^2\,dM'.$$

From (3.30) we obtain, since S' is constant

(3.32) $$\int_M S^2\,dM \geqslant \int_{M'} S'^2\,dM'.$$

Given n, we can find two integers q_1, q_2 between 0 and n such that T_1 and T_2 are positive or negative in either case (3.31) and (3.30) imply $C = 0$ and the scalar curvature S is constant and $S = S'$.

If we take $q_1 = 0$ and $q_2 = 1$, then we obtain

(3.33) $$A_1 = 0, \quad A_2 = 0, \quad B_1 = 0, \quad B_2 = n-1.$$

Relation (3.31), by means of (3.33), takes the form

(3.34) $$\int_M [-\lambda\lambda_3(n-1)|C|^2 - \mu\mu_3(n-1)S^2]\,dM = \int_{M'} -\mu\mu_3 S'^2\,dM'.$$

From (3.20), (3.23), (3.24), (3.27) and (3.34) we have $|C|^2 = 0$ which implies $C = 0$. We also obtain $S' = S = \text{const}$.

References

[1] P. Gilkey, *Recursion relations and their asymptotic behavior of the eigenvalues of the Laplacian*, Compositio Math. 38 (1979), 201–240.

[2] V. K. Petodi, *Curvature and the fundamental solution of the heat operator*, J. Ind. Math. Soc. 34 (1970), 269–285.

[3] Gr. Tsagas and K. Kockinos, *The geometry and the Laplace operator on the exterior 2-forms on a compact Riemannian manifold*, Proc. Amer. Math. Soc. 73 (1979), 109–116.

[4] Gr. Tsagas, *The spectrum of the Laplace operator for a special Riemannian manifold*, Kodai Math. Sem. Rep. (to appear).

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