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Some properties of stationary parabolic mixed problems in an infinite cylinder

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Abstract. In this paper the regularity, uniqueness, asymptotic behaviours and the convergence of successive approximations are investigated for solutions of problem (1), (2), (3). All these questions are studied in spaces of Hölder continuous functions.

1. Introduction. In paper [2] the existence of solution of the stationary parabolic initial-boundary value problem is studied for the system of $p \geqslant 1$ equations with p unknown functions

$$D_t u - \sum_{|k|=2b} A_k(x) D_x^k u = F(x, t, \ldots, D_x^\gamma u, \ldots) \tag{1}$$

for $(x, t) \in \Omega \times \langle 0, \infty) = Q_\infty$, where $0 \leqslant |\gamma| \leqslant 2b-1$, $b \geqslant 1$, with data

$$u|_{t=0} = O, \quad x \in \Omega \tag{2}$$

$$\sum_{|k| \leqslant r_q} (B_k^{(q)}, D_x^k u)|_{\Gamma_\infty} = 0, \tag{3}$$

where $r_q \leqslant 2b-1$ and $q = 1, \ldots, bp$ and Ω is a bounded domain of the Euclidean space $\boldsymbol{R}_m$ with the boundary $\partial\Omega$ and $\Gamma_\infty = \partial\Omega \times \langle 0, \infty)$. The present paper is devoted to some other fundamental questions concerning problem (1), (2), (3). We investigate the regularity, uniqueness and asymptotic behaviour of the solution and the convergence of successive approximations.

Throughout this paper, appart of some new notation, we shall hold fast to the notions, notations and assumptions (A), (B), (D_{l+a}) introduced in paper [2].

Our considerations will be carried out in the following space of Hölder continuous vector functions: Consider the three classes of real functions $g_{i,k_0,k}\colon \langle 0, \infty) \to (0, \infty)$ for $i = 1, 2$ and $g_{3,k_0,k}\colon \langle 0, \infty) \times \langle 0, \infty) \to (0, \infty)$ (k_0 is a non-negative integer and k is a multiindex $(k_1, \ldots, k_m)$) with the property

(j) The functions $g_{i,k_0,k}$ for $i = 1, 2$ and $g_{3,k_0,k}$ are bounded and integrable on $\langle 0, T\rangle$ and on $\langle 0, T\rangle \times \langle 0, T\rangle$, respectively, for any real $T > 0$.

Then, if $l \geq 0$, $b \geq 1$ and $0 < a < 1$, the linear space $C_{x,t,g}^{l+a,(l+a)/2b}(Q_\infty)$ of vector functions $u(x,t) = (u_1(x,t), \ldots, u_p(x,t))$ is defined by the inequality

$$(4)\qquad \|u\|_{l+a,Q_\infty}^{g} \overset{\text{df}}{=} \max_{j=1,\ldots,p} \Big\{ \sum_{i=0}^{l} \sum_{h=i} \sup_{Q_\infty} [D_t^{k_0} D_x^k u_j(x,t) g_{1,k_0,k}^{-1}(t)] + \sum_{h=l} \sup_{\substack{(x,t),(y,t)\in Q_\infty \\ x\neq y}} [\langle D_t^{k_0} D_x^k u_j(x,t)\rangle_{a,x} g_{2,k_0,k}^{-1}(t)] + \sum_{0<l+a-h<2b} \sup_{\substack{(x,t)(y,t')\in Q_\infty \\ t\neq t'}} [\langle D_t^{k_0} D_x^k u_j(x,t)\rangle_{(l+a-h)/2b,t}\, g_{3,k_0,k}^{-1}(t,t')] \Big\} < \infty,$$

where $h = 2bk_0 + |k|$.

Note that the space $C_{x,t,f(B,\varkappa,\mu,\nu)}^{2b-1+a,(2b-1+a)/2b}(Q_\infty)$ is a special case of the space $C_{l,t,g}^{l+a,(l+a)/2b}(Q_\infty)$ just defined for $l = 2b-1$ and $g_{1,k_0,k}(t) = f_{B,\varkappa}(t)$, $g_{2,k_0,k}(t) = f_{B,\mu}(t)$ and $g_{3,k_0,k}(t,t') = f_{B,\nu}(|t-t'|) f_{B,\nu}(t^*)$ ($t^* = \max(t,t')$).

Remark 1. If $0 < \varrho < a < 1$, then there is $\delta \in \langle 1-(a-\varrho)/2b, 1)$ such that $C_{x,t,g}^{l+a,(l+a)/2b}(Q_\infty) \subset C_{x,t,\tilde{g}}^{l+\varrho,(l+\varrho)/2b}(Q_\infty)$, where $\tilde{g}_{1,k_0,k}(t) = g_{1,k_0,k}(t)$, $\tilde{g}_{2,k_0,k}(t) = g_{2,k_0,k}(t)$ and $\tilde{g}_{3,k_0,k}(t,t') = g_{3,k_0,k}(t,t') f_{B,\delta}(|t-t'|)$.

In addition to the estimations of Green function (6), (7), (8) and (6′) established in Theorem 1 and Remark 3 of [2] we shall use the following modification of the quoted estimation (7):

$$(5)\qquad |D_t^{k_0} D_x^k G(x,t;\xi,\tau) - D_t^{k_0} D_x^k G(y,t;\xi,\tau)| \leq K|x-y|^a (t-\tau)^{-\mu} |x^* - \xi|^{2b\mu-(m+2bk_0+|k|+a)} e^{A(t-\tau)} E_1$$

for $\mu \leq (m+2bk_0+|k|+a)/2b$, $A > 0$, $K > 0$ and $0 \leq \tau < t < \infty$, $\xi \neq x$ and $|x^*-\xi| = \min(|x-\xi|, |y-\xi|)$.

In locally convex topological spaces the following fixed point theorem is true

THEOREM 1 (Deleanu and Marinescu [1]). *Let (P,τ) be a locally convex sequentially complete space, let $\mathscr{F}$ be a saturated family of seminorms defining the topology τ, and let φ: $\mathscr{F} \to \mathscr{F}$ be such that $\varphi[\varphi(\sigma)] = \varphi(\sigma)$ for any $\sigma \in \mathscr{F}$. Suppose that S is a closed subset of P and that the operator $\mathfrak{A}$: $S \to S$ satisfies the conditions:*

(a) *For any $\sigma \in \mathscr{F}$ there is $q_\sigma > 0$ such that $\|\mathfrak{A}u - \mathfrak{A}v\|_\sigma \leq q_\sigma \|u-v\|_{\varphi(\sigma)}$ for any $u, v \in S$,*

(b) *$q_{\varphi(\sigma)} < 1$ for every $\sigma \in \mathscr{F}$.*

Then the operator $\mathfrak{A}$ has a unique fixed point in S.

This theorem is a generalization of Banach principle of contractive mappings to the case of locally convex spaces. In the following text the symbol L always denotes positive constants.

2. Smoothness of the solution.

THEOREM 2. *Let assumptions* (A), (B), (D_{2b+a}) *be satisfied. Let F be a continuous vector function bounded in the norm*

$$\|\cdot\|_{0\tilde{H}_\infty}\Big(\tilde{H}_\infty = Q_\infty \times \prod_{i=1}^{s}\prod_{j=1}^{r}\{-Rf_{A,\varkappa}(t) \leqslant u_j^i \leqslant Rf_{A,\varkappa}(t)\} \subset H_\infty\Big)$$

and let the Hölder condition

$$(6)\qquad |F(x,t,\ldots,u^\gamma,\ldots)-F(y,t',\ldots,v^\gamma,\ldots)| \leqslant \Big\{q(t)|x-y|^\beta + p(t)|t-t'|^{\beta'/2b} + \sum_{i=0}^{2b-1}\sum_{|\gamma|=i}(q^\gamma, |u^\gamma - v^\gamma|^{\beta_\gamma})\Big\}J$$

be satisfied on $\tilde{H}_\infty$, where $q^\gamma(t) = (q_1^\gamma(t), \ldots, q_p^\gamma(t))$ and $q_j^\gamma(t) \geqslant 0$ for $j = 1, \ldots, p$ and $q(t) > 0$, $p(t) > 0$ are bounded and integrable real functions on $\langle 0, T\rangle$ for every real $T > 0$. Then to each β, β' and β_γ belonging to the interval $(0,1\rangle$ and for $0 < \varrho < \min(\beta, \beta', a\beta_\gamma)$ (< 1) there are functions $g_{i,k_0,k}$, $i = 1,2,3$, with property (j) *such that the solution of* (1), (2), (3) *belongs to $C_{x,t,g}^{2b+\varrho,(2b+\varrho)/2b}(Q_\infty^*)$, where $Q_\infty^* = \Omega^* \times \langle 0, \infty)$ and Ω^* is an arbitrary subdomain of Ω with closure contained in Ω, i.e., $\overline{\Omega^*} \subset \Omega$.*

Proof. According to Theorem 4 of [2], the solution u of (1), (2), (3) belongs to the space $C_{x,t,f(A,\varkappa,\mu,\nu)}^{2b-1+a,(2b-1+a)/2b}(Q_\infty)$ and in view of Remark 1 it appertains to $C_{x,t,\tilde{g}}^{2b-1+\varrho,(2b-1+\varrho)/2b}(Q_\infty)$ and satisfies the operator equation $u = \mathfrak{A}(x,t)u$ on Q_∞. (The operator $\mathfrak{A}(x,t)$ is defined in formula (9) of [2].) Now, we have to show that the expression $\|u\|_{2b+\varrho,Q_\infty^*}^g$ given by (see (4))

$$(7)\qquad \max_{j=1,\ldots,p}\Big\{\sum_{i=0}^{2b}\sum_{|k|=i}\sup_{Q_\infty^*}[|D_x^k u_j(x,t)|g_{1,0,k}^{-1}(t)] + \sup_{Q_\infty^*}[|D_t u_j(x,t)|g_{1,1,0}^{-1}(t)] +$$
$$+\sum_{|k|=2b}\sup_{\substack{(x,t),(y,t)\in Q_\infty^*\\ x\neq y}}[\langle D_x^k u_j(x,t)\rangle_{\varrho,x}g_{2,0,k}^{-1}(t)] +$$
$$+\sup_{\substack{(x,t),(y,t)\in Q_\infty^*\\ x\neq y}}[\langle D_t u_j(x,t)\rangle_{\varrho,x}g_{2,1,0}^{-1}(t)] +$$
$$+\sum_{i=1}^{2b}\sum_{|k|=i}\sup_{\substack{(x,t),(x,t')\in Q_\infty^*\\ t\neq t'}}[\langle D_x^k u_j(x,t)\rangle_{(2b+\varrho-|k|)/2b,t}\,g_{3,0,k}^{-1}(t,t')] +$$
$$+\sup_{\substack{(x,t),(x,t')\in Q_\infty^*\\ t\neq t'}}[\langle D_t u_j(x,t)\rangle_{\varrho/2b,t}\,g_{3,1,0}^{-1}(t,t')]$$

defines a norm in $C_{x,t,g}^{2b+\varrho,(2b+\varrho)/2b}(Q_\infty^*)$, that is, (7) is finite.

Put $\inf_{\substack{\xi\in\partial\Omega\\ x\in\Omega^*}}|x-\xi| = w > 0$ and $|x^*-\xi| = \min(|x-\xi|, |y-\xi|)$ for $\xi \in \Omega$

and $(x, t), (y, t), (x, t') \in Q_\infty^*$, $t < t'$. Using the Green formula, estimation (6' of [2], inequality (5) (with the exponent $\alpha = \varrho$) and assumption (6), we obtain for $|k| = 2b$

$$|D_x^k u(x,t)| \leqslant L \int_0^t d\tau \left| \int_\Omega D_x^k G(x,t;\xi,\tau)\,d\xi \right| J + \int_0^t d\tau \int_\Omega |D_x^k G(x,t;\xi,\tau)|\,|F[\xi,\tau,\ldots,D_x^\gamma u(\xi,\tau),\ldots] - F[x,\tau,\ldots,D_x^\gamma u(x,\tau),\ldots]|\,d\xi$$

$$\leqslant L\Big\{\operatorname{mes}(\partial\Omega) w^{2b-\mu-m-2b+1} f_{A,\mu}(t) + \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} e^{A(t-\tau)} |x-\xi|^{2b\mu-(m+2b)} P_\mu(x,\xi,\tau)\,d\xi\Big\} J$$

and

$$|D_x^k u(x,t) - D_x^k u(y,t)| \leqslant L \int_0^t d\tau \left| \int_\Omega [D_x^k G(x,t;\xi,\tau) - D_x^k G(y,t;\xi,\tau)]\,d\xi \right| J +$$

$$+ \int_0^t d\tau \int_\Omega |D_x^k G(x,t;\xi,\tau) - D_x^k G(y,t;\xi,\tau)|\,|F[\xi,\tau,\ldots,D_x^\gamma u(\xi,\tau),\ldots] - F[x^*,\tau,\ldots,D_x^\gamma u(x^*,\tau),\ldots]|\,d\xi$$

$$\leqslant L|x-y|^\varrho \Big\{\operatorname{mes}(\partial\Omega) w^{2b\mu-(m+2b-1+\varrho)} f_{A,\mu}(t) + \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} e^{A(t-\tau)} |x^*-\xi|^{2b\mu-(m+2b+\varrho)} P_\mu(x,\xi,\tau)\,d\xi\Big\} J,$$

where

$$P_\mu(x,\xi,\tau) = q(\tau)|x-\xi|^\beta + \sum_{i=0}^{2b-2} \sum_{|\gamma|=i} \big(q^\gamma(\tau), f_{A,\mu}^{\beta_\gamma}(\tau)|x-\xi|^{\beta_\gamma} J\big) + \sum_{|\gamma|=2b-1} \big(q^\gamma(\tau), f_{A,\mu}^{\beta_\gamma}(\tau)|x-\xi|^{\alpha\beta_\gamma} J\big).$$

Hence, if we take μ such that $\max[(2b-\beta)/2b, (2b-\alpha\beta_\gamma)/2b] \leqslant \mu < 1$ and $\max\{[2b-(\beta-\varrho)]/2b, [2b-(\alpha\beta_\gamma-\varrho)]/2b\} < \mu < 1$ for $|\gamma| = 0, 1, \ldots, 2b-1$, then

$$|D_x^k u(x,t)| \leqslant L g_{1,0,k}(t) J,$$

$$|D_x^k u(x,t) - D_x^k u(y,t)| \leqslant L|x-y|^\varrho g_{2,0,k}(t) J,$$

where the functions $g_{1,0,k}$ and $g_{2,0,k}$ are uniquely determined by the preceding estimations.

Equation (1) and assumption (D_ϱ) together with the last estimations yield

$$|D_t u(x,t)| \leqslant |F[x,t,\ldots,D_x^\gamma u(x,t),\ldots]| + \sum_{|k|=2b} |A_k(x)||D_x^k u(x,t)| \leqslant L g_{1,1,0}(t)$$

and

$$\begin{aligned} &|D_t u(x,t) - D_t u(y,t)| \\ &\quad \leqslant |F[x,t,\ldots,D_x^\gamma u(x,t),\ldots] - F[y,t,\ldots,D_x^\gamma u(y,t),\ldots]| + \\ &\quad + \sum_{|k|=2b} |A_k(x)|\,|D_x^k u(x,t) - D_x^k u(y,t)| + \sum_{|k|=2b} |A_k(x) - A_k(y)|\,|D_x^k u(x,t)| \\ &\quad \leqslant L|x-y|^\varrho g_{2,1,0}(t) J. \end{aligned}$$

For $|k| = 1, \ldots, 2b$ we have

$$\begin{aligned} &|D_x^k u(x,t) - D_x^k u(x,t')| \\ &\quad \leqslant \Big\{L \int_0^t d\tau \Big| \int_\Omega [D_x^k G(x,t;\xi,\tau) - D_x^k G(x,t';\xi,\tau)]d\xi\Big| + \\ &\qquad + \int_t^{t'} d\tau \Big| \int_\Omega D_x^k G(x,t';\xi,\tau)d\xi\Big|\Big\} J + \\ &\qquad + \int_0^t d\tau \int_\Omega |D_x^k G(x,t;\xi,\tau) - D_x^k G(x,t';\xi,\tau)| \times \\ &\quad \times |F[\xi,\tau,\ldots,D_x^\gamma u(\xi,\tau),\ldots] - F[x,\tau,\ldots,D_x^\gamma u(x,\tau),\ldots]|d\xi + \\ &\qquad + \int_t^{t'} d\tau \int_\Omega |D_x^k G(x,t';\xi,\tau)| \times \\ &\quad \times |F[\xi,\tau,\ldots,D_x^\gamma u(\xi,\tau),\ldots] - F[x,\tau,\ldots,D_x^\gamma u(x,\tau),\ldots]|d\xi. \end{aligned}$$

Denote the integrals in this inequality by I_1, I_2, I_3 and I_4, as they appear. By the Green formula and the mean value theorem and assumption (6) one obtains

$$I_1 \leqslant L(t'-t) f_{A,\mu}(t') E_1,$$
$$I_2 \leqslant L f_{A,\lambda}(t'-t) E_1.$$

To estimate I_3, we put $S_3 = \{\xi \in \Omega\colon\ |\xi - x| > (t'-t)^{1/2b}\}$, $S_4 = \Omega - S_3$. Then there is $\tilde{t} \in (t,t')$ for which

$$\begin{aligned} I_3 \leqslant L\Big\{&(t'-t)\int_0^t d\tau \int_{S_3} e^{A(\tilde{t}-\tau)}(\tilde{t}-\tau)^{-\mu}|x-\xi|^{2b\mu-(m+2b+|k|)}P_\mu(x,\xi,\tau)d\xi + \\ &+ \int_0^t d\tau \int_{S_4} (t-\tau)^{-\mu} e^{A(t-\tau)}|x-\xi|^{2b\mu-(m+|k|)}P_\mu(x,\xi,\tau)d\xi + \\ &+ \int_0^t d\tau \int_{S_4} (t'-\tau)e^{A(t'-\tau)}|x-\xi|^{2b\mu-(m+|k|)}P_\mu(x,\xi,\tau)d\xi\Big\} J. \end{aligned}$$

Further,

$$I_4 \leqslant L\Big\{\int_t^{t'} d\tau \int_\Omega (t'-\tau)^{-\lambda} e^{A(t'-\tau)} |x-\xi|^{2b\lambda-(m+|k|)} P_\mu(x, \xi, \tau)\, d\xi\Big\} J.$$

If $\max[(2b+\varrho-|k|)/2b,\ (4b+\varrho-\beta)/4b,\ (|k|-\beta)/2b,\ (|k|-a\beta_\gamma)/2b] \leqslant \mu < 1$ and $\max[(|k|-\beta)/2b,\ (|k|-a\beta_\gamma)/2b] < \lambda < (|k|-\varrho)/2b$ for $|\gamma| = 0, 1, \ldots, 2b-1$ and $|k| = 1, \ldots, 2b$, then the expressions

$$I_1^*(t'-t) = (t'-t)^{(|k|-\varrho)/2b} \Big(\int_0^{t'-t} z^{-\mu} e^{Az} dz\Big)^{-1},$$

$$I_3^*(t'-t) = (t'-t)^{(2b\mu-2b-\varrho+\beta)/2b} \Big(\int_0^{t'-t} z^{-\mu} e^{Az} dz\Big)^{-1}$$

are bounded and

$$I_2^*(t'-t) = I_4^*(t'-t) = (t'-t)^{-(2b+\varrho-|k|)/2b} \int_0^{t'-t} z^{-\lambda} e^{Az} dz \leqslant \int_0^{t'-t} z^{-\varkappa} e^{Az} dz$$

for $\varkappa = 1-(|k|-\varrho)/2b+\lambda$ and $t, t' \in (0, \infty), t < t'$.

In virtue of these assertions and of the previous estimations for I_1, I_2, I_3 and I_4 we have

$$\begin{aligned}
&I_1 \leqslant L(t'-t)^{(2b+\varrho-|k|)/2b} f_{A,\mu}(t') f_{A,\mu}(t'-t) E_1,\\
&I_2 \leqslant L(t'-t)^{(2b+\varrho-|k|)/2b} f_{A,\varkappa}(t'-t) E_1,\\
(8)\quad &I_3 \leqslant L(t'-t)^{(2b+\varrho-|k|)/2b} W(t, t') J,\\
&I_4 < \tilde{L}(t')(t'-t)^{(2b+\varrho-|k|)/2b} [1 + f_{A',\gamma}^{\beta_\gamma}(t') + f_{A',\lambda}^{\beta_\gamma}(t')] f_{A,\varkappa}(t'-t) J,
\end{aligned}$$

where W is a positive function with property (j) and $\tilde{L}$ is a positive non-decreasing function on $\langle 0, \infty)$. Then

$$|D_x^k u(x, t) - D_x^k u(x, t')| \leqslant L(t'-t)^{(2b+\varrho-|k|)/2b} g_{3,0,k}(t, t') J,$$

which gives the required inequality. The function $g_{3,0,k}$ is determined by (8).

In conclusion of this proof we establish an estimation for the difference

$$\begin{aligned}
|D_t u(x, t) - D_t u(x, t')| &\\
\leqslant |F[x, t, \ldots, D_x^\gamma u(x, t), \ldots] &- F[x, t', \ldots, D_x^\gamma u(x, t'), \ldots]| + \\
&+ \sum_{|k|=2b} |A_k(x)|\, |D_x^k u(x, t) - D_x^k u(x, t')|.
\end{aligned}$$

Since $u \in C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$, by (6) we get a majorant of the first member of the previous inequality in the form

$$\Big\{p(t)(t'-t)^{\beta'/2b} + \\ + L \sum_{i=0}^{2b-1} \sum_{|\gamma|=2b-1} (t'-t)^{\beta_\gamma(2b-1+a-|\gamma|)2b} \big(q^\gamma(t), f_{A,\nu}(t'-t) f_{A,\nu}(t') J\big)\Big\} J.$$

We easily verify that for $\max[1-(\beta-\varrho)(2b)^{-1}, 1-(2b-1-|\gamma|)\beta_\gamma(2b)^{-1}, 1-(a\beta_\gamma-\varrho)(2b)^{-1}] \leqslant \mu < 1$ and for $t, t' \in (0, \infty)$, $t < t'$, the functions $(t'-t)^{(\beta'-\varrho)/2b}\Big(\int\limits_0^{t'-t} z^{-\mu}e^{Az}dz\Big)^{-1}$ and $(t'-t)^{\beta_\gamma(2b-1-|\gamma|)/2b}\Big(\int\limits_0^{t'-t} z^{-\mu}e^{Az}dz\Big)^{-1}$ for $|\gamma| = 0, 1, \ldots, 2b-2$ and $(t'-t)^{(a\beta_\gamma-\varrho)/2b}\Big(\int\limits_0^{t'-t} z^{-\mu}e^{Az}dz\Big)^{-1}$ for $|\gamma| = 2b-1$ are bounded in t, t'.

Consequently

$$|D_t u(x, t) - D_t u(x, t')| \leqslant L(t'-t)^{\varrho/2b} g_{3,1,0}(t, t')J,$$

where

$$\begin{aligned} g_{3,1,0}(t, t') &= p(t)f_{A,\mu}(t'-t) + \\ &+ f_{A,\mu}^2(t'-t)\sum_{i=1}^{2b-2}\sum_{|\gamma|=i}\big(q^\gamma(t), f_{A,\nu}^{\beta_\gamma}(t'-t)f_{A,\nu}^{\beta_\gamma}(t')J\big) + \\ &+ f_{A,\mu}(t'-t)\sum_{|\gamma|=2b-1}\big(q^\gamma(t), f_{A,\nu}^{\beta_\gamma}(t'-t)f_{A,\nu}^{\beta_\gamma}(t')J\big) + g_{3,0,k}(t, t'). \end{aligned}$$

This finishes the proof.

Remark. If there exists a solution $u \in C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$ of (1), (2), (3), then for the smoothness of this solution it is sufficient to assume $|F| \leqslant g(t)J$ on $\tilde{H}_\infty$, where g is a bounded and integrable real function on every $\langle 0, T\rangle$ for any real number $T > 0$, instead of the assumption $\|F\|_{0,\tilde{H}_\infty} \leqslant L$ in Theorem 2.

3. Uniqueness. Using Theorem 1 we get the following uniqueness theorem

THEOREM 3. *Let assumptions* (A), (B), (D_{2b-1+a}) *be fulfilled and let F be a continuous vector function bounded in the norm $\|\cdot\|_{0,\tilde{H}_\infty}$. Let the Lipschitz condition*

$$(9)\quad |F(x, t, \ldots, u^\gamma, \ldots) - F(x, t, \ldots, v^\gamma, \ldots)| \leqslant \sum_{i=0}^{2b-1}\sum_{|\gamma|=i}(|r^\gamma|, |u^\gamma - v^\gamma|)J$$

be satisfied on $\tilde{H}_\infty$. The vector function $r^\gamma(t) = \big(r_1^\gamma(t), \ldots, r_p^\gamma(t)\big)$ for $|\gamma| = 0, 1, \ldots, 2b-1$ is integrable on Q_n and $\big(|r^\gamma(t)|, f_{A,\varkappa}(t)J\big) \leqslant L_n$ for $t \in \langle 0, n\rangle$, where $L_n > 0$ and $n = 1, 2, \ldots$ Further let $B_n = LL_np^2[2s + t(2b-1)] < 1$ for $n = 1, 2, \ldots$ (L is the constant from the estimations in Lemma 2 of [2]; *s and $t(r)$ are defined in the second part of* [2]). *Then the integro-differential equation $u = \mathfrak{A}(x, t)u$ has one and only one solution belonging to $C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$, and furthermore, the successive approximations $u_l = \mathfrak{A}(x, t)u_{l-1}$ tend to this unique solution in the norm $\|\cdot\|^{f(A,\varkappa,\mu,\nu)}_{2b-1+a,Q_\infty}$ for any $u_0 \in C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$.*

Proof. Consider the locally convex space $\big(P(Q_\infty),\ \tau\big) = \big(C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty);\ \tau\big)$ topologized by the family of seminorms $\sigma_n(u) = \|u\|^{f(A,\varkappa,\mu,\nu)}_{2b-1+a,Q_n}$ for $n = 1, 2, \ldots$ According to hypothesis (9) and Lemma 2 of [2], for any u, v from this space and for $(x, t) \in Q_n$ we have

$$|D^k_x\mathfrak{A}(x,t)u - D^k_x\mathfrak{A}(x,t)v| \leqslant \Big\{\int_0^t d\tau \int_\Omega |G(x,t;\xi,\tau)| \times$$
$$\times \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} \big(|r^\gamma(\tau)| f_{A,\varkappa}(\tau),\ |D^\gamma_x u(\xi,\tau) - D^\gamma_x v(\xi,\tau)| f^{-1}_{A,\varkappa}(\tau)\, d\xi\big)\Big\} J$$
$$\leqslant L_n p I_{1,k}(x,t)\sigma_n(u-v) J$$
$$\leqslant L L_n p^2 f_{A,\varkappa}(t)\sigma_n(u-v) J, \qquad |k| = 0, 1, \ldots, 2b-1,$$

and

$$|D^k_x\mathfrak{A}(x,t)u - D^k_x\mathfrak{A}(x,t)v - D^k_x\mathfrak{A}(y,t)u + D^k_x\mathfrak{A}(y,t)v|$$
$$\leqslant L_n p I_{2,k}(x,y,t)\sigma_n(u-v) J$$
$$\leqslant L L_n p^2 f_{A,\mu}(t)\sigma_n(u-v)|x-y|^a J, \qquad |k| = 2b-1$$

and

$$|D^k_x\mathfrak{A}(x,t)u - D^k_x\mathfrak{A}(x,t)v - D^k_x\mathfrak{A}(x,t')u + D^k_x\mathfrak{A}(x,t')v|$$
$$\leqslant L_n p\, I_{3,k}(x,t,t')\sigma_n(u-v) J$$
$$\leqslant L L_n p^2 f_{A,\nu}(t'-t) f_{A,\nu}(t')\sigma_n(u-v)(t'-t)^{(2b-1+a-|k|)/2b}$$

for $|k| = 0, 1, \ldots, 2b-1$. Hence we obtain

$$\sigma_n[\mathfrak{A}(x,t)u - \mathfrak{A}(x,t)v] \leqslant L L_n p^2 [2s + t(2b-1)]\sigma_n(u-v),$$

which gives conditions (a) and (b) of Theorem 1 if φ is the identity mapping.

The remining claims in Theorem 1 are evidently satisfied and so the uniqueness of solution of $u = \mathfrak{A}(x,t)u$ is proved.

Since the space $\big(P(Q_\infty), \tau\big)$ is complete and the sequence of the successive approximations $\{u_l\}_{l=1}^\infty$ is fundamental $(\sigma_n(u_l - u_{l+m}) \leqslant B^l_n(1 - B_n)^{-1}\sigma_n\big(u_0 - \mathfrak{A}(x,t)u_0\big)$ for $n = 1, 2, \ldots$ and $u_0 \in P(Q_\infty))$, there exists a vector function $u^* \in P(Q_\infty)$ such that $\lim_{l\to\infty} u_l = u^*$ in $\big(P(Q_\infty), \tau\big)$. For any $\varepsilon > 0$ and for a sufficiently large index $l > 0$ we have

$$\sigma_n\big(u^* - \mathfrak{A}(x,t)u^*\big) \leqslant \sigma_n(u^* - u_l) + B_n\sigma_n(u_{l-1} - u^*) < \varepsilon$$

whence $u^* = \mathfrak{A}(x,t)u^*$ on every Q_n, $n = 1, 2, \ldots$ Consequently u^* is the unique solution of $u = \mathfrak{A}(x,t)u$ on Q_∞ and the proof is complete.

Remark 3. If, in addition to the assumptions of Theorem 3, condition (6) holds, then the sequence $\{\mathfrak{A}(x,t)u_{n-1}\}_{n=1}^\infty$ converges to the solution of problem (1), (2), (3) in the norm $\|\cdot\|^{f(A,\varkappa,\mu,\nu)}_{2b-1+a,Q_\infty}$.

4. Asymptotic behaviour. According to the previous result problem (1), (2), (3) has a unique regular solution in the class of vector functions $C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$. However, in general, the boundedness of this solution is not ensured. The behaviour of the solution as $t\to\infty$ is described by

THEOREM 4. *Let the condition*

$$(10)\qquad |F(x,t,\ldots,u^\gamma,\ldots)-F(x,t',\ldots,v^\gamma,\ldots)|$$
$$\leqslant Le^{-At^*}\Big\{|t-t'|^{\beta'/2b}+\sum_{i=0}^{2b-1}\sum_{|\gamma|=i}e^{-A\beta_\gamma|t-t'|}e^{-A\beta_\gamma t^*}(J,|u^\gamma-v^\gamma|^{\beta_\gamma})\Big\}J$$

be satisfied on $\tilde{H}_\infty$ for $A, L>0$ and $\beta', \beta_\gamma\in(0,1\rangle$, and let $F(x,0,\ldots,0,\ldots)=0$ on Ω. Then for each solution $u\in C^{2b-1+a,(2b-1+a)/2b}_{x,t,f(A,\varkappa,\mu,\nu)}(Q_\infty)$ of problem (1), (2), (3) *with $2b-1+a-|\gamma|<(\mu|\beta_\gamma|)+\gamma$, where $|\gamma|=0,1,\ldots,2b-1$, there exists $t_0>0$ such that*

$$|D^k_x u(x,t)|=O(1/t^{a\beta_0})J,\qquad x\in\Omega,\ t>t_0$$

for $|k|=0,1,\ldots,2b-1$, where $0<a<\min(\mu-(\beta'/2b),\mu,\nu)$ and $\beta_0=\min\limits_{|\gamma|=0,1,\ldots,2b-1}\beta_\gamma$.

Proof. If $u\in C^{2b-1+a,(2b-1+a)/2b}_{x,t,f,(A,\varkappa,\mu,\nu)}(Q_\infty)$, then

$$|D^k_x u(x,t)|\leqslant\int_0^t d\tau\,|D^k_x G(x,t;\xi,\tau)|\,|F[\xi,t,\ldots,D^\gamma_x u(\xi,t),\ldots]|\,d\xi+$$
$$+\int_0^t d\tau\int_\Omega|D^k_x G(x,t;\xi,\tau)|\,|F[\xi,\tau,\ldots,D^\gamma_x u(\xi,\tau),\ldots]-$$
$$-F[\xi,t,\ldots,D^\gamma_x u(\xi,t),\ldots]|\,d\xi\overset{\mathrm{df}}{=}I_1+I_2$$

on Q_∞ for $|k|=0,1,\ldots,2b-1$. Using (10) and the estimations of the Green function one obtains

$$I_1\leqslant L\Big\{e^{-At}t^{\beta'/2b}+pe^{-At}\sum_{i=0}^{2b-1}\sum_{|\gamma|=i}e^{-2A\beta_\gamma t}\Big(\int_0^t e^{Az}t^{-\varkappa}dz\Big)^{\beta_\gamma}\Big\}\int_0^t e^{Az}z^{-\mu}dzJ$$

and

$$I_2\leqslant\Big\{Le^{-At}\int_0^t e^{Az}z^{-\mu}\Big[z^{\beta'/2b}+$$
$$+p\sum_{i=0}^{2b-1}\sum_{|\gamma|=i}e^{-A\beta_\gamma-A\beta_\gamma t}z^{\beta_\gamma(2b-1+a-|\gamma|)/2b}\Big(\int_0^t e^{Au}u^{-\nu}du\int_0^t e^{Av}v^{-\nu}dv\Big)^{\beta_\gamma}\Big]dz\Big\}J.$$

We easily obtain for sufficiently large $t_0>0$ and $t\geqslant t_0$

$$e^{-At}t^{\beta'/2b}\int_0^t e^{Az}z^{-\mu}dz\leqslant t^{-a};\qquad e^{-At}\int_0^t e^{Az}z^{-\mu}dz\leqslant t^{-a}$$

and

$$e^{-At}\int_0^t e^{Az}z^{-\varkappa}dz \leqslant 1.$$

Similarly

$$e^{-At}\int_0^t e^{Az}z^{\beta'/2b-\mu}dz \leqslant t^{-a}; \quad e^{-At}\int_0^t e^{Av}v^{-\nu}dv \leqslant t^{-a}$$

and

$$e^{-At}\int_0^t \Big[e^{Az-A\beta_\gamma z}z^{\beta_\gamma(2b-1+a-|\gamma|)/2b-\mu}\Big(\int_0^t e^{Au}u^{-\nu}du\Big)^{\beta_\gamma}\Big]dz \leqslant 1.$$

Hence

$$I_1 \leqslant Lt^{-a}J \quad \text{and} \quad I_2 \leqslant L\Big(t^{-a}+\sum_{i=0}^{2b-1}\sum_{|\gamma|=i}t^{-a\beta_\gamma}\Big)J$$

which proves our statement.

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