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The finiteness of compact varieties in C^n

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To the memory of Stefan Bergman

Abstract. Every analytic subvariety of C^n is a finite set. This well-known theorem is given a new, very simple, and completely elementary proof.

In this paper I present a totally elementary proof of the following

THEOREM. *Every compact analytic subvariety of C^n is a finite set of points.*

Although this must be quite old, I know of only two proofs in the literature, namely [1], p. 106, and [2], p. 55. In both of these, the theorem is obtained as a corollary of the maximum principle for holomorphic functions on analytic varieties, whose proof, in turn, depends on fairly deep results concerning the local structure of analytic varieties.

The proof given here uses nothing at all from the theory of several complex variables, beyond the necessary definitions, nor does it use any of the ring theory that is usually associated with this topic. The principal tool – and essentially the only one – that is used is the residue theorem for holomorphic functions of one variable.

DEFINITIONS. If Ω is an open subset of C^n (the space of n complex variables), the class of all complex holomorphic functions is denoted by $H(\Omega)$. For $f \in H(\Omega)$, $Z(f)$ denotes the set of all $w \in \Omega$ at which $f(w) = 0$. A set $V \subset \Omega$ is said to be an *analytic subvariety of* Ω if

(i) V is (relatively) closed in Ω, and

(ii) every point $p \in V$ has a neighbourhood $N(p) \subset \Omega$ such that

$$V \cap N(p) = Z(f_1) \cap \ldots \cap Z(f_r) \tag{1}$$

for some $f_1, f_2, \ldots, f_r \in H(N(p))$.

In this situation, we may say that "V is defined in $N(p)$ by $f_1, \ldots, f_r$".

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We shall occasionally drop the adjective "analytic" and will just talk about *subvarieties of* Ω.

Remarks. (i) Ω is a subvariety of Ω. (Take $r = 1$, $f_1 = 0$.)

(ii) If V_1 and V_2 are subvarieties of Ω, so are $V_1 \cap V_2$ and $V_1 \cup V_2$.

Proof. If $\{f_i\}$ defines V_1 and $\{g_k\}$ defines V_2 in $N(p)$, then $\{f_i\} \cup \{g_k\}$ defines $V_1 \cap V_2$ and $\{f_i g_k\}$ defines $V_1 \cup V_2$.

(iii) If a subvariety V of Ω is compact, then V is also a subvariety of C^n.

Notation. We shall work with a fixed coordinate system in C^n. Points $w = (w_1, \ldots, w_n) \in C^n$ will be written in the form

$$w = (w', w_n) \tag{2}$$

where $w' = (w_1, \ldots, w_{n-1}) \in C^{n-1}$, $w_n \in C$.

A *polydisc* in C^n is a cartesian product Δ of n open discs, one in each coordinate plane. We shall sometimes write

$$\Delta = \Delta' \times \Delta_n \tag{3}$$

where Δ' is a polydisc in C^{n-1} and Δ_n is a disc in the z_n-plane.

Finally, there is a projection π: $C^n \to C^{n-1}$ defined by

$$\pi(w', w_n) = w'. \tag{4}$$

In particular, $\pi(\Delta) = \Delta'$.

The following projection lemma will enable us to prove the theorem by induction on the dimension n.

PROJECTION LEMMA. *Let V be an analytic subvariety of a region $\Omega \subset C^n$, let $p = (p', p_n)$ be a point of V, and let*

$$L = \{(p', \lambda): \lambda \in C\}. \tag{5}$$

If p is an isolated point of $L \cap V$, then p is the center of a polydisc $\Delta \subset \Omega$ such that $\pi(V \cap \Delta)$ is an analytic subvariety of $\pi(\Delta)$.

Proof. Without loss of generality, assume that p is the origin of C^n and that Ω is a polydisc in which V is defined by holomorphic functions $f_1, \ldots, f_r$. We divide the proof into three steps.

Step I. Our assumption about $L \cap V$ shows that for (at least) one f_i, say for f_r, the origin in C is an isolated zero of $f_r(O', \cdot)$. To emphasize the special role played by this property of f_r, we write F in place of f_r.

There exists $\delta > 0$ such that $F(O', \lambda) \neq 0$ if $0 < |\lambda| \leqslant \delta$. Put $\Delta_n = \{\lambda: |\lambda| < \delta\}$, and let Δ' be a polydisc in C^{n-1}, centered at O', so small that

$$F(w', \xi) \neq 0 \quad \text{if } w' \in \Delta',\ |\xi| = \delta. \tag{6}$$

Let $\Delta = \Delta' \times \Delta_n$. By taking δ and Δ' small enough, we obtain $\bar{\Delta} \subset \Omega$. Let Ω_0 be any open set such that $\Delta \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega$.

By (6), we can associate to every $h \in H(\Omega_0)$ the integral

$$J_h(w') = \frac{1}{2\pi i} \int_{|\xi|=\delta} h(w', \xi) \left(\frac{D_n F}{F}\right)(w', \xi)\, d\xi \qquad (w' \in \Delta'), \tag{7}$$

where D_n denotes differentiation with respect to w_n.

Note that $J_h \in H(\Delta')$. (Easy proof: Morera's theorem, in any of the variables $w_1, \ldots, w_{n-1}$.)

When $h \equiv 1$, $J_h(w')$ is the number of zeros of $F(w', \cdot)$ in Δ_n. Thus J_h is an integer-valued function in Δ'. Being holomorphic, J_h is continuous, hence is a constant, say m. Since $F(0', 0) = 0$, $m > 0$. [We note, in passing, that this completes the proof of the lemma if $r = 1$, since then $\pi(\Delta \cap V) = \Delta'$. From now on, $r > 1$.] To every $w' \in \Delta'$ corresponds thus an (unordered) m-tuple of numbers $\alpha_k(w') \in \Delta_n$ $(1 \leqslant k \leqslant m)$. These are the zeros of $F(w', \cdot)$, counted with appropriate multiplicities.

If we now apply the residue theorem to the integral (7) we see that

$$J_h(w') = \sum_{k=1}^{m} h(w', \alpha_k(w')) \qquad (w' \in \Delta'). \tag{8}$$

We conclude that the sum (8) *is a holomorphic function in Δ' for every $h \in H(\Omega_0)$.*

Step II. Fix $h \in H(\Omega)$. Then h is bounded on Ω_0, hence there is an $\varepsilon > 0$ such that $\log(1-\lambda h) \in H(\Omega_0)$ for all $\lambda \in C$ with $|\lambda| < \varepsilon$. Apply the result of Step I to $\log(1-\lambda h)$ in place of h, to see that

$$\sum_{k=1}^{m} \log\left[1-\lambda h(w', \alpha_k(w'))\right] \tag{9}$$

is holomorphic in Δ'. Hence so is its exponential

$$\prod_{k=1}^{m} \left[1-\lambda h(w', \alpha_k(w'))\right], \tag{10}$$

for all λ with $|\lambda| < \varepsilon$. Consequently, the coefficient of every power λ^i of λ in (10) (after we multiply out) is in $H(\Delta')$. Taking $i = m$, we see that the *product*

$$P(w') = \prod_{k=1}^{m} h(w', \alpha_k(w')) \tag{11}$$

is holomorphic in Δ'.

Now fix some $w' \in \Delta'$. It is clear that $P(w') = 0$ if and only if some $\alpha_k(w')$ is also a zero of $h(w', \cdot)$, i.e., if and only if F and h have a common zero in Δ that lies "above" w'. Hence

$$\pi(\Delta \cap Z(F) \cap Z(h)) = Z(P). \tag{12}$$

Since P is holomorphic in Δ', we conclude: $\pi(\Delta \cap Z(F) \cap Z(h))$ *is an analytic subvariety of* $\Delta' = \pi(\Delta)$.

Step III. To complete the proof of the lemma, we return to the functions $f_1, \ldots, f_r$ (with $f_r = F$) that define V in Ω.

Let (c_{ij}) be a rectangular matrix, with $(r-1)m$ rows and $r-1$ columns, in which every square matrix of size $(r-1)\times(r-1)$ has determinant $\neq 0$. Define

$$h_i = \sum_{j=1}^{r-1} c_{ij} f_j \quad (1 \leqslant i \leqslant rm-m). \tag{13}$$

The result of Step II, applied to h_i in place of h, shows that each of the sets

$$E_i = \pi(\Delta \cap Z(F) \cap Z(h_i)) \tag{14}$$

is a subvariety of Δ'. We claim that

$$\pi(\Delta \cap V) = \bigcap_i E_i. \tag{15}$$

To prove one half of (15), let $w \in \Delta \cap V$. Then $w \in Z(h_i)$ for all i, and $w \in Z(F)$. Hence $\pi(w) \in E_i$ for all i. The left-hand side of (15) is thus a subset of the right.

For the opposite inclusion, take $w' \in \bigcap E_i$. To each of the $(r-1)m$ values of i corresponds then an $\alpha_k(w')$ such that

$$h_i(w', \alpha_k(w')) = 0. \tag{16}$$

This follows from (14). Since k runs over only m values, there is some k (fixed from now on) and some set I of $r-1$ distinct i's, for which (16) holds. The corresponding system of equations

$$\sum_{j=1}^{r-1} c_{ij} f_j(w', \alpha_k(w')) = h_i(w', \alpha_k(w')) = 0 \quad (i \in I) \tag{17}$$

has a unique solution, by our choice of (c_{ij}). Thus $f_j(w', \alpha_k(w')) = 0$ for all j, and therefore $w' \in \pi(\Delta \cap V)$.

This proves (15). Since each E_i is a subvariety of Δ', the same is true of their intersection. This completes the proof of the lemma.

Proof of the theorem. When $n = 1$, the theorem is true because zero-sets of non-constant holomorphic functions of one variable are discrete. Assume that $n \geqslant 2$ and that the theorem is true in $\boldsymbol{C}^{n-1}$. Let V be a compact subvariety of $\boldsymbol{C}^n$.

Pick $w' \in \pi(V)$ and define

$$L = \{(w', \lambda) \colon \lambda \in \boldsymbol{C}\}. \tag{18}$$

After an obvious identification of L with $\boldsymbol{C}$, we see that $L \cap V$ is a compact subvariety of $\boldsymbol{C}$, hence is finite. Let $p^{(i)}$ $(1 \leqslant i \leqslant q)$ be the points of $L \cap V$.

By the projection lemma, each $p^{(i)}$ is the center of a polydisc Δ_i in C^n such that $\pi(V \cap \Delta_i)$ is a subvariety of $\pi(\Delta_i)$. The part of V that is not covered by $\Delta_1 \cup \ldots \cup \Delta_q$ is compact and hence has positive distance from L. Hence w' is center of a polydisc $\Delta' \subset \bigcap_i \pi(\Delta_i)$, so small that all points of V that project into Δ' lie in $\bigcup \Delta_i$. In other words,

$$\Delta' \cap \pi(V) = \Delta' \cap \bigcup_{i=1}^{q} \pi(V \cap \Delta_i). \tag{19}$$

Thus $\Delta' \cap \pi(V)$ is a subvariety of Δ'. Since Δ' is a neighbourhood of the arbitrarily chosen point $w' \in \pi(V)$, and since $\pi(V)$ is compact (hence closed), it follows that $\pi(V)$ is a subvariety of C^{n-1}.

Hence $\pi(V)$ is a finite set, by our induction hypothesis. Since each point of $\pi(V)$ is the π-image of only finitely many points of V, we conclude that V is the union of finitely many finite sets.

Remarks. (i) The variety $V = \{w_1 - w_2 w_3 = 0\}$ $(n = 3,\ p = (0, 0, 0))$ shows that the hypothesis on $L \cap V$ cannot be removed from the projection lemma.

(ii) The variety $V = \{w_1 w_2 - 1 = 0\} \subset C^2$ satisfies the hypotheses of the projection lemma at every point, but nevertheless the conclusion holds only locally, not globally: $\pi(V) = C \setminus \{0\}$ is not a subvariety of C.

(iii) Take $h(w', w_n) = w_n$ in Step II. The remark that follows (10) shows then that

$$G(w', w_n) = \prod_{k=1}^{m} \big(w_n - \alpha_k(w')\big) \tag{20}$$

is holomorphic in Δ. This establishes the essential part of the Weierstrass preparation theorem since it is easy to see that F/G has removable singularities in Δ.

References

[1] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall 1965.

[2] R. Narasimhan, *Introduction to the theory of analytic spaces*, Springer Lecture Notes No. 25, 1966.

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