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Boundary behaviour of quasiconformal mappings in normed spaces

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Abstract. In this paper we introduce the concept of boundary elements in a normed space, representing a generalization of Carathéodory's prime ends and which correspond to n-dimensional boundary elements of V. A. Zorič. The quasiconformal mappings are defined by means of a new concept of the module used instead of the module (meaningless in a normed space) involved in one of Väisälä's definitions for quasiconformal mappings. Our module corresponds to the ∞-module (i.e., the p-module with $p = \infty$) *of* $\boldsymbol{R}^n$. We show that this definition of the quasiconformal mappings f is equivalent to that obtained by means of the relative distance of 2 sets contained in the domain of definition of f. If f is defined on the unit ball B of a normed space, then we show that f induces a correspondence between the points of the unit sphere and the boundary elements of $f(B)$. Also other boundary properties of f are obtained.

As is well known, a finite-dimensional homeomorphism between two Jordan domains induces a homeomorphism between their boundaries and this is no longer true if at least one of the domains is not a Jordan one. Thus, for instance, C. Carathéodory [10] observed that, if f is a conformal mapping of a disk onto a simply connected domain D of the complex plane, then to each point of the circumference there corresponds not a point of the boundary ∂D of D but a whole set, called by him "Primende". He established that the correspondence between the points of the circumference and the prime ends of D is a bijection. V. A. Zorič [13], [14] generalized the prime ends in the case of qc (quasiconformal mappings) in a Euclidean n-space $\boldsymbol{R}^n$. He called the corresponding concept a boundary element (a term used by P. Koebe [12] in the case of the analytic functions of a complex variable). We considered in [2]–[7] the problem of the estimate of some exceptional sets of the unit sphere S relatively to a qc of the unit ball B onto a domain D^* ($B, D^* \subset \boldsymbol{R}^n$) such as for instance the points of S corresponding to the boundary elements of D^* inaccessible by rectifiable arcs.

In the present paper, we define a class of qc f: $D \rightleftarrows D^*$ (D, D^* domains in a normed space X) by means of the module of an arc family (introduced by us in [8]). Next we show that if $E_0, E_1 \subset D$ and the relative distance $d_D(E_0, E_1)$ is equal to 0, then also $d_{D^*}(E_0^*, E_1^*) = 0$, where $E_k^* = f(E_k)$

($k = 0, 1$), and conversely $d_{D^*}(E_0^*, E_1^*) = 0 \Rightarrow$ ("implies") $d_D(E_0, E_1) = 0$. Next, we propose a concept of boundary elements (corresponding to Carathéodory's prime ends) and show that if $f\colon B \rightleftarrows D^*$ is *qc* then there is a bijection between the boundary elements of B and D^*. Among the different results relatively to the boundary behaviour of a *qc* in a normed space, we establish that the exceptional set of the boundary elements inaccessible by rectifiable arcs is empty. We point out that this last result is not a direct consequence of the fact that the corresponding space X is infinite-dimensional but follows from the concept of the arc family module, which we had to use in the case of such a space. Thus, for instance, the same property holds also in $\boldsymbol{R}^n$ if we define the *qc* in it by means of the corresponding module (which we call "∞-module" [9]). Finally, we observe that in spite of terminological similarity, in the infinite-dimensional case there are some additional specific difficulties. Thus, for example, since such a space is not locally compact, we have to consider a new category of boundary elements, which do not exist in $\boldsymbol{R}^n$, i.e., degenerated ones.

And now let us recall the difinition of the module of an arc family in a normed space X.

Let Γ be a family of arcs γ contained in an open set $D \subset X$ and let $F^D(\Gamma)$ be the corresponding class of admissible functions ϱ, i.e., such that $\varrho \geqslant 0$ are continuous and bounded in D, 0 in the complement CD of D and $\int_\gamma \varrho dH^1 \geqslant 1\ \forall \gamma \in \Gamma$ ("for every γ belonging to Γ"), where H^1 is a Hausdorff linear measure defined by means of the diameter $d(E) = \sup_{x,y\in E} \|x-y\|$, $\|\cdot\|$ being the norm of X. Then the *module* M^D of Γ is

$$M^D\Gamma = \inf_{\varrho\in F^D(\Gamma)} \sup_D \varrho(x);$$

$\Gamma = \emptyset \Rightarrow M^D\Gamma = 0$ and $F^D(\Gamma) = \emptyset \Rightarrow M^D\Gamma = \infty$.

Let us denote by $\Gamma(E_0, E_1, E)$ the family of open arcs $\gamma \subset E$ — an open arc being a homeomorphic image of a open linear interval $(0, 1)$ — such that the closure $\bar{\gamma}$ of γ is a homeomorphic image of a closed linear interval $[0, 1]$, the endpoints of $\bar{\gamma}$ belonging to E_0 and E_1, respectively. We recall that the *relative distance* (with respect to a set E) *between two points* x, y is given by $d_E(x, y) = \inf_{\gamma\in\Gamma(x,y,E)} H^1(\gamma)$ and the *relative distance* (with respect to E) *between two sets* E_0, E_1 is given by $d_E(E_0, E_1) = \inf_{\gamma\in\Gamma(E_0,E_1,E)} H^1(\gamma)$. If $\Gamma(E_0, E_1, E) = \emptyset$, then $d_E(E_0, E_1) = \infty$.

As a direct consequence of the definition of the module we have

LEMMA 1. $d_D(E_0, E_1) = 0 \Rightarrow M^D\Gamma(E_0, E_1, D) = \infty$.

Indeed, there is no bounded ϱ verifying the condition $\int_\gamma \varrho dH^1 \geqslant 1$ for every $\gamma \in \Gamma(E_0, E_1, D)$, since this family contains arcs of length (Hausdorff linear measure) as small as one wishes.

COROLLARY. $d(E_0, E_1) = 0 \Rightarrow M^B\Gamma(E_0, E_1, B) = \infty$.

This is a direct consequence of the preceding lemma and of the fact that $d_B(E_0, E_1) = d(E_0, E_1)$.

LEMMA 2. $\Gamma \subset \bigcup_{k=1}^{n} \Gamma_k \subset D \Rightarrow M^D \Gamma \leqslant \sum_{k=1}^{n} M^D \Gamma_k$.

Indeed, let $\varrho_k \in F^D(\Gamma_k)$ $(k = 1, .., n)$ and $\varrho_0(x) = \sum_{k=1}^{n} \varrho_k(x)$. If $\gamma \in \Gamma$, then there exists a k $(1 \leqslant k \leqslant n)$ such that $\gamma \in \Gamma_k$, so that $\int_\gamma \varrho_0 \, dH^1 \geqslant \int_\gamma \varrho_k \, dH^1 \geqslant 1$; hence $\varrho_0 \in F^D(\Gamma)$ and then

$$M^D \Gamma = \inf_{\varrho \in F^D(\Gamma)} \sup_D \varrho(x) \leqslant \sup_D \varrho_0(x) = \sup_D \sum_{k=1}^{n} \varrho_k(x) \leqslant \sum_{k=1}^{n} \sup_D \varrho_k(x);$$

taking the infimum over each $\varrho_k \in F^D(\Gamma_k)$ $(k = 1, \ldots, n)$, we obtain the desired inequality.

Let us recall that an arc family Γ is said to be *minorized by an arc family* Γ' if: $\forall \gamma \in \Gamma$ there exists an arc $\gamma' \in \Gamma'$ such that $\gamma' \subset \gamma$. Then we denote $\Gamma' < \Gamma$.

LEMMA 3. $\Gamma' < \Gamma \Rightarrow M^D \Gamma \leqslant M^D \Gamma'$.

$\varrho' \in F^D(\Gamma') \Rightarrow \varrho' \in F^D(\Gamma)$ since: $\forall \gamma \in \Gamma$ there exists a $\gamma' \in \Gamma'$ such that $\gamma' \subset \gamma$; hence $\int_\gamma \varrho' \, dH^1 \geqslant \int_{\gamma'} \varrho' \, dH^1 \geqslant 1$. Then, however,

$$M^D \Gamma = \inf_{\varrho \in F^D(\Gamma)} \sup_D \varrho(x) \leqslant \sup_D \varrho'(x) \Rightarrow M^D \Gamma \leqslant M^D \Gamma'.$$

LEMMA 4. *If the sets* $E_0, E_1 \subset \bar{D}$, $E_k \cap D$ $(k = 0, 1)$ *are closed relatively to* D *and such that* $E_0 \cap E_1 \cap D = \emptyset$, *then*

$$\begin{aligned} M^D \Gamma(E_0, E_1, D) &= M^\Delta \Gamma(E_0, E_1, D) \\ &= M^\Delta \Gamma(E_0, E_1, \Delta) = M^D \Gamma(E_0, E_1, \Delta), \end{aligned} \tag{1}$$

where $\Delta = D - (E_0 \cup E_1)$.

Clearly, $\Gamma(E_0, E_1, \Delta) \subset \Gamma(E_0, E_1, D)$; hence, on account of Lemma 2,

$$\begin{aligned} M^D \Gamma(E_0, E_1, \Delta) &\leqslant M^D \Gamma(E_0, E_1, D), \\ M^\Delta \Gamma(E_0, E_1, \Delta) &\leqslant M^\Delta \Gamma(E_0, E_1, D). \end{aligned} \tag{2}$$

And now let us show that $\Gamma(E_0, E_1, \Delta) < \Gamma(E_0, E_1, D)$. Indeed, let $\gamma_0 \in \Gamma(E_0, E_1, D)$. By definition, $\bar{\gamma}_0 = \varphi([0, 1])$, where φ is a homeomorphism, $\varphi(0) \in E_0$ and $\varphi(1) \in E_1$. The sets $\bar{\gamma}_0 \cap E_k$ $(k = 0, 1)$ are closed so that the sets $\varphi^{-1}(\bar{\gamma}_0 \cap E_k) = \tilde{E}_k$ $(k = 0, 1)$ are also closed, implying the existence of a smallest value t_1 of $\tilde{E}_1$. Clearly, also $\tilde{E}_0 \cap [0, 1]$ is closed, so that it has a largest value $t_0 < t_1$. If we denote, for instance, $x_k = \varphi(t_k)$ $(k = 0, 1)$, then the subarc $\gamma_0' = \varphi[(t_0, t_1)] \subset \gamma_0 \subset \Delta$ joins the points $x_k \in E_k$ $(k = 0, 1)$ in Δ, yielding $\gamma_0' \in \Gamma(E_0, E_1, \Delta)$, whence $\Gamma(E_0, E_1, \Delta) < \Gamma(E_0, E_1, D)$, and, on account of the preceding lemma, this implies

$$M^D\Gamma(E_0, E_1, \Delta) \geqslant M^D\Gamma(E_0, E_1, D), \quad M^\Delta\Gamma(E_0, E_1, \Delta) \geqslant M^\Delta\Gamma(E_0, E_1, D),$$

which, together with (2), yields

$$\begin{aligned} M^D\Gamma(E_0, E_1, \Delta) &= M^D\Gamma(E_0, E_1, D), \\ M^\Delta\Gamma(E_0, E_1, \Delta) &= M^\Delta\Gamma(E_0, E_1, D). \end{aligned} \tag{3}$$

Finally, let us establish that

$$M^D\Gamma(E_0, E_1, D) = M^\Delta\Gamma(E_0, E_1, D). \tag{4}$$

First, we observe that $\forall\varepsilon > 0$ there is a $\varrho \in F^\Delta[\Gamma(E_0, E_1, D)]$, such that $M^\Delta\Gamma(E_0, E_1, D)+\varepsilon > \sup\limits_\Delta \varrho(x) = c$; if $\varrho_c(x) = c \forall x \in D$ and $\varrho_c(x) = 0$ in cD, then, $M^\Delta\Gamma(E_0, E_1, D)+\varepsilon > c = \sup\limits_D \varrho_c(x) > M^D\Gamma(E_0, E_1, D)$, hence, letting $\varepsilon \to 0$,

$$M^D\Gamma(E_0, E_1, D) \leqslant M^\Delta\Gamma(E_0, E_1, D). \tag{5}$$

Now, let $\varrho \in F^D[\Gamma(E_0, E_1, D)]$; then the restriction $\varrho|_\Delta \in F^\Delta[\Gamma(E_0, E_1, D)]$ implies

$$\begin{aligned} M^\Delta\Gamma(E_0, E_1, D) = \inf_{\varrho\in F^\Delta[\Gamma(E_0,E_1,D)]} \sup_\Delta \varrho(x) \leqslant \sup_\Delta \varrho|_\Delta(x) \leqslant \sup_D \varrho(x) \\ \Rightarrow M^\Delta\Gamma(E_0, E_1, D) \leqslant M^D\Gamma(E_0, E_1, D), \end{aligned}$$

which, together with (5), yields (4) and this, together with (3), implies (1) as desired.

Lemma 5. *$E_0, E_1 \subset \bar{D}$ implies*

$$M^\Delta\Gamma(E_0, E_1, \Delta) = \frac{1}{d_\Delta(E_0, E_1)}, \tag{6}$$

$$M^D\Gamma(E_0, E_1, D) = \frac{1}{d_D(E_0, E_1)}, \tag{7}$$

where $\Delta = D-(E_0 \cup E_1)$.

Indeed, $\forall\varrho \in F^\Delta[\Gamma(E_0, E_1, \Delta)]$ and $\forall\gamma \in \Gamma(E_0, E_1, \Delta)$, if $d_\Delta(E_0, E_1) > 0$ and $\Gamma(E_0, E_1, \Delta) \neq \emptyset$,

$$1 \leqslant \int_\gamma \varrho\, dH^1 \leqslant \sup_\Delta \varrho(x) H^1(\gamma);$$

hence,

$$1 \leqslant \inf_{\varrho\in F^\Delta[\Gamma(E_0,E_1,\Delta)]} \sup_\Delta \varrho(x) \inf_{\gamma\in\Gamma(E_0,E_1,\Delta)} H^1(\gamma) = M^\Delta\Gamma(E_0, E_1, \Delta)\, d_\Delta(E_0, E_1),$$

which may be written also as

$$M^\Delta\Gamma(E_0, E_1, \Delta) \geqslant \frac{1}{d_\Delta(E_0, E_1)}. \tag{8}$$

On the other hand, since

$$\varrho_0(x) = \begin{cases} \dfrac{1}{d_\Delta(E_0, E_1)} & \text{for } x \in \Delta, \\ 0 & \text{elsewhere,} \end{cases}$$

belongs to $F^\Delta[\Gamma(E_0, E_1, \Delta)]$, it follows that

$$M^\Delta \Gamma(E_0, E_1, \Delta) = \inf_{\varrho \in F^\Delta[\Gamma(E_0, E_1, \Delta)]} \sup_\Delta \varrho(x) \leqslant \sup_\Delta \varrho_0(x) = \frac{1}{d_\Delta(E_0, E_1)},$$

which, together with (8), implies (6) in the case $d_\Delta(E_0, E_1) > 0$ and $\Gamma(E_0, E_1, \Delta) \neq \emptyset$.

If $\Gamma(E_0, E_1, \Delta) = \emptyset$, then, by definition, $d_\Delta(E_0, E_1) = \infty$ and $M^\Delta \Gamma(E_0, E_1, \Delta) = 0$.

Finally, if $d_\Delta(E_0, E_1) = 0$, then, on account of Lemma 1, $M^\Delta \Gamma(E_0, E_1, \Delta) = \infty$.

By exactly the same argument, we also obtain (7).

Remark. It is well known that in the finite-dimensional case the module (and also the conformal capacity) are conformally invariant. However, the module defined at the beginning of this paper is not invariant for the inversions, which in the case of a Hilbert space are conformal mappings, for instance according to Gehring's metric definition). Thus, for instance, if $D = \{x;\ 0 < \|x\| < \infty\}$, f is an inversion with center 0 defined in D, $S(r) = \{x;\ \|x\| = r\}$ and $A_r = \{x;\ r < \|x\| < 2r\}$, then, on account of the preceding lemma,

$$M^{A_r} \Gamma[S(r), S(2r), A_r] = \frac{1}{d_{A_r}[S(r), S(2r)]} = \frac{1}{2r - r} = \frac{1}{r},$$

$$M^{A_r^*} \Gamma[S(1/r), S(1/2r), A_r] = \frac{1}{d_{A_r^*}[S(1/r), S(1/2r)]} = \frac{1}{1/r - 1/2r} = 2r,$$

where $A_r^* = f(A_r) = \{x^*;\ 1/2r < \|x^*\| < 1/r\}$; hence, the module is not invariant with respect to the inversions, moreover, the ratio of the corresponding modules is not bounded in D (if $r \to \infty$, it becomes as large as one wishes).

It is easy to see that the module defined in this paper is invariant with respect to translations since the norm and thus also the Hausdorff linear measure corresponding to it are invariant with respect to translations, implying the invariance with respect to translations of the class $F(\Gamma)$ of admissible functions and thus also of the module.

However, the module is only quasiinvariant with respect to the homotheties. Indeed, if $x^* = ax$ with $a \in R$ (the real line), then $\|x^*\| = \|ax\|$

$= |a| \, \|x\|$ and thus $dH^1(\gamma^*) = |a| \, dH^1(\gamma)$; now, if $\forall \varrho \in F(\Gamma)$ and we consider $\varrho^*(x^*) = \frac{1}{|a|} \varrho(x)$, it follows that

$$\int_{\gamma^*} \varrho^*(x^*) \, dH^1(\gamma^*) = \int_{\gamma} \frac{\varrho(x)}{|a|} |a| \, dH^1(\gamma) = \int_{\gamma} \varrho(x) \, dH^1(\gamma) \geqslant 1;$$

hence $\varrho^* \in F(\Gamma^*)$ and thus

$$M^{D^*} \Gamma^* \leqslant \sup_{D^*} \varrho^*(x^*) = \sup_{D} \frac{\varrho(x)}{|a|} = \frac{1}{|a|} \sup_{D} \varrho(x) \Rightarrow M^{D^*} \Gamma^* \leqslant \frac{M^D \Gamma}{|a|}. \tag{9}$$

On the other hand, if $\forall \varrho^* \in F(\Gamma^*)$ we choose ϱ of the form $\varrho(x) = |a| \, \varrho^*(x^*)$, then

$$M^D \Gamma \leqslant \sup_{D} \varrho(x) = \sup_{D^*} |a| \, \varrho^*(x^*) \Rightarrow M^D \Gamma \leqslant |a| \, M^{D^*} \Gamma^*,$$

which, together with (9), implies $M^{D^*} \Gamma^* = M^D \Gamma / |a|$; hence, if $|a| \geqslant 1$,

$$\frac{M^D \Gamma}{|a|} = M^{D^*} \Gamma^* \leqslant M^D \Gamma \leqslant |a| \, M^D \Gamma,$$

and, if $|a| \leqslant 1$,

$$\frac{M^D \Gamma}{|a|} = M^{D^*} \Gamma^* \geqslant M^D \Gamma \geqslant |a| \, M^D \Gamma,$$

so that, in the two cases, we have the double inequality

$$\frac{M^D \Gamma}{K} \leqslant M^{D^*} \Gamma^* \leqslant K M^D \Gamma, \tag{10}$$

where $K = |a|$ if $|a| \geqslant 1$ and $K = 1/|a|$ if $|a| \leqslant 1$. Thus, homotheties which are conformal mappings, for instance according to the analytic definition, verify only a double inequality of form (10).

We recall that a homeomorphism $f\colon D \rightleftarrows D^*$ is said to be *conformal according to the analytic definition* if $\Lambda_f(x) = \lambda_f(x)$ in D, where

$$\Lambda_f(x) = \overline{\lim_{y \to x}} \frac{\|f(x) - f(y)\|}{\|x - y\|}, \qquad \lambda_f(x) = \underline{\lim_{y \to x}} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

We observe that the property of the module to be or not to be a conformal invariant is only an indirect consequence of the fact that the corresponding space is finite- or infinite-dimensional. Indeed, the same definition of the module as in this paper may be given also in $\boldsymbol{R}^n$ (the ∞-module, see our paper [9]) and there it is not a conformal invariant either, while the module defined by F. Gehring [11] in $\boldsymbol{R}^n$: $\operatorname{mod} \Gamma = \inf_{\varrho \in F(\Gamma)} (\int_D \varrho^n \, dm)^{1/n}$

is a conformal invariant. Thus, the conformal invariance of the module is a direct consequence of its expression. The only influence of the property of the space to be finite- or infinite-dimensional is that in an infinite-dimensional space the module cannot have the above expression (which becomes meaningless) (for a more detailed discussion of our definition of the module, see our paper [8]).

Now let us define the *qc* by means of the module in a normed space.

A homeomorphism $f\colon D \rightleftharpoons D^*$ is K-*qc* if $\forall E_0, E_1 \subset D$ the following double inequality holds:

$$\text{(11)} \qquad \frac{M^D \Gamma(E_0, E_1, D)}{K} \leqslant M^{D^*} \Gamma(E_0^*, E_1^*, D^*) \leqslant K M^D \Gamma(E_0, E_1, D),$$

where $E_k^* = f(E_k)$ $(k = 0, 1)$. A *qc* is a K-*qc* with a non-specified K.

In the rest of the paper, *qc* is understood only in the sense of this definition.

LEMMA 6. *If $f\colon D \rightleftharpoons D^*$ is qc and $E_0, E_1 \subset D$, then $d_D(E_0, E_1) = 0 \Leftrightarrow$ ("is equivalent to") $d_{D^*}(E_0^*, E_1^*) = 0$, where $E_k^* = f(E_k)$ $(k = 0, 1)$.*

This is a direct consequence of the preceding definition and of Lemma 1.

LEMMA 7. *$f\colon D \rightleftharpoons D^*$ is K-qc iff $\forall E_0, E_1 \subset D$ the double inequality*

$$\text{(12)} \qquad \frac{d_D(E_0, E_1)}{K} \leqslant d_{D^*}(E_0^*, E_1^*) \leqslant K d_D(E_0, E_1)$$

holds.

If f is K-*qc* and $d_D(E_0, E_1) = 0$, then $d_{D^*}(E_0^*, E_1^*) = 0$ (on account of the preceding lemma) and (12) is satisfied.

If $d_D(E_1, E_1) > 0$, then (12) is a direct consequence of Lemma 5 and of the K-quasiconformality of f.

Now, suppose that f verifies (12). If $d_D(E_0, E_1) = 0$, then $d_{D^*}(E_0^*, E_1^*) = 0$ and (11) is a consequence of Lemma 1. If $d_D(E_0, E_1) > 0$, then $d_{D^*}(E_0^*, E_1^*) > 0$ and (11) is a consequence of Lemma 5. Thus f is K-*qc*, as desired.

Remark. This lemma gives a new characterization for the K-*qc*. Its equivalence to the definition by means of the modules may be considered as an additional justification of the concept of the module and of the corresponding characterization of the *qc* proposed by us.

And now, in order to establish some boundary properties of the *qc*, let us introduce some new concepts.

If an (open) arc $\gamma \subset D$ and its closure $\bar{\gamma}$ is homeomorphic to $[0, 1]$ while its endpoints ξ_0, x_1 belong to ∂D and D respectively, then we say that γ is an *endcut of D from ξ_0*. If γ has only one endpoint $x_0 \in D$ and $d(\gamma, \partial D) = 0$, then we call γ a *wandering arc* (this term is used in functional analysis). Two endcuts $\gamma_1, \gamma_2 \subset D$ are said to be *equivalent* if they have a common endpoint $\xi \in \partial D$ and, for each neighbourhood U_ξ of ξ, $d_D(\gamma_1 \cap U_\xi, \gamma_2 \cap U_\xi) = 0$.

A couple (ξ, γ) consisting of a boundary point ξ of D and an endcut γ of D from ξ is called an *accessible boundary point* of D. Two accessible boundary points (ξ_1, γ_1), (ξ_2, γ_2) of D are consideed *identical* if $\xi_1 = \xi_2$ and γ_1, γ_2 are D-equivalent.

From the preceding definition it follows that we may identify an accessible boundary point (ξ, γ) with the class of D-equivalent endcuts of D from ξ. In this way, an accessible boundary point (ξ, γ), as well as the corresponding class of D-equivalent arcs, are completely determined by an arc of this class.

LEMMA 8. *Let f: $B \rightleftarrows D^*$ be a qc and $\gamma_1, \gamma_2 \subset B$ two endcuts of B from the same boundary point $\xi_0 \in S$. If*

$$\lim_{\substack{x \to \xi_0 \\ x \in \gamma_k}} f(x) = \xi_k^* \quad (k = 1, 2), \tag{13}$$

it follows that γ_1^* and γ_2^* are D^*-equivalent.

Let us suppose first that $\gamma_1 \cap \gamma_2 \neq \emptyset$. Then there are two possibilities: $d(\gamma_1 \cap \gamma_2, \xi_0) = 0$ and $d(\gamma_1 \cap \gamma_2, \xi_0) = d > 0$. In the first case there is a sequence $\{x_n\} \subset \gamma_1 \cap \gamma_2$ such that $x_n \to \xi_0$, so that (13) implies, in particular, $\lim_{n \to \infty} f(x_n) = \xi_k^*$ $(k = 1, 2)$; hence $\xi_1^* = \xi_2^*$. Let us denote the common point by ξ_0^*. Since f is a homeomorphism, we have in this case $d(\gamma_1^* \cap \gamma_2^*, \xi_0^*) = 0$, implying $d_{D^*}(\gamma_1^* \cap U^*, \gamma_2^* \cap U^*) = 0$ for any neighbourhood U^* of ξ_0^*, i.e., γ_1^*, γ_2^* are D^*-equivalent. In the second case, we may choose two disjoint arcs $\gamma_k' \subset \gamma_k \cap B(\xi_0, \frac{1}{2}d)$ $(k = 1, 2)$ representing endcuts of B from ξ_0. Thus we may suppose without loss of generality that $\gamma_1 \cap \gamma_2 = \emptyset$.

We first observe that $\xi_k^* \in \partial D^*$ $(k = 1, 2)$ since if at least one of ξ_k^*, say ξ_1^* belongs to D^*, then considering a sequence $\{x_n^*\} \subset \gamma_1^*$ and $x_n^* \to \xi_1^*$ the property of f of being a homeomorphism implies $x_n = f^{-1}(x_n^*) \to f^{-1}(\xi_1^*) = \xi_0 \in B$, contradicting the hypothesis $\xi_0 \in S$. Next, on account of Lemma 6, $d(\gamma_1, \gamma_2) = 0 \Rightarrow d_{D^*}(\gamma_1^*, \gamma_2^*) = 0$ and because ξ_0 is the only common endpoint of γ_1 and γ_2, it follows that $\xi_1^* = \xi_2^*$. Let us denote this point by ξ_0^*. Now we establish the D^*-equivalence of γ_1^* and γ_2^* by *reductio ad absurdum*. Suppose to prove it is false that there exists at least a neighbourhood U^* of ξ_0^* such that $d_{D^*}(\gamma_1^* \cap U^*, \gamma_2^* \cap U^*) > 0$. Then there are two arcs $\tilde{\gamma}_k^* \subset \gamma_k^* \cap U^*$ representing two endcuts of D^* from ξ_0^*, since otherwise, we must have at least a sequence, say $\{x_n^*\} \subset \gamma_1^* \cap CU^*$, such that $x_n^* \to \xi_0^*$ (where we have denoted by CU^* the complement of U^*), contradicting the fact that U^* is a neighbourhood of ξ_0^* and thus must contain all x_n^* except a finite number of them. But then $\tilde{\gamma}_k = f^{-1}(\tilde{\gamma}_k^*) \subset \gamma_k$ $(k = 1, 2)$ will be endcuts of B from ξ_0, so that $d(\tilde{\gamma}_1, \tilde{\gamma}_2) = 0$, yielding (on account of Lemma 6) $d_{D^*}(\gamma_1^*, \gamma_2^*) = 0$; hence

$d_{D^*}(\gamma_1^* \cap U^*, \gamma_2^* \cap U^*) \leqslant d_{D^*}(\gamma_1^*, \gamma_2^*) = 0$. This contradiction allows us to conclude that γ_1^* and γ_2^* are D^*-equivalent, as desired.

Now in order to introduce the concept of a boundary element, let us generalize the notion of regular sequence of subdomains (for the n-dimensional case see V. A. Zorič [14]).

A sequence of domains $\{U_n\}$, $U_n \subset D$ ($n = 1, 2, \ldots$) is said to be *regular* if

(a) $U_{n+1} \subset U_n$ ($n \in N$),

(b) $F = \bigcap_{n=1}^{\infty} \bar{U}_n \subset \partial D$;

(c) $\sigma_n = \partial U_n \cap D$ (boundary of U_n relatively to D) ($n \in N$) is a connected set;

(d) $d_D(\sigma_n, \sigma_{n+1}) > 0$ ($n \in N$);

(e) there is at most an accessible boundary point of D which is an accessible boundary point for each domain of the sequence $\{U_n\}$.

In the infinite-dimensional case, we have two kinds of regular sequences of domains: *non-degenerate* if $F \neq \emptyset$ and *degenerate* if $F = \emptyset$. The latter possibility corresponds in the case of a compactified n-dimensional space to the case $F = \{\infty\}$.

A sequence $\{U_n\}$ of domains is said to be *imbedded* in a sequence $\{U'_n\}$ if each term U'_n of $\{U'_n\}$ contains all the terms of $\{U_m\}$ with sufficiently large indices. This is denoted by $\{U_m\} < \{U'_n\}$. Two sequences of domains which are imbedded in each other are called *equivalent*. Thus, if $\{U_m\}$ is equivalent to $\{U'_n\}$, this is denoted by $\{U_m\} \sim \{U'_n\}$.

A regular sequence of domains is *minimal* if it is equivalent to each regular sequence of domains imbedded in it.

Now let us generalize the concept of boundary element for a normed space (it was introduced in $\mathbf{R}^n$ by V. A. Zorič [14]).

A *boundary element* of a domain $D \subset X$ is a pair $(F, \{U_n\})$ consisting of a minimal regular sequence of domains $\{U_n\}$ and of a continuum $F = \bigcap_{n=1}^{\infty} \bar{U}_n$ called the *impression* of the corresponding boundary element.

As in $\mathbf{R}^n$, we shall distinguish, also in a normed space, the following kinds of points of the impression of a boundary element: the accessible boundary points (introduced above), the principal points and the subsidiary points.

A point $\xi \in F$ is called a *principal point* of the boundary element $(F, \{U_n\})$ if there exists a regular sequence of domains (belonging to the class of equivalent regular sequences of domains involved in the definition of a boundary element of a domain D) with the property that any ball $B(\xi, r)$ contains at least a cross-cut σ_n (in other words, if σ_n converge to ξ). Clearly, an accessible boundary point is a principal one. A point $\xi \in F$, which is not principal is said to be a *subsidiary* one.

A boundary element is non-degenerate if $F \neq \emptyset$ and *degenerate* if $F = \emptyset$. Two boundary elements $(F, \{U_m\})$ and $(F', \{U'_n\})$ are considered *identical* if $\{U_m\} \sim \{U'_n\}$.

LEMMA 9. *If* $\{U_n\}$ *is a sequence of subdomains of the unit ball satisfying conditions* (a), (b), (d) *from above (involved in the definition of a regular sequence) and* $\{x_m\}$ *is a sequence of points of* B *converging to a point of* F, *then each domain* U_n *contains all points* $x_p \in \{x_m\}$ *from a sufficiently large index* $p(n)$ *onwards.*

The same proof as in R^n (see our monograph [1], Part 3, Chapter 4, Lemma 1, p. 388).

LEMMA 10. *If* $\{U_n\}$ *is a sequence of subdomains of* B *satisfying* (a), (b), (d) *(from above), then all the points of* F *are accessible boundary points of the sequence.*

The same proof as in R^n ([1], Part 3, Chapter 4, Corollary 1 of Lemma 1, p. 388).

COROLLARY. *If* $\{U_n\}$ *is a regular sequence of subdomains of* B, *then* F *consists of at most one point.*

This is a direct consequence of the preceding lemma and of condition (e) involved in the definition of the regularity of subdomain sequences.

Remark. V. A. Zorič ([13], Lemma 5) established in R^n the following result (generalized by the preceding corollary): "A regular sequence of subdomains of B shrinks into only one point". We recall first that a sequence of subdomains is said to shrink into a point ξ_0 if a ball $B(\zeta_0, r)$ of radius r as small as one wishes contains all the domains of the sequence except a finite number. But in the infinite-dimensional case this result is no longer true since if, for instance, the sequence is degenerate, it does not shrink into a point. But, what is more, we shall provide an example showing that there exist non-degenerate regular sequences of domains of a ball which do not shrink into a point. We start with an example of a degenerated regular sequence.

EXAMPLE 1. Let B be a unit ball in a Hilbert space and let us consider balls $B(x_n, \frac{1}{2})$, where $x_n = (0, \ldots, 0, 1, 0, 0, \ldots)$ (i.e., all the coordinates of x_n are 0 except the n-th one). Next, let $B_n^1 = B(x_n, \frac{1}{2}) \cap B$ and let us join in B two successive domains B_n^1, B_{n+1}^1 by an open circular cylinder C_n^1 with a basis of codimension 1 and radius $r < \frac{1}{2}$. Let us write $U_1 = \bigcup_{n=1}^{\infty} (B_n^1 \cup C_n^1)$ and $U_2 = \bigcup_{n=2}^{\infty} (B_n^2 \cup C_n^2)$, where $B_n^2 = B(x_n, \frac{1}{3}) \cap B$ and C_n^2 is a cylinder with the same axis as C_n^1 but with radius $\frac{1}{2}r$; in general, $U_k = \bigcup_{n=k}^{\infty} (B_n^k \cup C_n^k)$, where $B_n^k = B(x_n, 1/(k+1)) \cap B$ and C_n^k has the same axix as C_n^1 but the radius is r/k. Let us denote $\sigma_k = \partial U_k \cap B$. Clearly, $d(\sigma_k, \sigma_{k+1}) > 0$, but $F = \bigcap_{k=1}^{\infty} \bar{U}_k = \emptyset$, and then condition (e) (involved in the definition of regular sequences) is verified. Thus, in a Hilbert space, there exist regular sequences of subdomains of a ball which do not shrink into a point but have $F = \emptyset$.

EXAMPLE 2. And now, let us show that there exist non-degenerate regular sequences of domains of the ball which do not shrink into a point. Let us choose $\xi_0 \in S$ and, on $B \subset S(\xi_0, r_0)$ $(r_0 < \frac{1}{2})$, an infinite sequence $\{x_p\}$ which does not contain a convergent subsequence. Next, let us suppose that $d(\{x_p\}, S) = d > 0$ and let us consider the domains $B(x_p, \frac{1}{2}d)$, the cylinders C_p^1 with a circular basis of codimension 1 and radius $\varrho < \frac{1}{2}d$ joining $B(x_p, \frac{1}{2}d)$ and $B(\xi_0, \varrho_1) \cap B$, where $\varrho_1 < \frac{1}{2}d$, and finally the domain

$$V_1 = \bigcup_{p=1}^{\infty} [B(x_p, \tfrac{1}{2}d) \cup C_p^1] \cup [B(\xi_0, \varrho_1) \cap B]$$

and, in general,

$$V_m = \bigcup_{p=m}^{\infty} [B(x_p, d/(m+1)) \cup C_p^m] \cup [B(\xi_0, \varrho_m) \cap B],$$

where C_p^m is a cylinder with the same axis as C_p^1 but with radius ϱ/m and $\varrho_1 > \varrho_2 > \dots$ such that $\varrho_m \to 0$. Clearly, $\sigma_m = \partial V_m \cap B$ satisfies $\partial(\sigma_m, \sigma_{m+1}) > 0$. It is easy to see that $\{V_m\}$ is a non-degenerate regular sequence of domains of the ball with $F = \{\xi_0\}$ and which do not shrink into ξ_0. Besides, the sequence of the domains $B_n = B(\xi_0, r_n) \cap B$, where $1 > r_1 > r_2 > \dots$ and $r_n \to 0$, is regular too but shrinks to ξ_0 and is not equivalent to $\{V_m\}$.

LEMMA 11. *If $\{U_m\}$, $\{U'_n\}$ are two equivalent non-degenerate regular sequences of subdomains of B, then the corresponding sets F, F' come to the same point.*

From the definition of equivalent regular sequences of subdomains it follows that each U_m contains all U'_n with sufficiently large indices. But then each $\bar{U}_m$ will contains all the corresponding $\bar{U}'_n$, so that, on account of the preceding corollary, $\bar{U}_m \supset \bigcap_{n=n_0}^{\infty} \bar{U}'_n = \bigcap_{n=1}^{\infty} \bar{U}'_n = F' = \{\xi_0\}$, i.e., $\bar{U}_m \supset \xi_0$ and, since this holds for every $\bar{U}_m$, it follows that $F' = \{\xi_0\} \subset \bigcap_{m=1}^{\infty} \bar{U}_m = F$; hence and by the preceding corollary $F = \{\xi_0\} = F'$.

Remark. The corresponding result obtained by V. A. Zorič ([13], Lemma 5) in $\mathbf{R}^n$ was the following; "Two regular sequences of subdomains of B are equivalent iff they shrink into the same boundary point of S". However, this is not true in the infinite-dimensional case, because not only it is necessary for the regular sequences considered to be also non-degenerate, but, on the model of the preceding example, it is possible to obtains two equivalent non-degenerate regular sequences of domains which do not shrink into a point. As one may see from this discussion, in the infinite-dimensional case it is necessary to assume that the regular sequences considered are also non-degenerate and minimal. In that case we have

LEMMA 12. *A minimal non-degenerate regular sequence* $\{U_n\}$ *of subdomains of* B *shrinks into a point and two such sequences are equivalent iff they shrink into the same point of* S.

Indeed, since the sequence is non-degenerate, on account of the preceding corollary, it follows that the corresponding set F comes to a point $\xi_0 \in S$. On the other hand, the sequence $\{B_m\}$ of the domains $B_m = B(\xi_0, r_m) \cap B$ with $r_m \to 0$ has the corresponding set $F = \bigcap_{m=1}^{\infty} \bar{B}_m = \{\xi_0\}$. But $\{B_m\}$ is regular, non-degenerate and $\{B_m\} < \{U_n\}$, where $\{U_n\}$ has been supposed to be minimal, so that $\{B_m\} \sim \{U_n\}$ and the imbedding $\{U_n\} < \{B_m\}$ implies that $\{U_n\}$ shrinks into ξ_0.

In order to establish the second part of the lemma, we observe that, if two minimal non-degenerate regular sequences $\{U_m\}$, $\{U'_n\}$ are equivalent, then, by the preceding lemma, the corresponding sets $F = F'$ are equal to $\{\xi_0\}$, and from the first part of the proof we deduce that $\{U_m\}$ and $\{U'_n\}$ shrink into ξ_0. In order to establish the opposite implication, we remark that $\{U_m\}$ and $\{U'_n\}$ are each of them equivalent to $\{B_p\}$ from above; hence $\{U_m\} \sim \{B_p\} \sim \{U'_n\}$ and, since the equivalence is a transitive relation, we have $\{U_m\} \sim \{U'_n\}$ as desired.

Arguing as in the first part of the proof of the preceding lemma, we obtain the following

COROLLARY. *If* $\{U_m\}$, $\{U'_n\}$ *are two regular sequences of subdomains of* B *and the corresponding sets* $F = F' = \{\xi_0\} \subset S$, *then there is a sequence of spherical domains* $\{B_p\}$ *(as in the proof of the preceding lemma) such that, for each pair of domains* (U_m, U'_n), *the intersection* $U_m \cap U'_n$ *contains all* B_p *from a sufficiently large index* $p(m, n)$ *on.*

Now, in order to establish the bijection between the boundary elements of a ball and a domain induced by a *qc*, let us consider the relation between the regular sequences of these domains.

LEMMA 13. *A qc* $f\colon B \rightleftharpoons D^*$ *carries regular sequences* $\{U_n\}$ *of subdomains of* B *into sequences* $\{U_n^*\}$ *of subdomains of* D^* *satisfying conditions* (a), (b), (c), (d), *the pre-image of a regular sequence* $\{U_n^*\}$ *of subdomains of* D^* *is a sequence of subdomains of* B *satisfying conditions* (a), (b), (c), (d), *and the equivalent regular sequences of subdomains of* B *are carried into equivalent sequences.*

If $\{U_n\}$ is a regular sequence, then, since f is a homeomorphism, (a), (b), (c) will also be trivially satisfied by $\{U_n^*\}$. Also condition (d) holds for $\{U_n^*\}$ since from Lemma 6 we deduce that $d(\sigma_n, \sigma_{n+1}) > 0 \Rightarrow d_{D^*}(\sigma_n^*, \sigma_{n+1}^*) > 0$. The same argument allows us to establish also the opposite implication. Finally, the assertion that equivalent regular sequences are carried into equivalent sequences is a direct consequence of the property of f to be a homeomorphism.

Remark. The corresponding result in R^n, established by V. A. Zorič ([13], Theorem 2), is that regular sequences of subdomains of B are carried into regular sequences and the pre-images of regular sequences of subdomains of D^* are again regular sequences. However, in the infinite-dimensional case, this is not true. Indeed, if $\{U_n\}$ is regular and (ξ_1^*, γ_1^*), (ξ_2^*, γ_2^*) are two accessible boundary points of D^*, where $\xi_1^*, \xi_2^* \in F^* = \bigcap_{n=1}^{\infty} \bar{U}_n^*$, then it is possible that at least one of the arcs $\gamma_k = f^{-1}(\gamma_k^*)$ $(k = 1, 2)$ is a wandering arc, and in that case condition (e) and hence the regularity of $\{U_n\}$ would not be contradicted. This is an additional reason to consider minimal regular sequences. In that case, the conclusions of the preceding lemma may be improved as follows:

THEOREM 1. *A qc f: $B \rightleftharpoons D^*$ carries minimal regular sequences $\{U_n\}$ of subdomains of B into minimal regular sequences $\{U_n^*\}$ of subdomains of D^*, the pre-image of a minimal regular sequence $\{U_n^*\}$ of subdomains of D^* is a minimal regular sequence of B, and equivalent minimal regular sequences of B are carried into equivalent minimal regular sequences of D^*.*

Let $\{U_n\}$ be a minimal regular sequence of B. Then, according to the preceding lemma, $\{U_n^*\}$ verifies conditions (a), (b), (c), (d). Now, let us show by *reductio ad absurdum* the $\{U_n^*\}$ also satisfies condition (e). Suppose to the contrary that there exist at least two different accessible boundary points (ξ_1^*, γ_1^*), (ξ_2^*, γ_2^*) of D^* accessible for every U_n^* $(n \in N)$. Assume additionally that $\xi_1^* \neq \xi_2^*$. Then there exist two neighbourhoods $V_{\xi_1^*}$, $V_{\xi_2^*}$ of those points such that $d(V_{\xi_1^*}, V_{\xi_2^*}) > 0$. It is easy to see that the arcs γ_k^* contained by an arc $\tilde{\gamma}_k^* \subset \gamma_k^* \cap V_{\xi_k^*}$ representing an endcut of D^* from ξ_k^* $(k = 1, 2)$. But then

$$d_{D^*}(\tilde{\gamma}_1^*, \tilde{\gamma}_2^*) \geqslant d_{D^*}(V_{\xi_1^*}, V_{\xi_2^*}) \geqslant d(V_{\xi_1^*}, V_{\xi_2^*}) > 0. \tag{14}$$

Next, since the accessible boundary points (ξ_k^*, γ_k^*) of D^* are accessible boundary points for each U_n^*, it follows that there exist two arcs $\gamma_{kn}^* \subset \tilde{\gamma}_k^* \cap U_n^*$ representing endcuts of U_n^* from ξ_k^* $(k = 1, 2)$. Now let us consider the arcs $\gamma_{kn} = f^{-1}(\gamma_{kn}^*)$ $(k = 1, 2)$. If $d(\gamma_{1n}, \gamma_{2n}) = 0$, then, according to Lemma 6, $d_{D^*}(\gamma_{1n}^*, \gamma_{2n}^*) = 0$; hence, on account of (14), $0 = d_{D^*}(\gamma_{1n}^*, \gamma_{2n}^*) \geqslant d_{D^*}(\tilde{\gamma}_1^*, \tilde{\gamma}_2^*) > 0$. From this absurdity it follows that, in this case, we must have $\xi_1^* = \xi_2^*$. It remains to consider the case $d(\gamma_{1n}, \gamma_{2n}) > 0$. Let us show by *reductio ad absurdum* that $d(\gamma_{kn}, \sigma_n) > 0$. Indeed, suppose to the contrary that $d(\gamma_{kn}, \sigma_n) = 0$. We may consider that $d(\gamma_{k,n+1}, \sigma_n) = 0$ too, since otherwise we may take $\gamma_{k,n+1}$ as γ_{kn}. But then, $d(\gamma_{k,n+1}, \sigma_{n+1}) = 0$ too, since σ_{n+1} separates σ_n from $\gamma_{k,n+1}$. But for the same reason $d(\sigma_n, \sigma_{n+1}) = 0$, contradicting the regularity of $\{U_n\}$ so that we are allowed to conclude that $d(\gamma_{kn}, \sigma_n) > 0$

$(k = 1, 2;\ n \in N)$. Now let us denote

$$\varrho_n = \frac{1}{n}\min[d(\gamma_{1n}, \gamma_{2n}), d(\gamma_{1n}, \sigma_n), d(\gamma_{2n}, \sigma_n), (n-1)\varrho_{n-1}] \qquad (n = 2, 3, \ldots),$$

where $\varrho_1 = \min[d(\gamma_{11}, \gamma_{21}), d(\gamma_{11}, \sigma_1), d(\gamma_{21}, \sigma_1)]$ and $\tilde{V}_{kn} = \{x \in U_n;\ d(x, \gamma_{kn}) < \varrho_n\}$ $(k = 1, 2;\ n \in N)$. It is easy to see that the sequence $\{\tilde{V}_{kn}\}$ $(k = 1, 2)$ satisfies conditions (a), (b), (d). If (c) does not hold, i.e., if $\tilde{\sigma}_{kn} = \partial\tilde{V}_{kn} \cap B$ is not connected, there is one of the components of $\tilde{\sigma}_{kn}$ – let us denote it by σ_{kn} – separating γ_{kn} from σ_n; then we consider the component V_{kn} of $B - \sigma_{kn}$ which contains γ_{kn}. The new sequence $\{V_{kn}\}$ satisfies (a), (b), (c), (d), so that for its regularity it is enough to verify also condition (e). But $\bar{V}_{kn} \subset \bar{U}_n$, whence $\bigcap_{n=1}^{\infty} \bar{V}_{kn} \subset \bigcap_{n=1}^{\infty} \bar{U}_n = F$, where $F = \emptyset$ or $F = \{\xi_0\}$ since $\{U_n\}$ is a regular sequence of subdomains of B (corollary to Lemma 10). But then, $\bigcap_{n=1}^{\infty} \bar{V}_{kn} = \emptyset$ or $\bigcap_{n=1}^{\infty} \bar{V}_{kn} = \{\xi_0\}$ and thus, condition (e) also holds. Next, by construction, $\{V_{nk}\} < \{U_n\}$ and, since $\{U_n\}$ was supposed to be also minimal, it follows that $\{V_{kn}\} \sim \{U_n\}$ $(k = 1, 2)$, which is absurd since $V_{11} \cap V_{21} = \emptyset$. This contradiction implies $\xi_1^* = \xi_2^*$. Let us denote this unique point by ξ_0^*. But then, since (by the hypothesis at the beginning of the proof) (ξ_0^*, γ_1^*), (ξ_0^*, γ_2^*) are different accessible boundary points, we deduce the existence of a neighbourhood U^* of ξ_0^* such that $d_{D^*}(U^* \cap \gamma_1^*, U^* \cap \gamma_2^*) > 0$, implying the existence of two arcs $\tilde{\gamma}_k^* \subset \gamma_k^* \cap U^*$ $(k = 1, 2)$ representing endcuts of D^* from ξ_0^* and such that $d_{D^*}(\tilde{\gamma}_1^*, \tilde{\gamma}_2^*) \geqslant d_{D^*}(U^* \cap \gamma_1^*, U^* \cap \gamma_2^*) > 0$. Next, by arguing as in the first part of the proof, if $d(\tilde{\gamma}_1, \tilde{\gamma}_2) = 0$, where $\tilde{\gamma}_k = f^{-1}(\tilde{\gamma}_k^*)$ $(k = 1, 2)$, then Lemma 6 yields $d_{D^*}(\tilde{\gamma}_1, \tilde{\gamma}_2^*) = 0$, which is absurd, implying that (ξ_0^*, γ_1^*) and (ξ_0^*, γ_2^*) are D^*-equivalent. If $d(\tilde{\gamma}_1, \tilde{\gamma}_2) > 0$, then again, by the same part of the proof, we obtain a similar absurdity, allowing us to conclude that (ξ_0^*, γ_1^*) and (ξ_0^*, γ_2^*) are D^*-equivalent and thus represent the same accessible boundary point. This contradicts the hypothesis made of the beginning of the proof and thus establishes condition (e) and consequently the regularity of $\{U_n^*\}$.

Next, let us show that $\{U_n^*\}$ is minimal. Assume that $\{V_m^*\}$ is a regular sequence of subdomains of D^* imbedded in $\{U_n^*\}$. By the preceding lemma, $\{V_m\}$, where $V_m = f^{-1}(V_m^*)$ $(m \in N)$, satisfies conditions (a), (b), (c), (d) and since f is a homeomorphism, $\{V_m\} < \{U_n\}$. But $\{U_n\}$ verifies by hypothesis condition (e), implying the existence of at most one point of S which is an accessible boundary point for each U_n and, since $\bigcap_{m=1}^{\infty} \bar{V}_m \subset \bigcap_{n=1}^{\infty} \bar{U}_n = F$, it follows that $\{V_m\}$ also has at most one point of S with the same property, i.e., $\{V_m\}$ also satisfies condition (e) and thus is a regular sequence. But $\{U_n\}$ was supposed to be minimal, so that $\{V_m\} < \{U_n\} \Rightarrow \{V_m\} \sim \{U_n\}$ and, by the preceding lemma, $\{V_m^*\} \sim \{U_n^*\}$, allowing us to conclude that $\{U_n^*\}$ is minimal.

Now, in order to establish the opposite implication, let $\{U_n^*\}$ be a minimal regular sequence of subdomains of D^*. By the preceding lemma, $\{U_n\}$ verifies (a), (b), (c), (d). We shall establish condition (e) again by *reductio ad absurdum*. Assume, to prove it is false, that there exist at least two points $\xi_1, \xi_2 \in S$, which are accessible boundary points for every U_n. Then let us denote $d_n = \min[\frac{1}{2}d(\xi_1, \xi_2), d(F, \sigma_n)]$ and $B_n^k = B(\xi_k, d_n) \cap B$ $(k = 1, 2)$. Clearly, $\{B_n^k\}$ $(k = 1, 2)$ are two disjoint regular sequences imbedded in $\{U_n\}$. The preceding lemma implies that $\{B_n^{*k}\}$, where $B_n^{*k} = (B_n^k)$ $(k = 1, 2;\ n \in N)$, satisfies (a), (b), (c), (d) and, since f is a homeomorphism, $\{B_n^{*k}\} < \{U_n^*\}$, hence $\bigcap_{n=1}^{\infty} \bar{B}_n^{*k} \subset \bigcap_{n=1}^{\infty} \bar{U}_n^*$ $(k = 1, 2)$. But $\{U_n^*\}$ is regular by hypothesis, and thus, in particular, it satisfies condition (e) implying that also $\{B_n^{*k}\}$ verify (e) and hence each of them is a regular sequence imbedded in $\{U_n^*\}$, which was assumed to be minimal, yielding $\{B_n^{*k}\} \sim \{U_n^*\}$ $(k = 1, 2)$. But this contradicts the property of $\{B_n^{*k}\}$ $(k = 1, 2)$ of being disjoint. This contradiction shows that $\{U_n\}$ satisfies condition (e) and hence is a regular sequence. Now let us show that $\{U_n\}$ is also minimal. Indeed, let $\{V_m\}$ be a regular sequence imbedded in $\{U_n\}$. Since f is a homeomorphism, $\{V_m^*\} < \{U_n^*\}$; but, arguing as in the first part of the proof and taking into account the preceding lemma, we infer that $\{V_m^*\}$ is regular; hence and since $\{U_n^*\}$ was supposed to be minimal, it follows that $\{V_m^*\} \sim \{U_n^*\}$, the preceding lemma implying $\{V_m\} \sim \{U_n\}$; hence $\{U_n\}$ is minimal too.

Next, if $\{U_m\}$ and $\{V_n\}$ are two equivalent minimal regular sequences of B, then, according to the first part of the proof, $\{U_m^*\}$ and $\{V_n^*\}$ will be two minimal regular sequences of D^* and, since f is homeomorphic, it follows that also $\{U_m^*\} \sim \{V_n^*\}$, as desired.

This theorem may be stated as follows in terms of boundary elements (by taking into account that there is a bijection between the points of S and non-degenerate boundary elements of B):

COROLLARY 1. *By a qc $f\colon B \rightleftharpoons D^*$, to each point of S there corresponds a boundary element of D^*, and the pre-image of each boundary element $(F^*, \{U_n^*\})$ of D^* is a point of S determined by the non-degenerate minimal regular sequence $\{U_n\} = f^{-1}(\{U_n^*\})$ if $F \neq \emptyset$ or a degenerate minimal regular sequence if $F = \emptyset$.*

COROLLARY 2. *A qc $f\colon B \rightleftharpoons D^*$ induces a bijection between the boundary elements of B and D^*.*

Remarks 1. The above concept of boundary element generalizes that of C. Carathéodory's [10] *prime ends* ("Primende"); instead of the condition of not having proper divisors involved in the definition of prime ends, we have the condition of being minimal. This condition does not exist in the definition of V. A. Zorič [13], [14] for the n-dimensional case, but in our case it was necessary for Theorem 1 and Corollary 2. We observe also that Carathéodory's prime ends are not assumed to satisfy condition (e); however, he proves that "every prime end has at most an accessible boundary point", so that, in the plane, condition (e) is a consequence of the other conditions.

2. In R^n, V. A. Zorič ([13], Theorem 3) established a bijection between the points of S and the boundary elements of $D^* = f(B)$, where f is a qc. In a normed space, we have succeeded in proving the existence of a bijection between the boundary elements of B and D^*; however, we have only found that to each point of S there corresponds a boundary element of D^* (which may be degenerate or not), while to a boundary element of D^* there may correspond a point of S or a degenerate boundary element of B, and this seems to happen independently of whether the corresponding boundary element of D^* is degenerate or not. However, up to now we have not found a more appropriate definition of a boundary element of a domain of a normed space.

Now let us give also some other results on the boundary behaviour of qc in a normed space generalizing the corresponding ones in R^n.

LEMMA 14. *$\Gamma_1 \subset \Gamma_2 \subset \ldots$ and $\Gamma = \bigcup\limits_{n=1}^{\infty} \Gamma_n \subset D \Rightarrow \lim\limits_{n\to\infty} M^D \Gamma_n = M^D \Gamma$.*

Let us suppose to the contrary that $a = \lim\limits_{n\to\infty} M^D \Gamma_n < M^D \Gamma = b$ and let $\varepsilon < (b-a)/2$. Then, for each $n \in N$ let us choose a $\varrho_n \in F^D(\Gamma_n)$ such that $\sup\limits_D \varrho_n(x) < a+\varepsilon$ and let us write $\varrho_0(x) = \sup\limits_n \varrho_n(x)$. Then, $\varrho_0 \in F(\Gamma)$ since, if $\gamma \in \Gamma = \bigcup\limits_{n=1}^{\infty} \Gamma_n$, there exists an $n \in N$ such that $\gamma \in \Gamma_n$; hence

$$\int\limits_\gamma \varrho_0\, dH^1 = \int\limits_\gamma \sup\limits_n \varrho_n\, dH^1 \geqslant \int\limits_\gamma \varrho_n\, dH^1 \geqslant 1,$$

and thus, $M^D \Gamma \leqslant \sup\limits_D \varrho_0(x) = \sup\limits_D \sup\limits_n \varrho_n(x) = \sup\limits_n \sup\limits_D \varrho_n(x) \leqslant a+\varepsilon < b$. This contradiction establishes our lemma.

THEOREM 2. *Let $f\colon B \rightleftarrows D^*$ be a qc and $\{\gamma_n\}$ a sequence of arcs $\gamma_n \subset B$ with the endpoints $a_n, b_n \in B$ $(n \in N)$ such that $a_n \to a \in S$, $b_n \to b \in S$. If $H^1(\gamma_n^*) \to 0$, where $\gamma_n^* = f(\gamma_n)$, then $a = b$.*

Suppose, to prove it is false, that $a \neq b$. But then we may assume without loss of generality (possibly appealing to subsequences) that $d(\{a_n\}, \{b_n\}) > 0$. On the other hand, according to the definition of the relative distance and on account of Lemma 7, it follows that there is a K $(1 \leqslant K < \infty)$ such that for every $\varepsilon > 0$, $d(\{a_n\}, \{b_n\}) \leqslant K d_{D^*}(\{a_n^*\}, \{b_n^*\}) \leqslant K H^1(\gamma_m^*) < \varepsilon$ for m sufficiently large; hence, since ε may be as small as one wishes, we obtain $d(\{a_n\}, \{b_n\}) = 0$. This contradiction establishes our lemma.

THEOREM 3. *Let $f\colon B \rightleftarrows D^*$ be a qc and $\{\gamma_n\}$ a sequence of arcs $\gamma_n \subset B$ with the endpoints $a_n, b_n \in B$ $(n \in N)$ such that $a_n \to a \in S$. If $H^1(\gamma_n^*) \to 0$, where $\gamma_n^* = f(\gamma_n)$ $(n \in N)$, then $b_n \to a$.*

Suppose, to the contrary, that we do not have $\lim_{n\to\infty} b_n = a$. Then (appealing possibly to subsequences), we may assume (without loss of generality) that there exists an $r_0 > 0$, sufficiently small such that $\{b_n\} \cap B(a, r_0) = \emptyset$, since otherwise, $\lim_{n\to\infty} b_n = a$; but we may even suppose that $d(\{a_n\}, \{b_n\}) = d > 0$ since otherwise we may consider the subsequence $\{a_n\} \cap B(a, \frac{1}{2}r_0)$ instead of $\{a_n\}$. But then, on account of Lemma 7 and of the K-quasiconformality of f (with a non-specified K), for every $\varepsilon > 0$ and m sufficiently large, we have $0 < d = d(\{a_n\}, \{b_n\}) \leqslant K d_{D^*}(\{a_n^*\}, \{b_n^*\}) \leqslant K H^1(\gamma_m^*) < \varepsilon$, which is absurd since ε may be as small as one wishes while $d > 0$ does not depend on ε. This absurdity establishes our lemma.

COROLLARY. *Let f: $B \rightleftarrows D^*$ be a qc and γ_1, γ_2 two endcuts of B from a and b, respectively, where $a, b \in S$. If $\gamma_k^* = f(\gamma_k)$ and (ξ_k^*, γ_k^*) $(k = 1, 2)$ are two D^*-equivalent accessible boundary points of D^*, then $a = b$.*

THEOREM 4. *If f: $B \rightleftarrows D^*$ is qc and (ξ_1^*, γ_1^*), (ξ_2^*, γ_2^*) are two different accessible boundary points of D^*, then $\gamma_k = f^{-1}(\gamma_k^*)$ $(k = 1, 2)$ cannot be endcuts of B from the same point $\xi_0 \in S$.*

Let us first consider the case where $\xi_1^* \neq \xi_2^*$. Then, arguing as in Theorem 1, we consider two neighbourhoods $V_{\xi_1^*}$, $V_{\xi_2^*}$ such that $d(V_{\xi_1^*}, V_{\xi_2^*}) > 0$ and two arcs $\tilde{\gamma}_k^* \subset \gamma_k^* \cap V_{\xi_k^*}$ $(k = 1, 2)$ representing endcuts of D^* from ξ_k^*. Now suppose, to prove it is false, that γ_k $(k = 1, 2)$ are endcuts of B from the same point $\xi_0 \in S$. Then, $d(\tilde{\gamma}_1, \tilde{\gamma}_2) = d(\gamma_1, \gamma_2) = 0$, where $\tilde{\gamma}_k = f^{-1}(\tilde{\gamma}_k^*)$ $(k = 1, 2)$ and Lemma 6 yields

$$0 < d(V_{\xi_1^*}, V_{\xi_2^*}) \leqslant d_{D^*}(V_{\xi_1^*}, V_{\xi_2^*}) \leqslant d_{D^*}(\tilde{\gamma}_1^*, \tilde{\gamma}_2^*) = 0.$$

This absurdity implies $\xi_1^* = \xi_2^*$. But, also in this case, since γ_k^* $(k = 1, 2)$ belong to two different accessible boundary points of D^*, by definition there is a neighbourhood U^* of $\xi_0^* = \xi_1^* = \xi_2^*$ such that $d_{D^*}(U^* \cap \gamma_1^*, U^* \cap \gamma_2^*) > 0$, and now, if $\gamma_k = f^{-1}(\gamma_k^*)$ $(k = 1, 2)$ were endcuts of B from ξ_0, then $d[f^{-1}(U^* \cap \gamma_1^*), f^{-1}(U^* \cap \gamma_2^*)] = 0$ implying, on account of Lemma 6, $d_{D^*}(U^* \cap \gamma_1^*, U^* \cap \gamma_2^*) = 0$. By this contradiction our lemma is completely proved.

Remark. We point out that the arcs γ_k corresponding to γ_k^* $(k = 1, 2)$ in the preceding lemma may be two wandering arcs of B, or one of them an endcut of B from a point $\xi_0 \in S$ and the other a wandering arc of B.

THEOREM 5. *The wandering arcs of a domain D of a Banach space are not rectifiable.*

Indeed, let γ be such an arc and suppose, to prove it is false, that it is rectifiable, i.e., that its linear Hausdorff length (with respect to the corresponding norm) is finite. We may assume (without loss of generality) that γ is the homeomorphic image of $[0, 1)$, i.e., that φ: $[0, 1) \rightleftarrows \gamma$ and $\varphi(0)$ is an endpoint of γ. Next, let $\{t_n\}$ be a sequence of points of $[0, 1)$ such that

$t_n \to 1$ and let $\{x_n\}$ be the sequence of points $x_n = \varphi(t_n) \in \gamma$ $(n \in N)$. From the hypotheses of rectifiability, it follows that there exists a constant l such that

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| \leqslant l < \infty;$$

hence, given $\varepsilon > 0$, there is an $n_0 \in N$ sufficiently large such that $\|x_{n+1} - x_n\| < \varepsilon \forall n \geqslant n_0$. But the Banach space is complete (with respect to its norm), so that the sequence $\{x_n\}$ will converge to a point $x_0 \in \bar{D}$. This contradiction establishes our lemma.

The following results will be established again in a normed space.

LEMMA 15. *If $E_0, E_1 \subset \bar{D}$ and Γ_0 is the subfamily of $\Gamma(E_0, E_1, D)$ consisting of the arcs which are not rectifiable, then $M^D \Gamma_0 = 0$.*

For every $\varepsilon > 0$ there exists a continuous ϱ such that $0 < \varrho(x) < \varepsilon$ in $\bar{D}$ and $\varrho(x) = 0$ in $C\bar{D}$. Thus, for every $\gamma \in \Gamma_0$,

$$\int_{\gamma} \varrho \, dH^1 = \infty. \tag{15}$$

Indeed, suppose, to the contrary, that $\int_{\gamma} \varrho \, dH^1 = K_0 < \infty$ and let $m_\gamma = \inf_{\gamma} \varrho(x) > 0$ (since γ is compact and ϱ is continuous on γ). Then

$$K_0 = \int_{\gamma} \varrho \, dH^1 \geqslant m_\gamma H^1(\gamma);$$

hence $H^1(\gamma) \leqslant K_0/m < \infty$, contradicting the hypothesis that γ is not rectifiable. This contradiction establishes (15), which still holds for

$$\varrho_0(x) = \begin{cases} \varrho(x) & \text{if } x \in D, \\ 0 & \text{if } x \in CD. \end{cases}$$

But, clearly, $\varrho_0 \in F^D(\Gamma_0)$; hence

$$M^D \Gamma_0 = \inf_{\varrho \in F^D(\Gamma_0)} \sup \varrho(x) \leqslant \sup \varrho_0(x) < \varepsilon,$$

and thus, letting $\varepsilon \to 0$, we obtain $M^D \Gamma_0 = 0$, as desired.

Now, before giving the next result, let us specify certain terms. According to a definition given above, a boundary point ξ of a domain D^* is accessible by an arc γ if $\bar{\gamma}$ is the homeomorphic image of $[0, 1]$ and ξ is one of its endpoints. Now a boundary element $(\{U_n^*\}, F^*)$ is said to be accessible by an arc γ^* if $\gamma^* = \varphi([0, 1))$, where φ is a homeomorphism and each U_n^* contains an arc $\gamma_n^* \subset \gamma^*$ such that $\gamma_n^* = \varphi([r_n, 1))$, where $(0 < r_n < 1)$. In particular, if $\xi_0 \in S$ corresponds by a *qc* $f\colon B \rightleftarrows D^*$ to the boundary element $(\{U_n^*\}, F^*)$ and γ_0 is an endcut of B from ξ_0, then $(\{U_n^*\}, F^*)$ is said to be accessible in D^* by $\gamma_0^* = f(\gamma_0)$. According to this definition, any boundary element is accessible by some arcs.

The *cluster set $C(f, \xi)$ of a mapping $f\colon D \to X$ at a boundary point $\xi \in \partial D$*

is the set of the points $x^* \in X$ such that there exists a sequence $\{x_n\}$, $x_n \to \xi$, $x_n \in D$ $(n \in N)$ with $f(x_n) \to x^*$. If the arc γ is an endcut of D from ξ, then the *cluster set* $C_\gamma(f, \xi)$ *of* f *at* ξ *with respect to* γ is the set of all $x^* \in X$ for which there exists a sequence of points $\{x_n\}$ converging to ξ along γ, so that $f(x_n) \to x^*$. We observe that it is possible to have $C(f, \xi) = \emptyset$ or $C_\gamma(f, \xi) = \emptyset$, because the normed space X, in general, is not compact.

THEOREM 6. *If* $f\colon B \rightleftarrows D^*$ *is qc, then there is no point of* S *corresponding (by the bijection established in Corollary 2 of the preceding theorem) to a boundary element of* D^* *inaccessible by rectifiable arcs.*

The proof will again be by *reductio ad absurdum*. Suppose, to prove it is false, that there exists a point $\xi_0 \in S$ corresponding to a boundary element of D^* inaccessible to rectifiable arcs and let us consider the radius γ_0 of B with an endpoint at ξ_0. Next, let $\{x_n\} \subset \gamma_0$ be a sequence of points converging to ξ_0 and $\{\gamma_n\}$ the corresponding sequence of arcs $\gamma_n = [x_n, \xi_0)$ $(n \in N)$, $\gamma_0^* = f(\gamma_0)$ and $\gamma_n^* = f(\gamma_n)$. We observe that γ_0^* is, by hypothesis, not rectifiable and may be a wandering arc or have an endpoint $\xi_0^* \in \partial D^*$, or its cluster set $C_{\gamma_0}(f, \xi_0) \subset F^*$ may be a whole non-degenerate continuum. But, in all the cases, the argument will be the same. Clearly, $d[B(r_0), \gamma_n] < 1$, $n \in N$. This, combined with the quasiconformality of f (with a non-specified constant K) and Lemma 7, implies $d_{D^*}\{f[B(r_0)], \gamma_n^*\} \leqslant Kd[B(r_0), \gamma_n] < K$. But, since the boundary element $(\{U_n^*\}, F^*)$ is not accessible to rectifiable arcs, $\lim_{n \to \infty} d_{D^*}\{f[B(r_0)], \gamma_n^*\} = \infty$. This contradiction establishes our theorem.

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