ANNALES
POLONICI MATHEMATICI
XXVII (1973)

On meromorphic solutions of a certain class of functional-differential equations

by FRED GROSS* and CHUNG-CHUN YANG** (Washington)

1. Introduction. It has been shown [6] that a differential equation of the form

$$p\big(z, y(z), y'(z), \ldots, y^{(k)}(z)\big) = g\big(y(z)\big), \tag{1}$$

where p is a polynomial in variables $z, y, y', \ldots, y^{(k)}$ and $g(z)$ is a given transcendental entire function, cannot have non-constant entire solutions. In this paper, we investigate the rate of growth of functions meromorphic in the plane ($|z| < \infty$) which are solutions of differential equations of the form

$$p\big(z, y(z), y'(z), \ldots, y^{(k)}(z)\big) = y\big(g(z)\big). \tag{2}$$

For a meromorphic function the growth is measured by the Nevanlinna characteristic $T(r,f)$ (see Section 2).

The following is our main result.

THEOREM 1. *Let g be a given non-constant entire function and $p\big(z, y(z), y'(z), \ldots, y^{(k)}(z)\big)$ be a given polynomial in variables $z, y(z), \ldots, y^{(k)}(z)$. If $f(z)$ is a transcendental meromorphic solution of equation* (2), *then $g(z)$ must be a polynomial. Furthermore, if $g(z)$ is not linear, then the order of f is zero and $T\big(r, f(z)\big) = O(\log r)^{\beta}$ as $r \to \infty$ for some constant $\beta > 1$.*

The proof is based on Nevanlinna's theory of meromorphic functions and some comparison results of Clunie's on the composition of entire and meromorphic functions.

2. Preliminaries. In this section, for the reader's convenience we first review some of the usual notation used in Nevanlinna's theory of meromorphic functions. For a good account of the theory we refer the reader to Hayman [2].

* Dr. F. Gross is affiliated with the Mathematics Research Center, Naval Research Laboratory, Washington, D. C. 20375 and the University of Maryland, Baltimore County, Baltimore, Maryland.

** Dr. C. C. Yang is affiliated with the Mathematics Research Center, Naval Research Laboratory, Washington, D. C. 20375.

If $f(z)$ is a meromorphic function on the plane, the Nevanlinna characteristic $T(r,f)$ is defined as follows:

$$T(r,f) = m(r,f)+N(r,f),$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\,d\theta; \quad \log^+ x = \max(0, \log x)$$

and

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt + n(0,f)\log r;$$

$n(t,f)$ denotes the number of poles of f (counting multiplicity) in $|z| < \infty$.

The order ϱ of f is defined as

$$\varrho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Remark 1. We note that a meromorphic function f is a rational function if and only if $\limsup\limits_{r\to\infty} \dfrac{T(r,f)}{\log r} < +\infty$. We shall use $S(r,f)$ to denote any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside a set of r values of finite measure.

Now we quote some basic properties and fundamental results of the Nevanlinna theory which will be needed later.

LEMMA 1 ([2], p. 5). *Let $f_1, f_2, \dots, f_p$ be meromorphic functions. Then*

$$T\Big\{r, \sum_{\nu=1}^{p} f_\nu(z)\Big\} \leqslant \sum_{\nu=1}^{p} T\{r, f_\nu(z)\} + \log p, \tag{a}$$

$$T\Big\{r, \prod_{\nu=1}^{p} f_\nu(z)\Big\} \leqslant \sum_{\nu=1}^{p} T\{r, f_\nu(z)\}. \tag{b}$$

LEMMA 2 (Nevanlinna's first fundamental theorem; see e.g. [2], p. 5). *Let f be a meromorphic function. Then for every $a \neq \infty$,*

$$T(r,f) = T\left(r, \frac{1}{f-a}\right) + O(1). \tag{3}$$

It is frequently convenient to write $m(r,a,f)$, $N(r,a,f)$, $n(r,a,f)$, $T(r,a,f)$ instead of $m\left(r, \frac{1}{f-a}\right)$, $N\left(r, \frac{1}{f-a}\right)$, $n\left(r, \frac{1}{f-a}\right)$, $T\left(r, \frac{1}{f-a}\right)$.

LEMMA 3 (Nevanlinna [3]). *Let f be meromorphic with $f(0) \neq 0$.*

Then for $R > r$ we have

$$m\left(r, \frac{f'}{f}\right) < 4\log^+ T(R,f) + 3\log^+ \frac{1}{|f(0)|} + 4\log^+ R + + 3\log^+ \frac{1}{R-r} + 2\log^+ \frac{1}{r} + 24. \tag{4}$$

LEMMA 4 (Clunie [1], p. 78). *Let $f(z)$ be meromorphic and g be entire. Suppose that $f(z)$ and $g(z)$ are transcendental and at least one of them is of finite order. Then*

$$\lim_{r\to\infty} \frac{T(r, f(g))}{T(r,f)} = \infty. \tag{5}$$

LEMMA 5 (Clunie [1], p. 78). *Let $f(z)$ be meromorphic and g be entire and suppose that $f(z)$ and $g(z)$ are transcendental. Then*

$$\limsup_{r\to\infty} \frac{T(r, f(g))}{T(r,f)} = \infty. \tag{6}$$

Remark 2. If one examines Clunie's proof carefully one can conclude that under the same hypotheses of Lemma 4 the following is true-

$$\limsup_{\substack{r\to\infty \\ r\notin G}} \frac{T(r, f(g))}{T(r,f)} = \infty, \tag{7}$$

where G is any given set of r values of finite measure.

We shall use this remark later on.

LEMMA 6 (Nevanlinna [4]). *For any given ε with $0 < \varepsilon < \frac{1}{2}$, then we have for $r > r_0$*

$$N(r, a, f) \geqslant T(r,f) - 2[T(r,f)]^{(1+\varepsilon)/2} \tag{8}$$

for all values a except those which belong to a set $E(r_0, \varepsilon)$ of finite logarithmic capacity.

We note here that the Lebesgue measure of the set $E(r_0, \varepsilon)$ is zero [5]. Roughly Lemma 5 says that $N(r, a, f) \sim T(r,f)$ as $r \to \infty$ for all $a \notin E(r_0, \varepsilon)$.

3. Proof of Theorem 1. First we show that g must be a polynomial. Suppose that $g(z)$ is transcendental entire and $f(z)$ is a transcendental meromorphic solution of the equation; i.e., we have the following identity:

$$p(z, f, f', \ldots, f^{(k)}) = f(g). \tag{9}$$

We have by a result of Milloux (see e.g. [2], p. 55) that for $l \geqslant 0$

$$T(r, f^{(l)}) \leqslant (l+1)T(r,f) + S(r,f) \quad \text{as } r\to\infty. \tag{10}$$

It follows from this and Remark 1 that there exists a non-negative constant a such that

$$p(z, f, f', \ldots, f^{(k)}) \leqslant aT(r, f) + S(r, f). \tag{11}$$

Hence, outside a set G of r values of finite measure

$$\limsup_{r\to\infty} \frac{T\big(r, p(z, f, f', \ldots, f^{(k)})\big)}{T(r, f)} \leqslant a. \tag{12}$$

On the other hand by Lemma 5 and Remark 2 we have

$$\limsup_{\substack{r\to\infty \\ r\notin G}} \frac{T\big(r, f(g)\big)}{T(r, f)} = +\infty. \tag{13}$$

This and (12) show clearly that identity (9) cannot hold. Thus we conclude that g must be a polynomial.

Now suppose that $g(z)$ is non-linear. Let

$$g(z) = a_0(z) z^m + a_1(z) z^{m-1} + \ldots + a_m \qquad (a_0 \neq 0,\ m > 1). \tag{14}$$

In this case we can choose r_0 so large that for $|\omega| \leqslant \frac{|a_0|}{2r} r^m$ all m roots of $g(z) = \omega$ satisfy $|z| < r$, if $r > r_0$. It follows that if $r > r_0$, then to each c-point z_0 of f satisfying $|z_0| \leqslant \frac{a_0}{2} r^m$ there correspond m c-points $g^{-1}(z_0)$ of log in $|z| \leqslant r$. Thus

$$n\big(r, c, f(g)\big) \geqslant m \cdot n\left(\frac{a_0}{2} r^m, c, f\right). \tag{15}$$

We may assume without loss of generality that $f\big(g(0)\big) \neq c$.

Hence

$$N\big(r, c, f(g)\big) = \int_0^r \frac{n\big(t, c, f(g)\big)}{t}\, dt > \int_{r_0}^r \frac{n\big(t, c, f(g)\big)}{t}\, dt \tag{16}$$

$$\geqslant \int_{r_0}^r \frac{m \cdot n\left(\frac{a_0}{2} r^m, c, f\right)}{t}\, dt.$$

Making the substitution of variable $s = \frac{a_0}{2} t^m$ we obtain

$$N\big(r, c, f(g)\big) \geqslant \int_{\frac{a_0}{2} r_0^m}^{\frac{a_0}{2} r^m} \frac{n(s, c, f)}{s}\, ds = N\left(\frac{a_0}{2} r^m, c, f\right) + O(1). \tag{17}$$

Now according to Lemma 6 we can actually choose c such that

$$N(r,c,f) \sim T(r,f) \quad \text{as } r \to \infty. \tag{18}$$

Thus from this and (17) we have

$$T(r,f(g)) \geqslant N(r,c,f(g)) \geqslant (1-o(1))T\left(\frac{a_0}{2}r^m, c, f\right) \tag{19}$$

as $r \to \infty$.

On the other hand, we have by putting $R = 2r$ into Lemma 3 and Lemma 2 that

$$\begin{aligned} T\left(r,\frac{f'}{f}\right) &= m\left(r,\frac{f'}{f}\right) + N\left(r,\frac{f'}{f}\right) \\ &< 4\log^+ T(2r,f) + 4\log^+ 2r + O(1) + N\left(r,\frac{f'}{f}\right) \\ &< 4\log^+ T(2r,f) + 4\log^+ 2r + 2T(r,f) + O(1). \end{aligned} \tag{20}$$

It follows from this and Lemma 1 (b) that

$$\begin{aligned} T(r,f') = T\left(r,\frac{f'}{f}\cdot f\right) &\leqslant T\left(r,\frac{f'}{f}\right) + T(r,f) + \log 2 \\ &\leqslant 4\log^+ T(2r,f) + 4\log^+ 2r + 3T(r,f) + O(1). \end{aligned} \tag{21}$$

Similarly we can obtain for $l = 1, 2, \ldots$

$$T(r,f^{(l)}) < k_1\log^+ T(2^l r,f) + k_2\log^+ 2^l r + k_3 T(r,f) + O(1), \tag{22}$$

where constants k_1, k_2, and k_3 depend on l.

Since $T(r,f)$ is an increasing function of r, Lemma 1 and inequality (22) imply that the following estimate holds

$$T(r, p(z,f,f',\ldots,f^{(k)})) \leqslant k_4\log^+ T(2^k r,f) + k_5\log^+ r + k_6 T(r,f) + O(1), \tag{23}$$

where k_4, k_5, k_6 are constants depending on k.

It follows from this, (19) and (9) that for large values of r

$$k_6 T(r,f) + k_4\log^+ T(2^k r,f) + k_5 \log^+ r \geqslant \tfrac{1}{2} T\left(\frac{a_0}{2}r^m, c, f\right). \tag{24}$$

Hence

$$\begin{aligned} &\log T(r,f) + \log k_6 + \log\log^+ T(2^k r,f) + \log k_4 + \log^+\log^+ r + \log k_5 \\ &\qquad \geqslant \log\tfrac{1}{2} + \log T\left(\frac{a_0}{2}r^m, c, f\right) \geqslant \log\tfrac{1}{2} + \log T\left(\frac{a_0}{2}r^m, f\right) + O(1). \end{aligned} \tag{25}$$

The last inequality follows from Lemma 2.

Now suppose that

$$\limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log\log r} = +\infty. \tag{26}$$

Again since $T(r,f)$ is an increasing function of r it sollows that for large values of r

$$\tfrac{1}{2}\log^+ T\left(\frac{a_0}{2} r^m, f\right) \geqslant \log\log^+ T(2^k r, f). \tag{27}$$

From (26), (27), and (25) we have for a sequence of r_n values $\{r_n\}$, $r_n \uparrow \infty$:

$$\tfrac{1}{2}\log T(r_n, f) \geqslant \tfrac{1}{2}\log^+ T\left(\frac{a_0}{2} r_n^m, f\right) + O(1). \tag{28}$$

This inequality cannot hold for large values r_n. Therefore we have to conclude that

$$\limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log\log r} = \beta \quad (1 < \beta < +\infty). \tag{29}$$

β must be greater than one due to the fact that for a transcendental meromorphic function f, $\varliminf_{r\to\infty} T(r,f)/\log r = \infty$. Hence

$$\log^+ T(r,f) = O(1)(\log r)^\beta. \tag{30}$$

The theorem is thus proved completely.

In view of the above argument we can state the following more general result.

THEOREM 2. *Let $R(z, y, y', \ldots, y^{(k)})$ be a rational function in variables $z, y, y', \ldots, y^{(k)}$, and $g(z)$ be a given non-constant entire function. If $f(z)$ is a transcendental meromorphic solution of the equation*

$$R(z, y, y', \ldots, y^{(k)}) = y(g(z)), \tag{31}$$

then $g(z)$ must be a polynomial. Furthermore, if $g(z)$ is not linear, then the order of f is zero and

$$T(r, f(z)) = O(\log r)^\beta \quad \textit{as } r \to \infty$$

for some constant $\beta > 1$.

Final Remark. The argument used in Theorem 1 also enables us to treat similar types of equations with arbitrary meromorphic functions as the coefficients in $p(z, y, y', \ldots, y^{(k)})$ provided we restrict ourselves to the discussion of the meromorphic solutions which grow much faster than all the coefficients of the equation.

References

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Reçu par la Rédaction le 13. 11. 1971