CONJUGATION-INVARIANT MEANS

BY

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Let $G$ be a locally compact group with left Haar measure $dx$ and unit element $e$. For $x \in G$, the corresponding inner automorphism (conjugation) induces a mapping $\tau_x$ on $L^\infty(G)$ by $\tau_x f(y) = f(\text{xyz}^{-1})$. The adjoint map $\tau_x^*$ on $L^1(G)$ is given by $\tau_x u(y) = u(x^{-1} yx) \Delta(x)$ (where $\Delta$ denotes the Haar modulus of $G$). A non-negative linear functional $M$ on $L^\infty(G)$ satisfying $M(1) = 1$ (where on the left-hand side 1 denotes the function with constant value 1) is called a mean (see [6]).

Definition. 1) A mean $M$ on $L^\infty(G)$ is called conjugation-invariant (c.i.), if $M(\tau_x^* f) = M(f)$ for all $x \in G, f \in L^\infty(G)$. (In [4] Effros uses the term “inner-invariant”.)

2) A net $(u_a)$ in $L^1(G)$ is called asymptotically central (a.c.), if

$$\lim_{a} \frac{\|\tau_x u_a - u_a\|_1}{\|u_a\|_1} = 0 \quad \text{for all } x \in G.$$ 

(We assume that $u_a \neq 0$ and put $\|u\|_1 = \int_G |u(y)| \, dy$.)

Recall that the existence of non-trivial central elements in $L^1(G)$ is equivalent to the existence of a compact, conjugation-invariant neighbourhood of the identity in $G$ ([10]). This produces simple examples of c.i. means. A.c. approximate units and a certain subclass of c.i. means were studied in [8]. We show that the existence of a c.i. mean is equivalent to the existence of an a.c. net (Proposition 1). If $G$ is amenable, then there exists a (non-unique) c.i. mean (Proposition 2). If $G$ is connected, then the converse holds, i.e. existence of a c.i. mean implies amenability (Theorem 1).

In the case of discrete groups, $\delta_e$ (Dirac measure at $e$) furnishes a c.i. mean. Further examples come from finite conjugacy classes. If $G$ has Kazhdan’s property $T$, then all c.i. means arise in this way (Theorem 2, see also [1]). Other conditions for uniqueness were discussed earlier in [4] in the context of the property $\Gamma$ of the associated von Neumann algebra (see Proposition 3).
Proposition 1. The following assertions are equivalent:
(i) There exists a conjugation-invariant mean on \( L^p(G) \).
(ii) There exists an asymptotically central net \((u_\alpha)\) in \( L^1(G) \).
(iii) There exists a net \((v_\alpha)\) in \( L^1(G) \) such that \( v_\alpha \geq 0, \|v_\alpha\|_1 = 1 \) and \( \lim_{\alpha} \|\tau_x v_\alpha - v_\alpha\|_1 = 0 \) for all \( x \in G \).

Proof. (ii) \(\Rightarrow\) (iii): Put \( v_\alpha(x) = \frac{|u_\alpha(x)|}{\|u_\alpha\|_1} \).

(iii) \(\Rightarrow\) (i): The proof of this is similar to [8], Theorem 2. If \((v_\alpha) \subseteq L^1(G) \subseteq L^\infty(G) \)' is given as in (iii), then any \( w^* \)-cluster point \( M \) in \( L^\infty(G) \)' is a c.i. mean. Conversely, given a c.i. mean \( M \), it can be approximated in the \( w^* \)-sense by a net \((u_\alpha)\) in \( L^1(G) \) with \( u_\alpha \geq 0, \|u_\alpha\|_1 = 1 \). It follows that \( w^* \lim(\tau_x u_\alpha - u_\alpha) = 0 \) for all \( x \in G \). The \( w^* \)-topology induces the weak topology on \( L^1(G) \) and, since for convex sets the weak closure coincides with the norm closure, we can replace \((u_\alpha)\) by some convex combinations to get \( \lim_{\alpha} \|\tau_x u_\alpha - u_\alpha\|_1 = 0 \).

Remark. In the discrete case, a similar result was shown in [4]. The conditions (ii) and (iii) can be generalized to \( L^p(G) \) (1 \( \leq p < \infty \)) (compare [6], p. 46). By some manipulations it is possible to achieve \( \lim_{\alpha} \|\tau_x v_\alpha - v_\alpha\|_1 = 0 \) uniformly in \( x \) on compact subsets of \( G \).

Proposition 2. If \( G \) is amenable, then there exists a conjugation-invariant mean. This mean is not unique unless \( G = \{e\} \).

Proof. Any mean on \( L^\infty(G) \) that is invariant under left and right translations is clearly c.i. Such means exist if \( G \) is amenable by [6], p. 29. On the other hand, it was shown in [8] Theorem 3 that if \( G \) is amenable, there exists a c.i. mean on \( L^\infty(G) \) which coincides with \( \delta_e \) for bounded continuous functions.

Remark. Regarding uniqueness, the situation is slightly different from that in the case of translation-invariant means. If \( G \) is amenable as a discrete group, then by results of Granirer and Rudin the translation-invariant mean is not unique ([6], p. 91, [12]). But e.g. in the case of \( G = SO(n) \) (n \( \geq 5 \)) (or more generally when \( G \) has a dense subgroup, satisfying Kazhdan's property \( T \)), the left invariant mean is unique ([9]).

Theorem 1. Let \( G \) be a connected locally compact group. Then there exists a conjugation-invariant mean on \( L^\infty(G) \) iff \( G \) is amenable.

Remark. This result has been announced in [7].

Proof. One direction follows from Proposition 2. Now assume that there exists a c.i. mean \( M \) on \( L^\infty(G) \) and that \( G \) is not amenable. If \( H \) is a closed normal subgroup of \( G \), then \( L^\infty(G/H) \) is embedded into \( L^\infty(G) \) and \( M \) induces a c.i. mean on \( L^\infty(G/H) \). By Yamabe's theorem [11], Theorem 4.6, there exists a closed normal subgroup \( K \) of \( G \) such that \( G_1 = G/K \) is a Lie group. Let \( R \) be the radical of \( G_1 \) (i.e. the maximal solvable normal subgroup
of $G$). By [6], p. 53, $G_1/R$ is a non-compact semi-simple Lie group, it is connected and has trivial center (by the maximality of $R$). Hence it is sufficient to consider the case where $G$ is a connected semi-simple Lie group with trivial center. We will show that if $G$ is not compact, then property (iii) of Proposition 1 cannot hold.

Let $J_1, \ldots, J_r$ be a maximal system of pairwise non-conjugate Cartan subgroups of $G$. These are abelian, since $G$ has trivial center [13], I. 1.4.1.5, p. 111. Since $G$ is unimodular, each of the coset spaces $G/J_i$ ($1 \leq i \leq r$) carries a measure $d\tilde{z}$ that is invariant under $L_x g(\tilde{z}) = g((x^{-1} z))$ ($x \in G$). Here we write $\tilde{z} = zJ_i$. By [13], II. 8.1.2, p. 66, we have for $f \in L^1(G)$

$$\left\{ f(z) d\tilde{z} = \sum_{i=1}^r \int_{J_i} w_i(y) \int_{G/J_i} f(zyz^{-1}) d\tilde{z} dy. \right.$$ (Since $J_i$ is abelian, $zyz^{-1}$ depends only on the left coset $\tilde{z} = zJ_i$ of $z$; $w_i \geq 0$ signifies some weight function.) If property (iii) of Proposition 1 holds, then the following is true:

(2) Given $\varepsilon > 0$ and a finite subset $F$ of $G$, there exists $u \in L^1(G)$ with $u \geq 0$, $\|u\|_1 = 1$ such that $\sum_{x \in F} \|\tau_x u - u\|_1 < \varepsilon \|u\|_1$.

From (1), (2) we get (recall that $G$ is unimodular):

(3) $\sum_{i=1}^r \sum_{x \in F} \int_{J_i} w_i(y) \int_{G/J_i} |u(x^{-1} zyz^{-1} x) - u(zyz^{-1})| d\tilde{z} dy$

$$< \varepsilon \sum_{i=1}^r \int_{J_i} w_i(y) \int_{G/J_i} u(zyz^{-1}) d\tilde{z} dy.$$

Hence for some $i \in \{1, \ldots, r\}$ and some $y \in J_i$, we have:

(4) $\sum_{x \in F \setminus J_i} \int_{G/J_i} |u(x^{-1} zyz^{-1} x) - u(zyz^{-1})| d\tilde{z} < \varepsilon \int_{G/J_i} u(zyz^{-1}) d\tilde{z} < \infty$.

Put $g(\tilde{z}) = u(zyz^{-1})$. Then $g \in L^1(G/J_i)$ and (4) implies

(5) $\|L_x g - g\|_1 < \varepsilon \|g\|_1$ for all $x \in F$.

(Where $\|\|_1$ refers to the measure $d\tilde{z}$ on $G/J_1$.)

Since the pairs $(\varepsilon, F)$ form a directed set and there are only finitely many values of $i$, it is easy to see that the index $i$ in (5) can be chosen independently of $\varepsilon > 0$ and the finite subset $F$ of $G$. By [5], p. 28, this implies that $G/J_i$ is amenable. But since $J_i$ is abelian (hence amenable), it would follow that $G$ is amenable ([5], p. 16). This is a contradiction if $G$ is not compact.

Recall that a group $G$ satisfies Kazhdan's property $T$ if the trivial representation is isolated in the unitary dual $\hat{G}$ of $G$. $G$ is said to be an ICC-group, if all non-trivial conjugacy classes are infinite.
THEOREM 2. If \( G \) is a discrete group satisfying Kazhdan's property \( T \), then any conjugation-invariant mean on \( L^0(G) \) belongs to the \( w^* \)-closure of the center of \( L^1(G) \). In particular, if in addition \( G \) is an ICC group, then \( \delta_e \) is the unique conjugation-invariant mean.

Proof. Let \( M \) be a c.i. mean. As described in the proof of Proposition 1, we get a net \( (u_x) \subseteq L^1(G) \subseteq L^0(G) \) such that \( M = w^* \lim u_x \), \( u_x \geq 0 \), \( \|u_x\|_1 = 1 \) and \( \lim \|\tau_x u_x - u_g\|_1 = 0 \) for all \( x \in G \). Put \( v_x = u_x^{1/2} \); then we have \( \lim \|\tau_x v_x - v_g\|_2 = 0 \) for all \( x \in G \), where \( \tau_x \) is the unitary representation on \( L^2(G) \) induced by the inner automorphisms (compare [6], p. 46). Write \( v_x = v_x' + v_x'' \), where \( v_x' \) belongs to the subspace \( M \) of \( L^2(G) \) where \( \tau^{(2)} \) acts trivially and \( v_x'' \in M^\perp \). Then \( \lim \|\tau_x^{(2)} v_x'' - v_g''\|_2 = 0 \). If \( c = \lim sup \|u_x''\|_2 > 0 \), then for some subset of \( (v_x'') \) we get \( \tau_x^{(2)} v_x'' \to c^2 \) for all \( x \in G \) (where \( (\ , \ ) \) denotes the inner product on \( L^2(G) \)). It would follow that \( \tau^{(2)} \) contains the trivial representation weakly ([3], 3.4.10, p. 68), hence by property \( T \), \( \tau^{(2)} \) would contain the trivial representation strongly, contrary to the definition of \( M \). Thus \( c = 0 \), i.e. \( \lim \|v_x - v_g\|_2 = 0 \). Put \( u_x' = (v_x')^2 \); then \( u_x' \) belongs to the center of \( L^1(G) \) and \( \lim \|u_x - u_g\|_1 = 0 \) ([6], p. 47). Consequently, \( M = w^* \lim u_x' \).

EXAMPLES. \( SL(n, Z) \) has property \( T \) for \( n \geq 3 \) ([9], p. 234). The center \( Z \) consists of the scalar matrices. If \( n \) is odd, \( Z \) is trivial, if \( n \) is even, it has order 2. No other finite conjugacy classes do exist. Hence, if \( n \) is odd, \( \delta_e \) is the unique c.i. mean. If \( n \) is even, the same holds for \( PSL(n, Z) = SL(n, Z)/Z \).

Remarks. Discrete groups which have a c.i. mean different from \( \delta_e \) were called inner amenable in [4]. A related result was shown in [1], Theorem 11.

PROPOSITION 3. Let \( G \) be a discrete group which is the free product of groups \( H_1 \) and \( H_2 \), where \( H_1 \) has at least two and \( H_2 \) at least three elements. Then \( \delta_e \) is the unique conjugation-invariant mean.

Proof. This is essentially contained in [4]. We use the idea of von Neumann to construct “paradoxical” decompositions. Take \( a \in H_1 \setminus \{e\} \), \( b, c \in H_2 \setminus \{e\} \) with \( b \neq c \). Let \( D \) be the set of elements of \( G \) whose representation as a reduced word starts with an element of \( H_1 \). Then \( G = D \cup aDa^{-1} \cup \{e\} \) and \( D, bDb^{-1}, cDc^{-1} \) are disjoint. As shown in [4] this implies that any c.i. mean is supported by \( \{e\} \).

Remark. For a free group with at least two generators this was established by a different method in [2]. If \( H_1 = H_2 = Z_2 \), the free product is solvable, hence by Proposition 2 the c.i. mean is not unique.

EXAMPLE. \( PSL(2, Z) = SL(2, Z)/Z \) is the free product of \( Z_2 \) and \( Z_3 \).
REFERENCES


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