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ON A FUNCTIONAL INEQUALITY OF KEMPERMAN

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Some years ago Kemperman [4] raised the question whether there exists a non-measurable real-valued function of a real variable f such that, for all real x and all h in some right neighborhood of 0,

$$(*) \qquad 2f(x) \leqslant f(x+h)+f(x+2h).$$

Here and elsewhere "measurable" refers to Lebesgue measure. The motivation for this question was that Kemperman had shown [3] that a function satisfying $(*)$ and measurable on an interval was necessarily monotone.

If a non-measurable function f satisfying $(*)$ exists, then $(*)$ also has a non-measurable positive solution since if f satisfies $(*)$ and

$$g(x) \overset{\text{def}}{=} \exp[f(x)],$$

then

$$\begin{aligned} g(x+h)+g(x+2h) &= \exp[f(x+h)]+\exp[f(x+2h)] \\ &\geqslant 2\exp\left[\tfrac{1}{2}\big(f(x+h)+f(x+2h)\big)\right] \geqslant 2\exp[f(x)] = 2g(x) \end{aligned}$$

by the inequality between arithmetic and geometric means. Indeed, the same argument repeated shows that there would exist a non-measurable solution whose lower bound may be given in advance.

One might also consider the more general inequality (where similar results hold in the measurable case)

$$(**) \qquad 2f(x) \leqslant (1-u)f(x+h)+(1+u)f(x+th),$$

$(*)$ being the case $u = 0$, $t = 2$. Using an idea of Daroczy [2], Girod showed that Kemperman's question has a positive solution for certain values of t and u. In particular, if

(i) $-1 < u < 1$,

(ii) $t \neq (u-1)/(u+1)$,

(iii) both t and u are transcendental [1],

[1] Girod's construction actually also handles the case where t and u are algebraic irrational and t and $(u-1)/(u+1)$ have the same minimal polynomial, and satisfy (i) and (ii).

then there is a non-measurable f satisfying $(**)$ and, in fact, $f = A^2$, where A is a particularly constructed additive function.

When $t = -1$ and $u = 0$, we have a sort of weak convexity, and any Hamel function provides a non-measurable solution of $(*)$. Indeed, whenever $t = (u-1)/(u+1)$ $(u \neq +1)$, f satisfies a sort of weak convexity, and a non-measurable solution of $(**)$ can be constructed as follows.

If u is rational, $u \neq \pm 1$, then any Hamel function will serve; if u is algebraic irrational, any derivation of the reals over the rationals will serve; if u is transcendental, any derivation D of the reals over the rationals such that $D(t) \geqslant 0$ will serve. On the other hand, clearly, $u = -1$ or $u = +1$ reduces to monotonicity for any $t \neq 0$.

The case of original interest, however, where $t = 2$ and $u = 0$ remains untreated and surprisingly difficult. This note presents some partial results concerning $(*)$ which may, therefore, be of some interest.

THEOREM. *Suppose f is a non-measurable function satisfying $(*)$ for all real x and all $h \in [0, a)$, some right neighborhood of 0. Then*

(a) *f is not of the form $g \circ H$, where g is continuous and H additive* ($\circ$ *denotes functional composition*);

(b) *every interval I such that f restricted to I is non-measurable contains a set E_I of positive outer measure such that, for $x \in E_I$,*

$$D^+f(x) = D^-f(x) = +\infty \quad \text{and} \quad D_+f(x) = D_-f(x) = -\infty.$$

Before giving the proofs it is perhaps worth remarking that (a) exhibits the strong contrast between Kemperman's original problem and the cases of $(**)$ examined above and by Girod where the non-measurable function is of the form $g \circ H$. Also (b) holds for any non-measurable solution of $(**)$.

Proof of (a). Write $(*)$ in the form

$$f(x) \leqslant \frac{1}{2} f(x+h) + \frac{1}{2} f(x+2h) \quad \text{for all } x \text{ and all } h \in [0, a). \tag{1}$$

Now iterate by using (1) for $x+h$ instead of x, and substituting into (1). We have

$$f(x+h) \leqslant \frac{1}{2} f(x+2h) + \frac{1}{2} f(x+3h) \quad \text{for all real } x \text{ and all } h \in [0, a),$$

which put into (1) gives

$$f(x) \leqslant \frac{3}{4} f(x+2h) + \frac{1}{4} f(x+3h) \quad \text{for all real } x \text{ and all } h \in [0, a). \tag{2}$$

Repeat the procedure this time using (1) for $x+2h$ instead of x and substitute in (2) for $f(x+2h)$ to arrive at

$$f(x) \leqslant \frac{5}{8} f(x+3h) + \frac{3}{8} f(x+4h).$$

Continuing in this way it is easy to see by induction that one obtains

$$(3) \qquad f(x) \leqslant \big(f(x+nh) f(x+(n+1)h)\big) \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for all real x, all $h \in [0, a)$, and all integers $n \geqslant 1$. Now suppose $\alpha \in [0, a)$. Then $\alpha/n \in [0, a)$ for all integers $n \geqslant 1$. Hence, for all integers $n \geqslant 1$ and all $\alpha \in [0, a)$,

$$(4) \qquad f(x) \leqslant \left(f(x+\alpha) f\left(x+\alpha+\frac{\alpha}{n}\right)\right) \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now

$$\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}^n \to \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \text{as } n \to \infty,$$

and so (4) yields

$$(5) \qquad f(x) \leqslant \frac{2}{3} f(x+\alpha) + \frac{1}{3} \liminf_{n\to\infty} f\left(x+\alpha+\frac{\alpha}{n}\right)$$

for all real x and each $\alpha \in [0, a)$.

Now suppose, if possible, f is non-measurable and of the form $g \circ H$, where g is continuous and H additive. Then, by (5),

$$(6) \qquad g\big(H(x)\big) \leqslant \frac{2}{3} g\big(H(x+\alpha)\big) + \frac{1}{3} \liminf_{n\to\infty} g\left(H\left(x+\alpha+\frac{\alpha}{n}\right)\right)$$

for all x and all $\alpha \in [0, a)$.

But, by hypothesis,

$$\liminf_{n\to\infty} g\left(H\left(x+\alpha+\frac{\alpha}{n}\right)\right) = \liminf_{n\to\infty} g\left(H(x+\alpha) + \frac{1}{n} H(\alpha)\right) = g\big(H(x+\alpha)\big),$$

and so $f = g \circ H$ would be monotone, contradicting f non-measurable.

Proof of (b). We use the following Theorem of Denjoy-Young-Saks (D-Y-S) (see [5], Chapter 9, Section 4):

THEOREM D-Y-S. *The set of points x at which none of the following four conditions* (α)-(δ) *hold has measure* 0 *for any finite-valued function F*:

(α) $D^+F(x) = D^-F(x) = +\infty$, *and* $D_+F(x) = D_-F(x) = -\infty$;

(β) $D^+F(x) = D_-F(x)$ *is finite*, $D_+F(x) = -\infty$, *and* $D^-F(x) = +\infty$;

(γ) $D_+F(x) = D^-F(x)$ *is finite*, $D_-F(x) = -\infty$, *and* $D^+F(x) = +\infty$;

(δ) $D_+F(x) = D^+F(x) = D^-F(x) = D_-F(x)$ *is finite.*

This theorem, which holds when the space in question is an interval as well as the whole real line, was proved by Denjoy for continuous F, by Grace Young for measurable F, and by Saks for arbitrary F.

First observe that for an f satisfying ($*$) we have

$$0 \leqslant \frac{f(x+h)-f(x)}{h} + 2\frac{f(x+2h)-f(x)}{2h} \quad \text{for } h \in [0, a).$$

Hence

$$0 \leqslant \liminf_{h\to 0+} \frac{f(x+h)-f(x)}{h} + 2\limsup_{h\to 0+} \frac{f(x+2h)-f(x)}{2h}$$
$$= D_+f(x) + 2D^+f(x).$$

Hence, if $D_+f(x) = -\infty$, then $D^+f(x) = +\infty$, and so case (β) of the Denjoy-Young-Saks Theorem holds nowhere for f.

We now use an argument of Banach [1] which we repeat here, since it is brief and for the sake of completeness. Suppose $D^-f(x)$ were finite a.e. Let $S = S_A = \{x\colon f(x) \leqslant A\}$, where A is any given real number. Let $\{x_n\}$ be an increasing sequence of real numbers, with $x_n \in S$, and let

$$x = \lim_{n\to\infty} x_n.$$

Suppose $x \notin S$. Then $f(x_n) \leqslant A$ for all x_n and $f(x) > A$, and hence

$$\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n - x} = +\infty,$$

and so $D^-f(x) = +\infty$. Then, by hypothesis, the set of such x has measure 0. Furthermore, as is well known, the set of points of S which are limit points from the right but not from the left is at most countable. Hence S differs from a closed set by a set of measure 0, and so is measurable. Therefore, f is a measurable function, contradicting f non-measurable. Hence D^-f must be infinite on a set of positive outer measure. But as observed above, alternative (β) of the Denjoy-Young-Saks Theorem holds nowhere for f, and so alternative (α) must hold on such a set. Since all arguments hold when the measure space is an interval, with Lebesgue measure restricted to it, as well as for the whole real line, it follows that f satisfies (b).

The same argument, clearly, holds for f a non-measurable solution of ($**$).

Added in proof. On discussion of the above results with Professor Kemperman, he was able to show that if there exists a non-measurable function f satisfying $(*)$ for all real x and all $h \in [0, a)$, then f is not of the form $g \circ H$, where H is additive and g is any function. The argument is quite different from the above which can only treat the case g continuous. Nevertheless, the above argument still seems of interest as inequality (5) shows (replacing x by $x-a$) that if such a non-measurable function f exists, then

$$\liminf_{n\to\infty} f\left(x+\frac{a}{n}\right) > f(x) \quad \text{for all } a \in [0, a),$$

where n runs through the positive integers, since otherwise f would be monotone.

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