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ON DIFFERENTIABILITY OF PEANO TYPE FUNCTIONS. II

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We call a function $F\colon R \to R^n$ (or R^ω), $n \geqslant 2$, a *Peano function* if $F(R) = R^n$ (R^ω resp.). In this paper we investigate the properties of such functions with respect to differentiation, generalizing results of [1].

For a cartesian product $\bigtimes_{\alpha\in A} M_\alpha$, let

$$P_\beta(t) = \{x = (x_\alpha)_{\alpha\in A} \in \bigtimes_{\alpha\in A} M_\alpha\colon\ x_\beta = t\}$$

for $t \in M_\beta$; λ_n will stand for the n-dimensional Lebesgue measure in R^n.

In this paper we consider only finite derivatives.

We shall use the following lemma which is a particular case of Theorem in [3] (p. 18):

MAIN LEMMA. *The Continuum Hypothesis* (CH) *is equivalent to the following statement*:

R^n can be expressed as the sum of n sets $S_1, \ldots, S_n$ such that $|P_k(t) \cap S_k| \leqslant \aleph_0$ for each $t \in R$, $k = 1, \ldots, n$.

The following lemma says that the analogous theorem is not valid in the infinite-dimensional case.

LEMMA. *R^ω cannot be expressed as the sum $R^\omega = \bigcup_{k=1}^{\infty} S_k$, where $|P_k(t) \cap S_k| \leqslant \aleph_0$ for each $t \in R$, $k = 1, 2, \ldots$*

Proof. Let us assume that sets $S_1, S_2, \ldots$ satisfy the above conditions. We shall define a point $x = (x_k)_{k=1}^{\infty} \notin \bigcup_{k=1}^{\infty} S_k$ by induction with respect to k.

Let x_1 be an arbitrary point in R and assume we have defined x_k. We have $|P_k(x_k) \cap S_k| \leqslant \aleph_0$ and we can choose x_{k+1} to satisfy

$$P_{k+1}(x_{k+1}) \cap P_k(x_k) \cap S_k = \emptyset.$$

One can easily see that $x = (x_k)_{k=1}^{\infty} \notin \bigcup_{k=1}^{\infty} S_k$.

THEOREM 1. *The existence of Peano function $F = (f_1, \ldots, f_n)$ such that for each $t \in R$ there exists at least one of the derivatives $f_k'(t)$, $k = 1, \ldots, n$, is equivalent to the Continuum Hypothesis.*

Let us assume CH. Let sets $S_1, \ldots, S_n$ be as in Main Lemma.

Put $\varphi(t) = t \sin t$ for $t \in R$ and $I_j^{(k)} = \langle 4\pi(nj+k), 4\pi(nj+k)+2\pi \rangle$, where j is an integer and $k = 1, 2, \ldots, n$.

Put

$$\varphi^{-1}(\{u\}) \cap \bigcup_j I_j^{(k)} = \{s(u, k, m)\colon m = 1, 2, \ldots\},$$

$$P_k(u) \cap S_k = \{x(u, k, m)\colon m = 1, 2, \ldots\},$$

where $x(u, k, m) = (x_i(u, k, m))_{i=1}^{n}$ (obviously $x_k(u, k, m) = u$).

For $t \notin \bigcup_{k \neq i} \bigcup_j I_j^{(k)}$ let $f_i(t) = \varphi(t)$.

Let $t \in \bigcup_j I_j^{(k)}$. Then $t = s(u, k, m)$ for some real u and some positive integer m. Put $f_i(t) = x_i(u, k, m)$ for $i \neq k$.

We have thus defined a function $F = (f_1, \ldots, f_n)\colon R \to R^n$ which is easily seen to have the desired properties.

The inverse implication can be proved, using Main Lemma, in the same way as it was done for the two-dimensional case in [1].

THEOREM 2. *Let $F = (f_1, \ldots, f_n)\colon R \to R^n$. Suppose that for each $t \in R$ the derivatives $f_i'(t)$, $f_j'(t)$ exist for at least two different indices $i, j \leqslant n$. Then $\lambda_n(F(R)) = 0$.*

Proof. For $i < j \leqslant n$ let us put $D_{ij} = \{t \in R\colon f_i'(t) \text{ and } f_j'(t) \text{ exist}\}$. By the assumption $\bigcup \{D_{ij}\colon i < j \leqslant n\} = R$. The functions f_i, f_j are VBG on D_{ij} ([2], Chap. VII, th. 10.1, p. 234), i.e. $D_{ij} = \bigcup \{A_k\colon k = 0, 1, \ldots\} = \bigcup \{B_k\colon k = 0, 1, \ldots\}$ and for each k the functions f_i, f_j are of bounded variation on A_k and B_k, respectively. Let $C_0, C_1, \ldots$ be a sequence of sets $A_k \cap B_m$, $k, m = 0, 1, \ldots$ It is obvious that $D_{ij} = \bigcup_k C_k$ and f_i, f_j are both of bounded variation on each C_k. We can extend the functions f_i, f_j restricted to C_k to the functions g_i, g_j being of bounded variation on R. The functions g_i, g_j are Borel measurable and hence $(g_i, g_j)(R)$ is an analytic subset of R^2 and therefore Lebesgue measurable. Using the fact that a function of bounded variation satisfies Banach's (T_2) condition ([2], Chap. IX, pp. 277, 279), we obtain by Theorem 2 in [2] $\lambda_2((g_i, g_j)(R)) = 0$. Hence $\lambda_2(f_i, f_j)(C_k) = 0$ and this implies $\lambda_n(F(C_k)) = 0$. Thus $\lambda_n(F(D_{ij})) = 0$ and finally $\lambda_n(F(R)) = 0$.

THEOREM 3. *Let $F = (f_1, \ldots, f_n)\colon R \to R^n$. Assume that: the function f_1 is measurable; for each $t \in R$ there exists at least one of the derivatives $f_k'(t)$, $k = 1, \ldots, n$; $F(R)$ is a Lebesgue measurable subset of R^n. Then $\lambda_n(F(R)) = 0$.*

Proof. Let $D_k = \{t \in R: f_k'(t) \text{ exists}\}$. By the same method as in the proof of Theorem 3 in [1] we obtain $\lambda_2((f_1, f_k)(D_k)) = 0$ and hence $\lambda_n(F(D_k)) = 0$ for $k > 1$. Thus $F(D_1)$ is measurable because $F(R) = \bigcup_{i=1}^{n} F(D_i)$. Repeating the argument of the proof of Theorem 2 in [1], one can easily see that $\lambda_n(F(D_1)) = 0$. This completes the proof.

THEOREM 4. *There exists no Peano function* $F = (f_1, f_2, \ldots): R \to R^{\omega}$ *such that for each* $t \in R$ *the derivative* $f_k'(t)$ *exists for at least one index* k.

Proof. One can prove, using the method of the proof of Theorem 1 in [1], that the existence of such a function is contradictory with the statement of Lemma.

Let us formulate without proof the version of Theorem 3 which omits the assumption of Lebesgue measurability of $F(R)$.

THEOREM 3'. *Let* $F = (f_1, \ldots, f_n): R \to R^n$. *Let the function* f_1 *be measurable and suppose that for each* $t \in R$ *there exists at least one of the derivatives* $f_k'(t)$, $k = 1, \ldots, n$. *Then* $\lambda_{n*}(F(R)) = 0$, *where* λ_{n*} *denotes the inner* n*-dimensional Lebesgue measure in* R^n.

This theorem can be proved by the same methods as Theorem 3.

REFERENCES

[1] M. Morayne, *On differentiability of Peano type functions*, Colloquium Mathematicum, this fascicle, p. 129–132.

[2] S. Saks, *Theory of the Integral*, Warszawa–Lwów 1937.

[3] R. Sikorski, *A characterization of Alephs*, Fundamenta Mathematicae 38 (1951), p. 18–22.

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