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*A NOTE ON THE CHROMATIC NUMBER OF THE ALTERNATIVE NEGATION OF TWO GRAPHS**

BY

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Borowiecki [1] gives several results on the chromatic number of product graphs in terms of the chromatic numbers of the factors. At the end of his paper, he conjectures a formula for the chromatic number of one particular type of graph product called the alternative negation (the conjecture is also listed as Problem 783 in the Colloquium Mathematicum problem series). We show that neither inequality implied by Borowiecki's formula holds.

As in [1], we restrict our attention to undirected graphs with no self-loops and no multiple edges. The *alternative negation* $G = G_1 | G_2$ of two graphs G_1 and G_2 is defined by

$$V(G) = V(G_1) \times V(G_2)$$

and for every pair of distinct vertices (x_1, y_1), $(x_2, y_2) \in V(G)$

$$(x_1, y_1)-(x_2, y_2) \in E(G) \quad \text{if and only if} \quad x_1 - x_2 \notin E(G_1) \text{ or } y_1 - y_2 \notin E(G_2).$$

Following the notation in [1], let p_1 be the number of vertices of G_1, and let p_2 be the number of vertices of G_2. For any graph G, let $\bar{G}$ be the complement of G, let $\chi(G)$ be the chromatic number of G, and let $I(G)$ be the set of isolated vertices of G. If (x, y) is a vertex of $G_1 | G_2$, we say that x is its *first coordinate* and that y is its *second coordinate*. For any vertex $y \in G_2$ the set of vertices $\{(x, y) | \ x \in V(G_1)\}$ is called the *copy of* G_1 *associated with* y.

Borowiecki conjectures that

$$\chi(G_1 | G_2) = \max(d_1^*, d_2^*),$$

where

$$d_1^* = \chi(\bar{G}_2)(p_1 - |I(G_1)|) + p_2 |I(G_1)|, \quad d_2^* = \chi(\bar{G}_1)(p_2 - |I(G_2)|) + p_1 |I(G_2)|.$$

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We first exhibit a counterexample to show that the inequality

$$\chi(G_1|G_2) \leqslant \max(d_1^*, d_2^*) \tag{1}$$

does not hold. Consider the graphs G_1 and G_2 drawn in Fig. 1. The graph G_1 has 4 vertices; the graph G_2 has 8 vertices. The reader can verify that $\chi(\bar{G}_1) = 2$. Observe that in $\bar{G}_2$ the vertices $\{y_1, y_2, y_3, y_4, y_5\}$ form a clique, so they must all be colored differently; if we assign them the colors a, b, c, d, e, respectively, we can then assign vertex y_6 the color c, and assign vertices y_7 and y_8 the color e to complete a 5-coloring of $\bar{G}_2$. Inequality (1) would require that $\chi(G_1|G_2) \leqslant 20$.

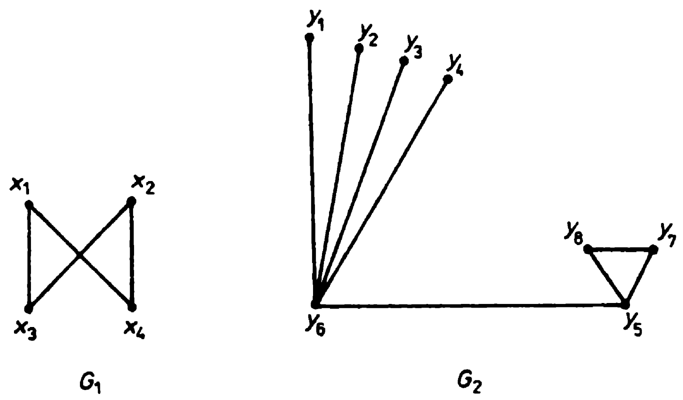

Fig. 1. The graphs G_1 and G_2

In $G_1|G_2$ the graph induced by the five copies of G_1 associated with $\{y_1, y_2, y_3, y_4, y_5\}$ is a clique of size 20, and each of those vertices must be assigned a different color. The vertices in the copies associated with y_7 and y_8 are connected to all of the vertices of the copies associated with y_1, y_2, y_3, y_4; hence if we do not want to use any more colors, both of those copies must be colored with the four colors assigned to

$$\{(x_1, y_5), (x_2, y_5), (x_3, y_5), (x_4, y_5)\}.$$

However, in $G_1|G_2$ the vertices

$$\{(x_1, y_5), (x_2, y_5), (x_1, y_7), (x_2, y_7), (x_1, y_8), (x_2, y_8)\}$$

induce a 6-clique. They all need different colors, but only 4 of the initial 20 colors are available for them.

An entire class of similar counterexamples can be constructed as follows. Assume that in every case $d_1^* \geqslant d_2^*$; if there are no isolated vertices in either factor, then this is equivalent to $p_1 \chi(\bar{G}_2) \geqslant p_2 \chi(\bar{G}_1)$. Suppose $\bar{G}_2$ has a relatively large clique C' that forces its chromatic number to be precisely the size of this clique. Then we make G_1 some almost complete graph so that the chromatic number of its complement is small, and we can satisfy $d_1^* \geqslant d_2^*$. Although G_1 is dense, we rely on the fact that it has two vertices, x_1 and x_2, that are not neighbors. Finally, we need enough vertices left over in G_2 to form a clique C of size greater than $p_1/2$, including exactly one of the vertices, y_c, of the clique C' in $\bar{G}_2$. We must use d_1^* different colors for the copies of G_1 associated with the vertices of C', with p_1 colors for each copy. All of the copies associated with the vertices of C must use the p_1 colors in the copy associated with y_c. However, there are more than p_1 vertices in these copies of G_1 whose second coordinate is x_1 or x_2; all of these vertices form a clique in $G_1|G_2$, so we need more colors to color them. One can usually add isolated vertices to such graphs without destroying the argument, as long as the inequality $d_1^* \geqslant d_2^*$ is preserved.

We now exhibit a counterexample to the opposite inequality, i.e.,

$$\chi(G_1|G_2) \geqslant \max(d_1^*, d_2^*). \tag{2}$$

Let H_1 be the complete graph on two vertices; let H_2 be a 5-cycle as drawn in Fig. 2.

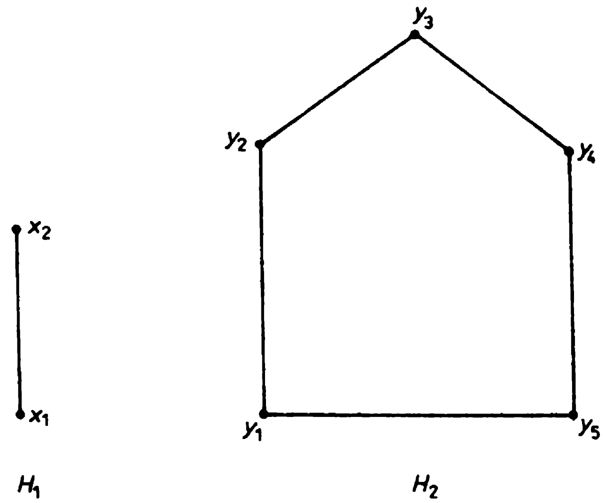

Fig. 2. The graphs H_1 and H_2

The graph H_2 is isomorphic to its complement; hence $\chi(\bar{H}_2) = 3$, and

$$\max(d_1^*, d_2^*) = \max(3 \times 2, 1 \times 5) = 6.$$

The graph $(H_1|H_2)$ can be 5-colored as follows: assign the color a to (x_1, y_1) and (x_2, y_5), assign the color b to (x_1, y_2) and (x_2, y_1), assign the color c to (x_1, y_3) and (x_2, y_2), assign the color d to (x_1, y_4) and (x_2, y_3), and assign the color e to (x_1, y_5) and (x_2, y_4). One can see that this coloring is proper by observing that, for each pair of vertices with the same color, the first coordinates are adjacent in H_1, and the second coordinates are adjacent in H_2.

A lower bound on $\chi(G_1|G_2)$ different from the one given by Borowiecki can be obtained by considering the k-chromatic number which is discussed by Lovász [2] and Stahl [3]. The *k-chromatic number* $\chi_k(G)$ of a graph G is the minimum number of colors needed to assign k distinct colors to each vertex so that no two adjacent vertices share any color. If G_1 is the complete graph on p_1 vertices, then

$$\chi(G_1|G_2) = \chi_{p_1}(\bar{G}_2).$$

Deleting edges from G_1 can only increase $\chi(G_1|G_2)$. Thus we have

$$\chi(G_1|G_2) \geqslant \chi_{p_1}(\bar{G}_2) \quad \text{and} \quad \chi(G_1|G_2) \geqslant \chi_{p_2}(\bar{G}_1).$$

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