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A REPRESENTATION THEOREM FOR POST-LIKE ALGEBRAS

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The algebraic system associated with n-valued propositional calculus is a *Post algebra* of the type n (so for $n = 2$ we have the ordinary two-valued calculus and a Boolean algebra). Using e.g. the Epstein's representation theorem [3] for Post algebras of the type n, we see that Post algebras of that type are precisely those algebras which belong to the smallest equational class containing the algebra $\mathfrak{P}_n = \langle P_n, f_0, f_1, \ldots\rangle$, where $P_n = \{0, 1, \ldots, n-1\}$ and $f_0, f_1, \ldots$ are all finitary operations on P_n (cf. [4]). We generalize the notion of Post algebras in this direction.

Definition. Let $\mathfrak{m}$ be a cardinal number (finite or not). By $\boldsymbol{P}(\mathfrak{m})$ we denote the *class of all Post-like algebras of type* $\mathfrak{m}$ which we define as *the smallest equational class* containing the algebra $\mathfrak{P}_\mathfrak{m} = \langle P_\mathfrak{m}, f_0, f_1, \ldots, f_\alpha, \ldots\rangle_{\alpha<\gamma}$, where $P_\mathfrak{m}$ has the cardinality $\mathfrak{m}$ and $f_0, \ldots, f_\alpha, \ldots$, $\alpha < \gamma$, are all finitary operations on $P_\mathfrak{m}$.

The aim of this paper is to give a uniform proof for representation theorems for all such classes. For cases $\mathfrak{m} \geqslant \aleph_0$ this theorem seems to be new although the idea of the proof is contained in [5], [7] and [8].

For some other generalizations of the notion of Post algebras see [1], [2], [4], and [6].

To prove our representation theorem we use the Birkhoff's characterization of equational classes; so

$$\boldsymbol{P}(\mathfrak{m}) = H\big(S\big(P(\mathfrak{P}_\mathfrak{m})\big)\big),$$

where P denotes the operation of taking direct products, S — of subalgebras and H — of homomorphic images.

1. Homomorphisms. Let $\mathfrak{M} = \langle A, 0, 1, \cap, \cup, \boldsymbol{n}, \boldsymbol{r}\rangle$ be an algebra such that:

(1.1) $\langle A, 0, 1 \cap, \cup\rangle$ is a lattice with the greatest element 1 and the smallest element 0 with $0 \neq 1$;

(1.2) $\boldsymbol{n}(0) = 1$, $\boldsymbol{n}(1) = 0$ and $\boldsymbol{n}(x) \neq 0, 1$ for $x \neq 0, 1$;

(1.3) $\boldsymbol{r}(x, x) = 1$ and $\boldsymbol{r}(x, y) = 0$ for $x \neq y$.

LEMMA 1.1. *Let* $\mathfrak{N} \in H(S(P(\mathfrak{M})))$. *Then the following three conditions are equivalent*:

(i) $\mathfrak{N}$ *is a one-element algebra*;

(ii) $\mathfrak{N} \models 0 = 1$;

(iii) *there is an element* $b \in \mathfrak{N}$ *such that* $r(b, b) = 0$.

The proof is obvious.

Now let $\mathfrak{A}$ be an algebra with the carrier A which has the algebra $\mathfrak{M}$ as a reduct. Let $\mathfrak{B} \subseteq \mathfrak{A}^I$. If D is a filter over I, then D determines a homomorphism $\mathfrak{A}^I \to \mathfrak{A}^I_D$ (¹). The image of $\mathfrak{B}$ under this homomorphism will be called a *reduced subalgebra* of $\mathfrak{A}^I$ and denoted by $\mathfrak{B}/D$. The aim of this section is to show that each element of $H(S(P(\mathfrak{A})))$ is a reduced subalgebra of a direct power of $\mathfrak{A}$. More precisely,

THEOREM A. *Let* $\mathfrak{B} \subseteq \mathfrak{A}^I$. *If there is a homomorphism of* $\mathfrak{B}$ *onto* $\mathfrak{C}$, *then there exists a filter* D *over* I *such that* $\mathfrak{B}/D$ *is isomorphic to* $\mathfrak{C}$.

Proof. Let h be a homomorphism of $\mathfrak{B} \subseteq \mathfrak{A}^I$ onto $\mathfrak{C}$. If $\mathfrak{C}$ is a one-element algebra, then we can take D to be the improper filter over I and all requirements of Theorem A are satisfied. So we may assume that $\mathfrak{C}$ is not one-element.

For every term f of the type of $\mathfrak{A}$, with n variables and with $a_0, a_1, \ldots, a_n \in B$ we put

(1) $$Z(f; a_0, a_1, \ldots, a_n) = \{i \in I\colon a_0(i) = f(a_1(i), \ldots, a_n(i))\}.$$

Let E be the family of all sets of the form (1) such that $Z(f; a_0, a_1, \ldots, a_n) \in E$ iff $\mathfrak{C} \models h(a_0) = f(h(a_1), \ldots, h(a_n))$, where f is any term and $a_0, \ldots, a_n \in B$ are arbitrary. Let D be the filter generated by E over I. We proceed to prove that

(2) the filter D is proper.

Suppose that for some $Z_1, \ldots, Z_s \in E$ we have $\bigcap_{p=1}^{s} Z_p = 0$. Then for each $i_0 \in I$ there is a p such that $i_0 \notin Z_p$. Let

$$Z_p = Z(f_p; a_0^{(p)}, a_1^{(p)}, \ldots, a_n^{(p)}).$$

If $i_0 \in Z_p$, then $a_0^{(p)}(i_0) \neq f_p(a_1^{(p)}(i_0), \ldots, a_n^{(p)}(i_0))$. Thus using the operation r we have

$$r\left(a_0^{(p)}(i), f_p\left(a_1^{(p)}(i_0), \ldots, a_n^{(p)}(i_0)\right)\right) = 0 \quad \text{for } i_0 \notin Z_p.$$

Let us consider the term

$$t(x_0^{(1)}, \ldots, x_n^{(s)}) = r\left(x_0^{(1)}, f_1(x_1^{(1)}, \ldots, x_n)\right) \cap \ldots \cap r\left(x_0^{(s)}, f_s(x_1^{(s)}, \ldots, x_n^{(s)})\right).$$

(¹) Let us recall that by $\mathfrak{A}^I_D$ we denote the reduced power of $\mathfrak{A}$, i.e. the quotient of $\mathfrak{A}^I$ by the equivalence relation $a \underset{D}{\sim} b$ iff $\{i \in I\colon a(i) = b(i)\} \in D$.

Since, for each $i \in I$, we have $\boldsymbol{t}(a_0^{(1)}(i), \ldots, a_n^{(s)}(i)) = 0$, so $\boldsymbol{t}(a_0^{(1)}, \ldots, \ldots, a_n^{(s)}) = 0$ and because h is a homomorphism, also $\boldsymbol{t}(h(a_0^{(1)}), \ldots, h(a_n^{(s)})) = 0$.

On the other hand, since each $Z_p \in E$, for each p we have

$$h(a_0^{(p)}) = f_p(h(a_1^{(p)}), \ldots, h(a_n^{(p)})),$$

which means that $\boldsymbol{t}(h(a_0^{(1)}), \ldots, h(a_n^{(s)})) = 1$. Thus we have $\mathfrak{C} \models 0 = 1$, whence we infer, by Lemma 1.1 ((i) $\leftrightarrow$ (iii)), that $\mathfrak{C}$ is a one-element algebra, contrary to our assumption. Thus (2) is proved.

In order to prove our theorem it suffices, in view of the construction of the filter D, to show that

(3) for each term f with n variables, if $a_0, \ldots, a_n \in B$ and $Z(f; a_0, \ldots, \ldots, a_n) \in D$, then $h(a_0) = f(h(a_1), \ldots, h(a_n))$.

In fact we will prove a stronger proposition that under assumption of (3) there is

(4) $$Z(f; a_0, \ldots, a_n) \in E.$$

Indeed, if $Z(f; a_0, \ldots, a_n) \in D$, then there exist $Z_1, \ldots, Z_s \in E$ such that

$$\bigcap_{p=1}^{s} Z_p \subseteq Z(f; a_0, \ldots, a_n) = Z.$$

Let $Z_p = Z(f_p; a_0^{(p)}, a_1^{(p)}, \ldots, a_m^{(p)})$ and consider the following term:

$$\boldsymbol{t}(x_0, \ldots, x_m^{(s)}) = \boldsymbol{r}(x_0, \boldsymbol{f}(x_1, \ldots, x_n)) \cup \boldsymbol{nr}(x_0^{(1)}, \boldsymbol{f}_1(x_1^{(1)}, \ldots, x_m^{(1)})) \cup \ldots$$
$$\ldots \cup \boldsymbol{nr}(x_0^{(s)}, \boldsymbol{f}_s(x_1^{(s)}, \ldots, x_m^{(s)})).$$

Since $I = Z \cup \bigcup_{p=1}^{s} (I - Z_p)$, we have $\boldsymbol{t}(a_0(i), \ldots, a_m^{(s)}(i)) = 1$ for each $i \in I$, thus $\boldsymbol{t}(a_0, \ldots, a_m^{(s)}) = 1$ and, consequently, since h is a homomorphism

(5) $$\boldsymbol{t}(h(a_0), \ldots, h(a_m^{(s)})) = 1.$$

Since each $Z_p \in E$, we have $h(a_0^{(p)}) = f_p(h(a_1^{(p)}), \ldots, h(a_m^{(p)}))$ for each p. Thus $\boldsymbol{r}(h(a_0^{(p)}), \boldsymbol{f}_p(h(a_1^{(p)}), \ldots, h(a_m^{(p)}))) = 1$ and, consequently, $\boldsymbol{nr}(h(a_0^{(p)}), \boldsymbol{f}_p(h(a_1^{(p)}), \ldots, h(a_m^{(p)}))) = 0$. But then

$$\boldsymbol{t}(h(a_0), \ldots, h(a_m^{(s)})) = \boldsymbol{r}(h(a_0), \boldsymbol{f}(h(a_1), \ldots, h(a_n))),$$

whence, by (5), $\boldsymbol{r}(h(a_0), \boldsymbol{f}(h(a_1), \ldots, h(a_n))) = 1$.

Now let us consider the following term:

$$\boldsymbol{u}(x_0, \ldots, x_n) = \boldsymbol{r}(x_0, \boldsymbol{f}(x_1, \ldots, x_n)).$$

By the definition of E we have $Z(\boldsymbol{u}; 1, a_0, \ldots, a_n) \in E$. But

$$\begin{aligned} Z(\boldsymbol{u}; 1, a_0, \ldots, a_n) &= \{i \in I\colon 1 = \boldsymbol{u}(a_0(i), \ldots, a_n(i))\} \\ &= \{i \in I\colon a_0(i) = \boldsymbol{f}(a_1(i), \ldots, a_n(i))\} \\ &= Z(\boldsymbol{f}; a_0, \ldots, a_n). \end{aligned}$$

This shows (4) and thus the proof of Theorem A is completed.

2. Subalgebras. Let A be a set and I be a non-void set. For any filter F over I^2 let us write (following Keisler [5])

$$A^I|F = \{f \in A^I : \{\langle i, j\rangle : f(i) = f(j)\} \in F\}.$$

Moreover, if $\mathfrak{A}$ is an algebra with the carrier A, then by $\mathfrak{A}^I|F$ we denote the subalgebra of $\mathfrak{A}^I$ with the carrier $A^I|F$.

The following theorem is due to Keisler [5]:

THEOREM 2.1. *A subset $B \subseteq A^I$ is of the form $B = A^I|F$ for a filter F over I^2 iff for each $f, g \in B$ and each $h \in A^I$, if $eq(f) \cap eq(g) \subseteq eq(h)$, then $h \in B$, where $eq(f) = \{\langle i, j\rangle : f(i) = f(j)\} \subseteq I^2$.*

THEOREM B. *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{P}_{\mathfrak{m}}^I$. Then there is a filter F over I^2 such that $\mathfrak{B} = \mathfrak{P}_{\mathfrak{m}}^I|F$.*

Proof. By Theorem 2.1 it suffices to show that for $a, b \in \mathfrak{B}$ and $c \in \mathfrak{P}_{\mathfrak{m}}^I$, if $eq(a) \cap eq(b) \subseteq eq(c)$, then $c \in \mathfrak{B}$.

And if $eq(a) \cap eq(b) \subseteq eq(c)$, define a function $f: A^2 \to A$ by the formula

$$f(a(i), b(i)) = c(i) \quad \text{for each } i \in I.$$

(In other cases f is defined in an arbitrary way.)

Since $eq(a) \cap eq(b) \subseteq eq(c)$, f if a well-defined binary operation on A. So there is an $\alpha < \gamma$ such that $f = f_\alpha$ and since $\mathfrak{B}$ is a subalgebra, there is $f_\alpha(a, b) = c \in \mathfrak{B}$. This completes the proof.

3. Representation theorem.

THEOREM C. *$\mathfrak{A} \in \boldsymbol{P}(\mathfrak{m})$ iff there are a non-void set I and two filters F over I^2 and D over I such that $\mathfrak{A}$ is an isomorph of $(\mathfrak{P}_{\mathfrak{m}}^I|F)/D$.*

Proof. Using Theorem B, we see that any algebra in $S(P(\mathfrak{P}_{\mathfrak{m}}))$ has a form $(\mathfrak{P}_{\mathfrak{m}}^I|F)$ for some $I \neq 0$ and some filter F over I^2. So, by Theorem A, we obtain $\mathfrak{A}$ as an isomorph of a reduced subalgebra of $\mathfrak{P}_{\mathfrak{m}}^I$, whence $\mathfrak{A}$ is an isomorph of $(\mathfrak{P}_{\mathfrak{m}}^I|F)/D$, q.e.d.

Remarks. (3.1) In the above theorem the reduction by a filter D cannot be removed, because for $\mathfrak{m} \geqslant \aleph_0$ no proper elementary extension of $\mathfrak{P}_{\mathfrak{m}}$ is of the form $(\mathfrak{P}_{\mathfrak{m}}^I|F)$.

(3.2) Let $\mathbf{2}$ be the two-element Boolean algebra. If $\mathfrak{A} \cong (\mathfrak{P}_{\mathfrak{m}}^I|F)/D$, then $(\mathbf{2}^I|F)/D$ is said to be a Boolean algebra associated with this representation of $\mathfrak{A}$. Such an algebra does not determine F, because for G being a filter over I^2 generated by finite decompositions of I we have $(\mathbf{2}^I|F)/D = (\mathbf{2}^I|(F \cap G))/D$ and $(\mathfrak{P}_{\mathfrak{m}}^I|F)/D \neq (\mathfrak{P}_{\mathfrak{m}}^I|(F \cap G))/D$ for $\mathfrak{m} \geqslant \aleph_0$. Evidently, for finite $\mathfrak{m}$ we have $(\mathfrak{P}_{\mathfrak{m}}^I|F)/D = (\mathfrak{P}_{\mathfrak{m}}^I|(F \cap G))/D$.

(3.3) For finite $\mathfrak{m}$ the reduction by a filter D can be removed in the way suggested in the proof of Theorem 4 in [7]. And since, for finite $\mathfrak{m}$, Post-like algebras are Post-algebras, our Theorem C yields in this case well-known cases of representation theorem.

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