ANNALES
POLONICI MATHEMATICI
XLI (1983)

Non-parametric convex hypersurfaces with a curvature restriction

by ROLF SCHNEIDER (Freiburg i. Br.)

Abstract. If the Gauss–Kronecker curvature K of the graph of a convex C^2 function, defined on $\{x \in R^n\colon \|x\| \leqslant R\}$, satisfies $K \geqslant 1/a^n$ with a constant $a > 0$, then $R \leqslant a$. This known result is improved in so far as only $K(x) \geqslant \varphi(\|x\|)$ is assumed, where φ is a function which satisfies a certain inequality. At the same time, the differentiability assumption is dropped.

Let f be a real function of class C^2, defined on the ball

$$B_R := \{x \in R^n\colon \|x\| \leqslant R\},$$

where $\|\cdot\|$ denotes the Euclidean norm on the n-dimensional real vector space R^n ($n \geqslant 1$). If the mean curvature H of the graph of f satisfies $H \geqslant 1/a$, where a is a positive constant, then $R \leqslant a$. This has been proved by Heinz [6] for $n = 2$ and by Chern [3] and Flanders [4] for $n \geqslant 2$. Now suppose that f is convex, and let K denote the Gauss–Kronecker curvature of the graph of f. If $K \geqslant 1/a^n$ with some $a > 0$, then $H \geqslant K^{1/n} \geqslant 1/a$; hence $R \leqslant a$. We wish to generalize this latter result under two aspects. Firstly, we merely assume that K is bounded below in a certain way, but not necessarily bounded away from zero. Secondly, we consider general convex functions, without a differentiability assumption, where the curvature restriction is modified appropriately.

We shall prove the following. If $f\colon B_R \to R$ is convex and of class C^2 and if the Gauss–Kronecker curvature $K(x)$ of the graph of f at the point $(x, f(x))$ satisfies

$$K(x) \geqslant \varphi(\|x\|) \quad \text{for } x \in B_R, \tag{1}$$

where φ is a continuous real function with

$$\int_0^a \varphi(r)\, r^{n-1}\, dr \geqslant 1/n \tag{2}$$

for some $a > 0$, then $R \leqslant a$ (in fact, $R < a$ in this case, as can be seen from the proof).

For $\varphi \equiv 1/a^n$ we get the result mentioned above. Suppose that $f(x) = z(\|x\|)$ with a convex C^2 function z on $[0, \infty[$. Define $\varphi(r) := K(x)$, where $\|x\| = r$. Then we have

$$\int_0^a \varphi(r) r^{n-1}\, dr = \int_0^a \frac{z''(r) z'(r)^{n-1}}{[1+z'(r)^2]^{1+n/2}}\, dr = \frac{1}{n}\left[\frac{z'(r)^n}{[1+z'(r)^2]^{n/2}}\right]_0^a.$$

In particular, if $z'(0) = 0$ and $\lim_{r\to 0} z'(r) = \infty$, we get

$$\int_0^{\infty} \varphi(r) r^{n-1}\, dr = 1/n.$$

This shows that our result is, in a certain sense, best possible. Below we shall use a symmetrization procedure to reduce the general case to the case of rotational symmetry.

To formulate a generalization without a differentiability assumption, we make use of the Gaussian curvature measure. For a closed convex set A in Euclidean space R^{n+1} and a Borel subset $\beta \subset R^{n+1}$, let $\varkappa(A, \beta)$ denote the spherical Lebesgue measure of the subset of the unit sphere in R^{n+1} which consists of all outer unit normal vectors to A at points of $\partial A \cap \beta$. Then $\varkappa(A, \cdot)$ is a measure over the Borel sets of R^{n+1} which was introduced by Aleksandrov [1], § 6 (see also [2], ch. V, § 2). If the boundary of A is a regular C^2 hypersurface, then

$$\varkappa(A, \beta) = \int_{\partial A \cap \beta} K\, d\mathscr{H}^n,$$

where K denotes the Gauss–Kronecker curvature of ∂A and $\mathscr{H}^n$ is the n-dimensional Hausdorff measure. Hence it is clear that the theorem below comprises the assertion made above. In the following we identify R^{n+1} with $R^n \times R$. If f is a convex function defined on B_R and if β is a Borel subset of B_R^0, the interior of B_R, we define

$$\varkappa(f, \beta) := \varkappa(\text{epigraph } f, \beta \times R),$$
$$\eta(f, \beta) := \mathscr{H}^n(\text{graph } f \cap (\beta \times R)).$$

THEOREM. *Let $n \geq 1$, $R > 0$ and $f\colon B_R \to R$ a convex function. Suppose that*

(3) $$\varkappa(f, \beta) \geq \int_\beta \varphi(\|x\|)\, d\eta(f, x) \quad \textit{for every Borel set } \beta \subset B_R^0,$$

where φ is some non-negative continuous function which satisfies (2) *with some $a > 0$. Then $R \leq a$.*

The main idea of the proof is taken from Diskant [4], but the symmetrization procedure here is different. It is based on the following lemma. Let SO_n denote the rotation group of R^n (leaving the origin fixed), and write

$$D := \{(\delta_1, \dots, \delta_k, \lambda_1, \dots, \lambda_k) \colon k \in N,\ \delta_i \in SO_n,\ \lambda_i \in \boldsymbol{R},\ \lambda_i \geqslant 0 \quad (i = 1, \dots, k),\ \sum_{i=1}^{k} \lambda_i = 1\}.$$

For $\Delta = (\delta_1, \dots, \delta_k, \lambda_1, \dots, \lambda_k) \in D$ and any real function f on B_R define the function Δf by

$$(\Delta f)(x) := \lambda_1 f(\delta_1 x) + \dots + \lambda_k f(\delta_k x) \quad \text{for } x \in B_R.$$

LEMMA. *Let $f\colon B_R \to \boldsymbol{R}$ be continuous. There exists a sequence $(\Delta_j)_{j \in N}$ in D such that the sequence $(\Delta_j f)_{j \in N}$ converges uniformly to a function g which is rotationally symmetric (i.e., satisfies $g(\delta x) = g(x)$ for $\delta \in SO_n$ and $x \in B_R$).*

Proof. For $0 \leqslant r \leqslant R$ let

$$S(f, r) := \operatorname*{Max}_{\|x\| = r} f(x) - \operatorname*{Min}_{\|x\| = r} f(x).$$

Clearly, $S(f, \cdot)$ is continuous. Write

$$S(f) := \int_0^R S(f, r)\, dr \quad \text{and} \quad s := \inf_{\Delta \in D} S(\Delta f).$$

Since the set $\{\Delta f \colon \Delta \in D\}$ is equicontinuous and uniformly bounded, there exists in D a sequence $(\Delta_j)_{j \in N}$ such that $(\Delta_j f)_{j \in N}$ converges uniformly to a (continuous) function g and $(S(\Delta_j f))_{j \in N}$ converges to s. It is easy to see that $S(g) = \lim S(\Delta_j f) = s$. Now assume that $s > 0$. Then $S(g, r_0) > 0$ for some r_0. It is easy to find $\Delta \in D$ for which $S(\Delta g, r_0) < S(g, r_0)$ (compare Pontrjagin [7], p. 214, from where the idea of the proof is taken). Since $S(\Delta g, r) \leqslant S(g, r)$ for $0 \leqslant r \leqslant R$ and $S(h, \cdot)$ is continuous, we deduce $S(\Delta g) < S(g) = s$. As $\Delta \Delta_j f \to \Delta g$ uniformly, we have $S(\Delta \Delta_k f) < s$ for some index k. Since $\Delta \Delta_k f = \Delta' f$ for some $\Delta' \in D$, this contradicts the definition of s. It follows that $s = 0$, hence $S(g, r) = 0$ for $0 \leqslant r \leqslant R$. Thus the function g is constant on each sphere $\{x \in \boldsymbol{R}^n \colon \|x\| = r\}$, which proves the lemma.

Proof of the theorem. First let f_1, f_2 be two convex functions on B_R which satisfy (3). Let $0 < t < 1$ and $f := (1-t) f_1 + t f_2$. An obvious modification of the argument used by Diskant [4], p. 612, together with the almost everywhere differentiability of indefinite integrals, shows that f also satisfies (3). By induction, every finite convex combination of convex functions satisfying (3) again satisfies (3).

Now suppose that $f\colon B_R \to \boldsymbol{R}$ is a convex function which satisfies (3). Then for $\Delta \in D$, the function Δf has the same property. By the lemma, there exists a sequence $(\Delta_j)_{j \in N}$ in D such that the sequence $(\Delta_j f)_{j \in N}$ converges uniformly to a rotationally symmetric function g. Clearly g is convex. The uniform convergence implies that the epigraphs of $\Delta_j f$ converge, in the

Hausdorff metric, to the epigraph of g. From the weak continuity of curvature measures (see, e.g., Schneider [8], Proposition (3.10)) it follows that the sequence of measures $(\varkappa(\Delta_j f, \cdot))_{j \in N}$, defined over the Borel subsets of B_R^0, converges weakly to the measure $\varkappa(g, \cdot)$. Similarly, defining

$$\nu(f, \beta) := \int_\beta \varphi(\|x\|)\, d\eta(f, x) \quad \text{for every Borel set } \beta \subset B_R^0,$$

we easily see (using the argument in [8], after (3.21)) that the sequence $(\nu(\Delta_j f, \cdot))_{j \in N}$ converges weakly to the measure $\nu(g, \cdot)$. We deduce that g also satisfies (3).

Now we can write $g(x) = z(\|x\|)$ for $x \in B_R$ with a convex function $z\colon (0, R] \to \boldsymbol{R}$. Let $x \in B_R^0$ be a normal point of g in the sense of Aleksandrov [1]. The Gauss–Kronecker curvature $K(x)$ of the graph of g at $(x, g(x))$, defined as the product of the squares of the reciprocal semiaxes of the indicatrix, exists and can be obtained from the Gauss curvature measure by means of differentiation with respect to the surface area measure ($\mathscr{H}^n$, restricted to graph g) by using suitable sequences of neighbourhoods (compare Aleksandrov [1] and the argument of Diskant [4], § 3). It follows from (3) that $K(x) \geqslant \varphi(\|x\|)$. On the other hand, $z'(r)$ and $z''(r)$ exist for $r = \|x\|$, and

$$K(x) = \frac{z''(r)\, z'(r)^{n-1}}{r^{n-1}[1 + z'(r)^2]^{1+n/2}}.$$

Now assume that $R > a$. The function z' is defined almost everywhere in $[0, R[$ and we may extend it to a non-decreasing finite function on all $[0, R[$. Then we get

$$\frac{1}{n} > \frac{1}{n}\left[\frac{z'(r)^n}{[1 + z'(r)^2]^{n/2}}\right]_0^a \geqslant \int_0^a \frac{z''(r)\, z'(r)^{n-1}}{[1 + z'(r)^2]^{1+n/2}}\, dr \geqslant \int_0^a \varphi(r)\, r^{n-1}\, dr \geqslant \frac{1}{n},$$

a contradiction, which shows that $R \leqslant a$.

References

[1] A. D. Aleksandrov, *Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it* (Russian), Uchenye Zapiski Leningrad. Gos. Univ., Math. Ser. 6 (1939), p. 3–35.

[2] —, *Die innere Geometrie der konvexen Flächen*, Akademie-Verlag, Berlin 1955.

[3] S. S. Chern, *On the curvatures of a piece of hypersurface in Euclidean space*, Abh. Math. Sem. Univ. Hamburg 29 (1965), p. 77–91.

[4] V. I. Diskant, *Stability of a sphere in the class of convex surfaces of bounded specific curvature*, Siberian Math. J. 9 (1968), p. 610–615; translated from Sibirskii Mat. Ž. 9 (1968), p. 816–824.

[5] H. Flanders, *Remark on mean curvature*, J. London Math. Soc. 41 (1966), p. 364–366.
[6] E. Heinz, *Über Flächen mit eindeutiger Projektion auf eine Ebene, deren Krümmungen durch Ungleichungen eingeschränkt sind*, Math. Ann. 129 (1955), p. 451–454.
[7] L. S. Pontrjagin, *Topologische Gruppen*, Teil 1, Teubner, Leipzig 1957.
[8] R. Schneider, *Curvature measures of convex bodies*, Ann. Mat. Pura Appl. 116 (1978), p. 101–134.

Reçu par la Rédaction le 3. 5. 1978