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# Error estimations for certain approximate solutions of a non-linear partial differential equation of the first order

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**§ 1.** Let us consider the equation

$$z_x = f(x, y, z, z_y), \tag{1.1}$$

where $f(x, y, z, q)$ is defined for $(x, y)$ belonging to the set

$$0 \leqslant x \leqslant a, \quad y \text{ arbitrary} \tag{2.1}$$

and for arbitrary $z, q$.

Suppose that $f(x, y, z, q)$ is continuous and has first derivatives $f_y, f_z, f_q$ which satisfy the Lipschitz condition

$$|f_i(x, y_1, z_1, q_1) - f_i(x, y_2, z_2, q_2)| \leqslant K(|y_1 - y_2| + |z_1 - z_2| + |q_1 - q_2|), \quad i = 1, 2, 3;\ f_1 = f_y,\ f_2 = f_z,\ f_3 = f_q, \tag{3.1}$$

and the inequalities

$$|f_y| \leqslant L, \quad |f_z| \leqslant L, \quad |f_q| \leqslant L, \tag{4.1}$$

where $K, L$ are constants. Let $\omega(y)$ be of class $C^2$ in $(-\infty, +\infty)$ and satisfy the inequalities

$$\left|\frac{d\omega}{dy}\right| \leqslant N, \quad \left|\frac{d^2\omega}{dy^2}\right| \leqslant M, \tag{5.1}$$

where $N, M$ are constants. Suppose finally that

$$K(1+M+N+2a)^2 \leqslant \tfrac{1}{2}, \quad L(M+a) \leqslant \tfrac{1}{2}, \quad L \leqslant \tfrac{1}{2}. \tag{6.1}$$

Under these assumptions E. Baiada ([1]) has shown that the sequence $S^{(n)}(x, y)$ defined in § 2 satisfies all the assumptions of Arzela's lemma and therefore contains a subsequence which converges almost uniformly to $S(x, y)$. Furthermore $S(x, y)$ satisfies equation (1.1) in the set (2.1) and the initial condition

$$S(0, y) = \omega(y). \tag{7.1}$$

The above-mentioned assumptions guarantees the uniqueness of the Cauchy problem for equation (1.1) ([2]). It easily follows that the sequence $S^{(n)}(x, y)$ converges almost uniformly to $S(x, y)$.

In the present paper we shall deal with the error estimations for approximate solution $S^{(n)}(x, y)$ of equation (1.1) (Theorem) (1).

**§ 2.** Now we shall define functions $S^{(n)}(x, y)$. Let $\delta = a/n$. Let the set (2.1) be divided into a finite number $n$ of such subsets that

$$\sigma_i\colon\ (i-1)\delta \leqslant x \leqslant i\delta,\ \ y \text{ arbitrary},\ \ i = 1, 2, \ldots, n.$$

We define functions

$$(1.2)\qquad \varphi^{(n,i+1)}(x, y) = \tfrac{1}{2}\left[\omega^{(n,i)}\left(y+\frac{x-i\delta}{2}\right)+\omega^{(n,i)}\left(y-\frac{x-i\delta}{2}\right)\right]+ + \int\limits_{y-(x-i\delta)/2}^{y+(x-i\delta)/2} f\big(i\delta, \xi, \omega^{(n,i)}(\xi), \dot{\omega}^{(n,i)}(\xi)\big)\,d\xi, \qquad i = 0, 1, \ldots, n-1,$$

where

$$(2.2)\qquad \omega^{(n,i)}(\xi) = \begin{cases} \omega(\xi) & \text{for} \quad i = 0, \\ \varphi^{(n,i)}(i\delta, \xi) & \text{for} \quad i = 1, 2, \ldots, n-1; \end{cases}$$

$$(3.2)\qquad \dot{\omega}^{(n,i)}(\xi) = \begin{cases} \dot{\omega}(\xi) & \text{for} \quad i = 0, \\ \varphi_y^{(n,i)}(i\delta, \xi) & \text{for} \quad i = 1, 2, \ldots, n-1. \end{cases}$$

Function $S^{(n)}(x, y)$ is defined in (2.1) as follows

$$(4.2)\qquad S^{(n)}(x, y) = \varphi^{(n,i)}(x, y) \quad \text{for} \quad (x, y) \in \sigma_i,\ i = 1, 2, \ldots, n.$$

**§ 3.** Lemma. *Let $U(x, y)$ be continuous in the set*

$$(1.3)\qquad 0 \leqslant x < a,\quad |y| \leqslant b - Mx,\quad a \leqslant b/M.$$

*Assume that $U(x, y)$ has a left derivative of the first order $D_x^- U(x, y)$ and a continuous derivative $U_y(x, y)$ on (1.3) for $x > 0$. Suppose finally that $U(x, y)$ satisfies the initial inequality*

$$(2.3)\qquad |U(0, y)| \leqslant D$$

*and the differential inequality*

$$(3.3)\qquad |D_x^- U(x, y)| \leqslant A|U(x, y)| + M|U_y(x, y)| + C$$

*in (1.3), where $M$, $C$, $D$ are non-negative constants and $A > 0$.*

*Under these assumptions the inequality*

$$(4.3)\qquad |U(x, y)| \leqslant De^{Ax} + \frac{C}{A}(e^{Ax}-1)$$

*is satisfied in (1.3).*

(1) The application of differential inequalities to these estimations was suggested by professor A. Pliś.

The proof of the lemma is very similar to a proof in [3] (where $U$ is supposed to be more regular) and hence is omitted.

From the lemma it is easy to prove the following

COROLLARY. *The lemma holds true if we replace the set* (1.3) *by the set* (2.1).

**§ 4.** THEOREM. *Suppose all the hypotheses from* § 1. *Suppose furthermore that* $|f(x,y,z,q)|\leqslant T$ *and that* $f(x,y,z,q)$ *has continuous derivative* $f_x$ *and* $|f_x|\leqslant L$ *in the set* $(0\leqslant x\leqslant a,\ y,z,q$ *arbitrary*), *where* $T$, $L$ *are constants, then the inequality*

$$|S(x,y)-S^{(n)}(x,y)|\leqslant\frac{C}{L}(e^{Lx}-1) \tag{1.4}$$

*holds true in* (2.1), *where*

$$C=\{L(L+T+\tfrac{3}{2})+L(L+\tfrac{1}{2})(N+a)+(L(L+\tfrac{1}{2})+\tfrac{1}{4})(M+a)\}\,\delta.$$

Proof. Put

$$U(x,y)=S(x,y)-S^{(n)}(x,y) \tag{2.4}$$

for $(x,y)$ in (2.1). It is obvious that $U(x,y)$ has the following properties

$P_1$. $U(x,y)$ is continuous and has a left derivative of the first order $D_x^-U(x,y)$ $(x>0)$ and a continuous derivative $U_y(x,y)$ in (2.1);

$P_2$. $|U(0,y)|=0$.

If $U(x,y)$ satisfies the inequality

$$|D_x^-U(x,y)|\leqslant L|U(x,y)|+L|U_y(x,y)|+C \tag{3.4}$$

in (2.1), then $U(x,y)$ satisfies all the hypotheses of the corollary in § 3 and (1.4) holds in (2.1).

Now, we are going to prove inequality (3.4) in (2.1). First we shall show that inequality (3.4) is satisfied in $\sigma_1$. In virtue of (1.2), (4.2), (2.4) we have

$$U(x,y)=S(x,y)-\varphi^{(n,1)}(x,y)$$

and hence

$$\begin{aligned} |D_x^-U(x,y) &= \Big|f(x,y,S,S_y)-\tfrac{1}{4}\big(\dot\omega(y+\tfrac{1}{2}x)-\dot\omega(y-\tfrac{1}{2}x)\big)-\\ &-\tfrac{1}{2}\Big(f\big(0,y+\tfrac{1}{2}x,\omega(y+\tfrac{1}{2}x),\dot\omega(y+\tfrac{1}{2}x)\big)+f\big(0,y-\tfrac{1}{2}x,\omega(y-\tfrac{1}{2}x),\dot\omega(y-\tfrac{1}{2}x)\big)\Big)\Big|\\ &\leqslant\tfrac{1}{4}|\dot\omega(y+\tfrac{1}{2}x)-\dot\omega(y-\tfrac{1}{2}x)|-\\ &-\tfrac{1}{2}\Big|f(x,y,S,S_y)-f\big(0,y+\tfrac{1}{2}x,\omega(y+\tfrac{1}{2}x),\dot\omega(y+\tfrac{1}{2}x)\big)\Big|+\\ &+\tfrac{1}{2}\Big|f(x,y,S,S_y)-f\big(0,y-\tfrac{1}{2}x,\omega(y-\tfrac{1}{2}x),\dot\omega(y-\tfrac{1}{2}x)\big)\Big|. \end{aligned} \tag{4.4}$$

By the mean value theorem we get

$$(5.4)\qquad |D_x^- U(x, y)| \leqslant \tfrac{1}{4}Mx + \tfrac{1}{2}L\big(x + \tfrac{1}{2}x + |S(x, y) - \omega(y + \tfrac{1}{2}x)| + |S_y(x, y) - \dot\omega(y + \tfrac{1}{2}x)|\big) + \tfrac{1}{2}L\big(x + \tfrac{1}{2}x + |S(x, y) - \omega(y - \tfrac{1}{2}x)| + |S_y(x, y) - \dot\omega(y - \tfrac{1}{2}x)|\big)$$
$$\leqslant \tfrac{1}{4}Mx + \tfrac{3}{2}Lx + L|U(x, y)| + L|U_y(x, y)| + \tfrac{1}{2}L\big(|\varphi^{(n,1)}(x, y) - \omega(y + \tfrac{1}{2}x)| + |\varphi^{(n,1)}(x, y) - \omega(y - \tfrac{1}{2}x)|\big) + \tfrac{1}{2}L\big(|\varphi_y^{(n,1)}(x, y) - \dot\omega(y + \tfrac{1}{2}x)| + |\varphi_y^{(n,1)}(x, y) - \dot\omega(y - \tfrac{1}{2}x)|\big).$$

By (1.2) we have

$$(6.4)\qquad \Big|2\varphi^{(n,1)}(x, y) - \omega(y + \tfrac{1}{2}x) - \omega(y - \tfrac{1}{2}x) = 2\Big|\int_{y-x/2}^{y+x/2} f\big(0, \xi, \omega(\xi), \dot\omega(\xi)\big)\, d\xi\Big| \leqslant 2Tx.$$

But

$$\omega(y + \tfrac{1}{2}x) = \omega(y - \tfrac{1}{2}x) + \dot\omega(y + \tfrac{1}{2}\theta_1 x),$$

where $0 < |\theta_1| < 1$ and hence

$$(7.4)\qquad |2\varphi^{(n,1)}(x, y) - \omega(y + \tfrac{1}{2}x) - \omega(y - \tfrac{1}{2}x)| = |2\varphi^{(n,1)}(x, y) - 2\omega(y - \tfrac{1}{2}x) - \dot\omega(y + \tfrac{1}{2}\theta_1 x)x| \geqslant 2|\varphi^{(n,1)}(x, y) - \omega(y - \tfrac{1}{2}x)| - Nx.$$

By (6.4) and (7.4) we obtain

$$(8.4)\qquad |\varphi^{(n,1)}(x, y) - \omega(y - \tfrac{1}{2}x)| \leqslant \tfrac{1}{2}(2T + N)x.$$

By the same argument we get

$$(9.4)\qquad |\varphi^{(n,1)}(x, y) - \omega(y + \tfrac{1}{2}x) \leqslant \tfrac{1}{2}(2T + N)x.$$

Similarly by (1.2) we have

$$(10.4)\qquad |2\varphi_y^{(n,1)}(x, y) - \dot\omega(y + \tfrac{1}{2}x) - \ddot\omega(y - \tfrac{1}{2}x)| = 2\big|f\big(0, y + \tfrac{1}{2}x, \omega(y + \tfrac{1}{2}x), \dot\omega(y + \tfrac{1}{2}x)\big) - f\big(0, y - \tfrac{1}{2}x, \omega(y - \tfrac{1}{2}x), \dot\omega(y - \tfrac{1}{2}x)\big)\big| \leqslant 2L(1 + N + M)x.$$

But

$$\dot\omega(y + \tfrac{1}{2}x) = \dot\omega(y - \tfrac{1}{2}x) + \ddot\omega(y + \tfrac{1}{2}\theta_2 x)x,$$

where $0 < |\theta_2| < 1$ and hence

$$(11.4)\qquad |2\varphi_y^{(n,1)}(x, y) - \dot\omega(y + \tfrac{1}{2}x) - \dot\omega(y - \tfrac{1}{2}x)| = |2\varphi_y^{(n,1)}(x, y) - 2\dot\omega(y - \tfrac{1}{2}x) - \ddot\omega(y + \tfrac{1}{2}\theta_2 x)x| \geqslant 2|\varphi_y^{(n,1)}(x, y) - \dot\omega(y - \tfrac{1}{2}x)| - Mx.$$

By (10.4) and (11.4) we obtain

$$(12.4)\qquad |\varphi_y^{(n,1)}(x, y) - \dot{\omega}(y - \tfrac{1}{2}x)| \leqslant \tfrac{1}{2}\big(2L(1+N+M)+M\big)\, x.$$

By the same argument we get

$$(13.4)\qquad |\varphi_y^{(n,1)}(x, y) - \dot{\omega}(y + \tfrac{1}{2}x)| \leqslant \tfrac{1}{2}\big(2L(1+N+M)+M\big)\, x.$$

By (5.4), (8.4), (12.4), (9.4) and (13.4) we have

$$|D_x^- U(x, y)| \leqslant L|U(x, y)| + L|U_y(x, y)| + \\ + \{L(L+T+\tfrac{3}{2}) + L(L+\tfrac{1}{2})N + (L(L+\tfrac{1}{2})+\tfrac{1}{4})M\}\,\delta,$$

which means that inequality (3.4) holds true in $\sigma_1$. Now, in the case of $\sigma_i$ $(i = 2, 3, \ldots, n)$ by (1.2) and (4.2) we have

$$U(x, y) = S(x, y) - \varphi^{(n,i)}(x, y).$$

But since (see [1])

$$|\varphi_y^{(n,i+1)}(i\delta, y)| = |\varphi_y^{(n,i)}(i\delta, y)| \leqslant N + i\delta,$$
$$|\varphi_{yy}^{(n,i+1)}(i\delta, y)| = |\varphi_{yy}^{(n,i)}(i\delta, y)| \leqslant M + i\delta,$$

by a similar calculation we get

$$|D_x^- U(x, y)| = |S_x(x, y) - \varphi_x^{(n,i+1)}(x, y)| \\ \leqslant L|U(x, y)| + L|U_y(x, y)| + \{L(L+T+\tfrac{3}{2}) + \\ + L(L+\tfrac{1}{2})(N+i\delta) + (L(L+\tfrac{1}{2})+\tfrac{1}{4})(M+i\delta)\}\,\delta,$$

which means that inequality (3.4) holds true in $\sigma_i$ $(i = 2, 3, \ldots, n)$ and hence completes the proof.

Remark 1. If we replace $|f_x| \leqslant L$ by the inequality

$$|f(x_1, y, z, q) - f(x_2, y, z, q)| \leqslant \psi(|x_2 - x_1|),$$

where $\psi(x)$ is a continuous function in $\langle 0, a\rangle$ and $\psi(0) = 0$, our theorem also holds in (2.1), but in this case

$$C = \{L(L+T+\tfrac{1}{2}) + L(L+\tfrac{1}{2})(N+a) + (L(L+\tfrac{1}{2})+\tfrac{1}{4})(M+a)\}\,\delta + \sup_{0\leqslant x\leqslant \delta}(x).$$

Remark 2. By inequality (1.4) we obtain also

$$\lim_{n\to\infty} S^{(n)}(x, y) = S(x, y).$$

Remark 3. As E. Baiada ([1]) has shown, if $f(x, y, z, q)$ and $\omega(y)$ satisfy the inequalities (3.1), (4.1) and (5.1) (without (6.1)) in (2.1), then equation (1.1) can be transformed (by $x = \varrho\tilde{x}$) to satisfy also (6.1).

## References

[1] E. Baiada, *Sul teorema di esistenza per le equazioni alle derivate parziali*, Ann. Sc. Norm. Sup. Pisa 12 (1943), pp. 135-145.

[2] J. Szarski, *Systèmes d'inégalités différentielles aux derivées partielles du premier ordre et leurs applications*, Ann. Polon. Math. 1 (1954), pp. 149-165.

[3] — *Differential inequalities*, Warszawa 1965.

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