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On orthogonal polynomials and second-order linear differential operators*

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§ 1. Introduction and results. It is well known that the "classical" orthogonal polynomials of Jacobi, Laguerre and Hermite are eigenfunctions of a second-order linear differential operator, and that the coefficients of such operator determine completely the polynomials and the measure of orthogonality (see [3], p. 32, Th. IV).

Our purpose is to generalize such result in the sense of giving a necessary and sufficient condition for a measure to be the orthogonality measure of a sequence of polynomials that are eigenfunctions of a second-order linear differential operator, and to determine completely the operator and the polynomials. The principal facts we are going to prove are stated a little below; we give first some notations.

Every function to be considered here will be supposed to be a real function.

L will denote always a second-order (ordinary) linear differential operator, with real coefficients,

$$Lx = a_0(t)x'' + a_1(t)x' + a_2(t)x .$$

We will say that *L admits a set Φ of functions*, when every $\varphi \in \Phi$ is an eigenfunction of L, that is, $L\varphi = \lambda\varphi$, for some eigenvalue λ.

Any polynomial $P_n(t)$ of degree n, $n = 0, 1, 2, \ldots$, will be denoted by

$$P_n(t) = \sum_{h=0}^{n} a_{nh} t^h .$$

We will suppose $a_{nn} = 1$, and will write a_h instead of a_{nh} when no confusion should arise. If L admits P_n, λ_n will denote the corresponding eigenvalue.

$d\sigma$ will denote any measure defined on the real line, and $M_k = \int t^k d\sigma$, $k = 0, 1, 2, \ldots$, its moments. We will say that $d\sigma$ is *trivial* when it is

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a finite union of point masses; if this is not the case we will say that $d\sigma$ is *non-trivial*.

We state now the principal theorems we are going to prove. The "sufficient" conditions of Theorems 1 and 2, as well as the (Rodrigues) formula for the P_n's in Theorem 3, when the conditions mentioned above hold, are known (see [2], Ch. VII, § 2). In this paper, emphasis will be made in proving the "necessary" conditions, and some consequences will be derived from them, especially Theorem 4 (in connection with this, see also [1]). The proof of Theorem 3 will be done using a generalization of the so-called Legendre's associated polynomials; we also give a few properties of such generalized polynomials.

THEOREM 1. *In order that there exists a sequence* $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, *of polynomials admitted by a second-order linear differential operator* L, *and orthogonal with respect to a non-trivial positive measure* $d\sigma$, *it is necessary and sufficient that the support of* $d\sigma$ *shall be an interval* $\langle a, b\rangle$ *of non-zero length, that* $d\sigma$ *shall be absolutely continuous on* $\langle a, b\rangle$, *and that the density* $\varrho(t)$ *of the measure, except for a constant factor, shall be the following*:

(a) $\varrho(t) = (b-t)^{q-1}(t-a)^{p-1}$, *with* $p > 0$, $q > 0$, *when* $\langle a, b\rangle$ *is finite*;

(b) $\varrho(t) = e^{\mu t}(t-a)^{p-1}$, *with* $\mu < 0$, $p > 0$, *when* a *is finite and* $b = +\infty$;

(c) $\varrho(t) = e^{-\mu t}(b-t)^{q-1}$, *with* $\mu < 0$, $q > 0$, *when* $a = -\infty$ *and* b *is finite*; *and*

(d) $\varrho(t) = e^{\frac{1}{2}\mu t^2+\nu t}$, *with* $\mu < 0$, ν *real, when* $a, b = (-\infty, +\infty)$.

For further reference, we define two constants μ and ν, for any non-zero $\langle a, b\rangle$ and $\varrho(t)$ as in Theorem 1, as follows. If $\langle a, b\rangle$ is finite, $\mu = -(p+q)$ and $\nu = aq+bp$; if only a is finite, $\nu = p-\mu a$; if only b is finite $\nu = -p-\mu b$.

THEOREM 2. *In order that a second-order linear differential operator*

$$Lx = a_0(t)x'' + a_1(t)x' + a_2(t)x$$

admits a sequence $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, *of polynomials, orthogonal with respect to some non-trivial positive measure* $d\sigma$ ($d\sigma$ *must satisfy necessarily the conditions imposed in Theorem* 1), *it is necessary and sufficient that* L *shall be of the form*

$$Lx = (\alpha t^2 + \beta t + \gamma)x'' + (\mu t + \nu)x' + \eta x\,,$$

where $\alpha, \beta, \gamma, \mu, \nu, \eta$ *are real constants that satisfy one of the following conditions*:

(1) $$\alpha\mu(\beta\nu - \gamma\mu) > \alpha^2\nu^2 \quad \textit{and} \quad \operatorname{sgn}\alpha = \operatorname{sgn}\mu;$$

or

(2) $$\alpha = 0\,, \quad \mu \neq 0 \quad \textit{and} \quad \beta\nu - \gamma\mu > 0\,.$$

If this is the case, then the density of the measure is, except for a constant factor,

$$\varrho(t) = a_0^{-1} \exp \int (a_1/a_0)\,dt\,,$$

and the orthogonality is fulfilled on the following intervals. If (1) *holds, in* $\langle t_1, t_2\rangle$, *where* t_1 *and* t_2 *are the (necessarily existent, because of* (1)) *real and different roots of* $a_0(t)$. *If* (2) *holds, in* $\langle t_0, +\infty)$ *if* $\operatorname{sgn}\beta = -\operatorname{sgn}\mu$, *in* $(-\infty, t_0\rangle$ *if* $\operatorname{sgn}\beta = \operatorname{sgn}\mu$, *and in* $(-\infty, +\infty)$ *if* $\beta = 0$, *where* $t_0 = -\gamma/\beta$ *is the root of* $a_0(t)$.

So, in every case it is $\mu \neq 0$ and the root $z_0 = -\nu/\mu$ of $a_1(t)$ lies in the interior of the interval of orthogonality, and there is orthogonality when $\operatorname{sgn} a_0(z_0) = -\operatorname{sgn} a_1'(z_0)$. (In particular, it is not difficult to see that the condition $\alpha\mu(\beta\nu-\gamma\mu) > \alpha^2\nu^2$ means $\alpha \neq 0$, $\beta^2 - 4\alpha\gamma > 0$ and $t_1 < z_0 < t_2$.)

THEOREM 3. *In order that the polynomials* $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, *shall be admitted by a second-order linear differential operator*

$$Lx = a_0(t)x'' + a_1(t)x' + a_2(t)x\,,$$

and the P_n*'s shall be orthogonal with respect to a non-trivial positive measure* $d\sigma$ ($d\sigma$ *must satisfy necessarily the conditions in Theorem* 1, *and with* $\varrho(t) = a_0^{-1}\exp\int(a_1/a_0)\,dt$ *by Theorem* 2), *it is necessary and sufficient that, except for a constant factor depending on* n, *the polynomials shall be given by the following generalized Rodriques formula:*

$$\begin{aligned} P_n(t) &= [\varrho(t)]^{-1}\frac{d^n}{dt^n}\{[a_0(t)]^n\varrho(t)\} \\ &= a_0 \exp\left\{-\int (a_1/a_0)\,dt\right\}\frac{d^n}{dt^n}\left\{a_0^{n-1}\exp\int (a_1/a_0)\,dt\right\}, \end{aligned}$$

where $a_0(t)$ *and* $a_1(t)$ *are subjected to the conditions imposed in Theorem* 2.

We observe then that, except for irrelevant constants, each system $\{P_n\}$, $n = 0, 1, 2, \ldots$, L and ϱ are determined by four real numbers a, b, μ, ν (or equivalently, a, b, p, q) where a and b may be eventually resp. $-\infty$ and $+\infty$. Given an interval $\langle a, b\rangle$, we obtain, varying μ and ν, all the orthogonal polynomials $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, admitted by some L. We conclude then:

THEOREM 4. *Except for a constant factor depending on* n, *the following are all the orthogonal (with respect to some non-trivial positive measure) polynomials* $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, *admitted by a second-order linear differential operator*:

Type A (*Jacobi*):

$$P_n(t) = (b-t)^{1-q}(t-a)^{1-p}\frac{d^n}{dt^n}\{(b-t)^{n+q-1}(t-a)^{n+p-1}\}\,, \qquad p > 0\,,\ q > 0\,;$$

Type B (*Laguerre*):

$$P_n(t) = e^{\mu t}(t-c)^{1-p}\frac{d^n}{dt^n}\{(t-c)^{n+p-1}e^{-\mu t}\}, \qquad \mu \neq 0\,,\ p > 0;$$

Type C (*Hermite*):

$$P_n(t) = e^{-\frac{1}{2}\mu t^2-\nu t}\frac{d^n}{dt^n}\{e^{\frac{1}{2}\mu t^2+\nu t}\}, \qquad \mu < 0\,.$$

The first ones are orthogonal on $\langle a, b\rangle$; *the second ones in* $\langle c, +\infty)$ *if* $\mu > 0$ *and on* $(-\infty, c\rangle$ *if* $\mu < 0$; *the third ones in* $(-\infty, +\infty)$. *The measure is given in Theorem* 1 *and the operator in Theorem* 2.

The polynomials of type A are the Jacobi polynomials; those of type B are the generalized Laguerre polynomials; and those of type C are esentially the Hermite polynomials. There are no other orthogonal (with respect to some non-trivial positive measure) polynomials admitted by a second-order linear differential operator.

A more detailed discussion about polynomials admitted by an L of second order, without considering the problem of orthogonality, can be seen in [1].

§ 2. Proof of Theorems 1 and 2.

PROPOSITION 1. *If a second-order linear differential operator* L *admits three polynomials* P_0, P_1 *and* P_2, *with* $\deg P_i = i$, *then it is of the type*

$$(*) \qquad Lx = (\alpha t^2+\beta t+\gamma)x''+(\mu t+\nu)x'+\eta x\,,$$

where $\alpha, \beta, \gamma, \mu, \nu, \eta$ *are* (*real*) *constants. Such* L *admits one and only one polynomial*

$$P_n(t) = \sum_{h=0}^{n} a_{nh}t^h\,, \qquad a_{nn} = 1\,,$$

for every $n = 0, 1, 2, \ldots$ *if and only if* $\mu \neq 0$ *and* $-\mu/\alpha$ *is not a natural number. In this case* (*and only in this case*) *the eigenvalues* λ_n *are all different.*

Proof. The fact that L is of type $(*)$ is immediate. Moreover, replacing P_n in $Lx = \lambda x$, we obtain

$$\lambda_n = \alpha n^2+(\mu-\alpha)n+\eta$$

and the recursion formula, for $h = n-1, \ldots, 0$ (writing $a_{n,n+1} = 0$),

$$(n-h)[\alpha(n+h-1)+\mu]a_{nh} = [\beta h^2+(\nu+\beta)h+\nu]a_{n,h+1}+\gamma(h^2+3h+2)a_{n,h+2},$$

and P_n is uniquely determined if and only if the coefficient of a_{nh} in this formula is non-zero for every h. The proposition follows easily from this.

A more detailed discussion about polynomials admitted by an L of second order can be seen in [1].

Our purpose is to study the relations between the P_n's admitted by some L and their orthogonality with respect to some measure.

We shall say that a measure $d\tilde{\sigma}$ is *equivalent* to $d\sigma$ when $d\tilde{\sigma}(t) = C\,d\sigma(At+B)$, A, B, C being real constants with $A \neq 0$ and $C \neq 0$.

We shall say that an operator $\tilde{L}$ is *equivalent* to an L, when there exist real constants A, B, M, N, with $A \neq 0$, $M \neq 0$, such that $\tilde{L}_t = ML_{At+B}+N$, where the index in the operator means the variable to which it applies. It is easy to verify that in order to study orthogonality relations between polynomials admitted by an L, it is irrelevant to substitute $d\sigma$ and L by equivalents.

If $d\sigma$ is not a point mass, in particular, if it is non-trivial, there exists, choosing suitably A, B and C, an equivalent measure $d\tilde{\sigma}$ normalized in the sense that $M_0 = 1$, $M_1 = 0$, $M_2 = 1$. So, we can suppose, with no restriction of generality, that $d\sigma$ is already normalized. Also, for each L of type (*), there exists, choosing N suitably, an equivalent operator such that $\eta = 0$.

In the next proposition we shall use the following fact ([4], p. 5, Th. 1.2): If $d\sigma$ is a positive measure, then $\Delta^n(M) = \det(M_{i+j})_{i,j=0}^{n} \geqslant 0$ for every non-negative integer n. If, in addition, $d\sigma$ is non-trivial, then the sign $\geqslant$ must be replaced by $>$.

PROPOSITION 2. *Suppose that a second-order linear differential operator L admits a sequence $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, of orthogonal polynomials with respest to a non-trivial positive measure $d\sigma$. The L is equivalent to*

$$Lx = (\alpha t^2+\beta t+1-\alpha)x''-tx', \qquad \text{with} \qquad \alpha \leqslant 0 .$$

Proof. We suppose, choosing suitably A, B, C, that $d\sigma$ is normalized. Moreover, since L admits P_0, P_1, P_2, we know that L is of type (*), and therefore, choosing N suitably, equivalent to

$$Lx = (\alpha t^2+\beta t+\gamma)x''+(\mu t+\nu)x' .$$

Since L admits $P_0 = 1$ and $P_1 = t+a_{10}$, it is easy to show that $\nu = \mu a_{10}$. The orthogonality relation $\int P_0 P_1 d\sigma = 0$ implies, $d\sigma$ being normalized, $a_{10} = 0$, and therefore it is

$$Lx = (\alpha t^2+\beta t+\gamma)x''+\mu t x' .$$

Now we prove that $\mu \neq 0$. In fact, if it were not so, it would be $\alpha \neq 0$, since on the contrary, because of the fact that L admits P_2, it would be $\alpha = \beta = \gamma = 0$, and L would not be of second order. Therefore, choosing M

suitably, L would be equivalent to $Lx = (t^2+\beta t+\gamma)x''$. Calculating P_i, $i = 0, 1, 2, 3$, from the orthogonality relations

$$\int P_0 P_1 d\sigma = \int P_0 P_2 d\sigma = \int P_1 P_2 d\sigma = \int P_1 P_3 d\sigma = 0$$

we obtain $\gamma = -1$, $M_3 = -\beta$ and $M_4 = \beta^2+1$, which implies $\Delta^2(M) = 0$, contradiction because $d\sigma$ is non-trivial. Therefore $\mu \neq 0$ and, choosing M suitably, L is equivalent to

$$Lx = (at^2+\beta t+\gamma)x'' - tx' .$$

Calculate now P_0, P_1, P_2. The existence of P_2 says that $a = 1$ implies $\gamma = 0$, and if $a \neq 1$, the orthogonality relation $\int P_0 P_2 d\sigma = 0$ says $1+\gamma(a-1)^{-1} = 0$. Thus in every case it is $a+\gamma = 1$ and L is

$$Lx = (at^2+\beta t+1-a)x'' - tx' .$$

It remains to prove that $a \leqslant 0$. In a first step we shall prove that $a < \frac{1}{2}$. In fact, calculating the first four polynomials, it is possible to verify the following assertions:

(a) the fact that L admits P_3 implies $a \neq \frac{1}{3}$;

(b) if $a = \frac{1}{2}$, then $\beta = 0$ because L admits P_2;

(c) if $a \neq \frac{1}{2}$, $\int P_1 P_2 d\sigma = 0$ gives $M_3 = -\beta(a-\frac{1}{2})^{-1}$;

(d) $\int P_1 P_3 d\sigma = 0$ gives $M_4 = -3$ if $a = \frac{1}{2}$, which is impossible, $d\sigma$ being positive, and

$$M_4 = \frac{3}{2}\cdot\frac{\beta^2}{(a-\frac{1}{4})(a-\frac{1}{2})} - \frac{1}{2}\cdot\frac{\beta^2}{(a-\frac{1}{4})(a-\frac{1}{3})} + \frac{a-1}{a-\frac{1}{3}} \quad \text{if} \quad a \neq \tfrac{1}{2},$$

where the first two terms must be suppressed if $a = \frac{1}{4}$.

Therefore, $a \neq \frac{1}{2}, \frac{1}{3}$, and

$$\begin{aligned}\Delta^2(M) &= M_4 - M_3^2 - 1\\ &= \left\{\frac{3}{2}\cdot\frac{\beta^2}{(a-\frac{1}{4})(a-\frac{1}{2})} - \frac{1}{2}\cdot\frac{\beta^2}{(a-\frac{1}{4})(a-\frac{1}{3})} - \frac{\beta^2}{(a-\frac{1}{2})^2}\right\} + \left\{\frac{a-1}{a-\frac{1}{3}} - 1\right\}\end{aligned}$$

suppresing the corresponding terms when $a = \frac{1}{4}$. The second $\{\ \}$ is always < 0. This implies $a < \frac{1}{2}$, since on the contrary the first $\{\ \}$ is < 0, as it is easy to verify, and consequently $\Delta^2(M) < 0$, contradiction. Our claim is then proved.

Consider now the recursion formula for the coefficients of $P_n = \sum_{h=0}^{n} a_h t^h$, $a_n = 1$. It is $\lambda_n = n[a(n-1)-1]$ and, writing $a_{n+1} = 0$, for $h = n-1, \dots, 0$,

$$(n-h)[a(n+h-1)-1]a_h = \beta(h+1)h a_{h+1} + (1-a)(h+2)(h+1)a_{h+2} .$$

In this formula the coefficient of a_{h+2} is positive, since $\alpha < \frac{1}{2} < 1$, and those of a_{h+1} has the same sign of β, except perhaps when $h = 0$, being zero. We will suppose now $\alpha > 0$ and arrive to a contradiction.

If we choose $n > \alpha^{-1}+1$, the coefficient of a_h is positive for every $h = n-1, \ldots, 0$. From this we obtain easily that $a_{n-2k} > 0$ and $\operatorname{sgn} a_{n-2k-1} = \operatorname{sgn}\beta$.

On the other hand, it is well known ([3], Th. V, p. 38) that the zeros of P_n are real, simple, and separated by those of P_{n-1}. This implies, using the fact that $P_1 = t$ has the zero $t = 0$, that each P_n, $n > 1$, has at least a negative zero and a positive one. If n is sufficiently large, the argument above shows that $\beta \geqslant 0$ implies $P_n > 0$ for $t > 0$, and $\beta \leqslant 0$ implies either $P_n > 0$ or $P_n < 0$ for $t < 0$ according n is either even or odd. In any case we get a contradiction. Consequently it is $\alpha \leqslant 0$ and the proposition is proved.

We shall see now that if L admits polynomials, such polynomials are, under certain circumstances, orthogonal with respect to some special measure. To see this, we define in case L is of type (*),

$$\varrho(t) = a_0^{-1}\exp\int (a_1/a_0)\,dt\,.$$

If $\alpha \neq 0$ and $a_0(t)$ has real and different zeros, we denote them by t_1 and t_2 $(t_1 < t_2)$; if $\alpha = 0$ and $\beta \neq 0$, we call t_0 the (real) zero of $a_0(t)$; if $\mu \neq 0$, we call z_0 the (real) zero of $a_1(t)$.

PROPOSITION 3. *If L admits the sequence $\{P_n\}$, $n = 1, 2, \ldots, \deg P_n = n$, of polynomials, with eigenvalues all different, then the polynomials are orthogonal with respect to the measure with density $\varrho = a_0^{-1}\exp\int (a_1/a_0)\,dt$, in the following cases, and on the interval mentioned in each case*:

(a) *if* $\alpha \neq 0$, $t_1 < z_0 < t_2$ *and* $\operatorname{sgn}\alpha = \operatorname{sgn}\mu$, *on* $\langle t_1, t_2\rangle$;

(b) *if* $\alpha = 0$, $\beta \neq 0$, $z_0 < t_0$ *and* $\operatorname{sgn}\beta = \operatorname{sgn}\mu$, *on* $(-\infty, t_0\rangle$;

(c) *if* $\alpha = 0$, $\beta \neq 0$, $z_0 > t_0$ *and* $\operatorname{sgn}\beta = -\operatorname{sgn}\mu$, *on* $\langle t_0, +\infty)$;

(d) *if* $\alpha = \beta = 0$, $\gamma \neq 0$ *and* $\operatorname{sgn}\gamma = -\operatorname{sgn}\mu$, *on* $(-\infty, +\infty)$.

Proof. ϱL is formally self-adjoint; so Green's formula says

$$(\varrho Lx, y) - (x, \varrho Ly) = [x, y](b) - [x, y](a)\,,$$

where

$$(x, y) = \int_a^b x(t)y(t)\,dt \quad \text{and} \quad [x, y] = \varrho a_0(x'y - xy')\,.$$

Appying this formula to two polynomials admitted by L, we see that it suffices to show, in the mentioned cases, that $\varrho a_0 = 0$ at a and b, in the sense that $\varrho a_0(t) \to 0$ when $t \to b-0$ and $t \to a+0$, more rapidly than any polynomial tends to infinity. This is an immediate consequence of the following formulas, obtainable by a direct calculation:

$$\varrho a_0(t) = \begin{cases} |t_2 - t|^{a_1(t_2)/a(t_2-t_1)} |t - t_1|^{-a_1(t_1)/a(t_2-t_1)} & \text{if} \quad a \neq 0; \\ e^{(\mu/\beta)t} |\beta t + \gamma|^{a_1(t_0)/\beta}, & \text{if} \quad a = 0,\ \beta \neq 0; \\ e^{(\mu/2\gamma)t^2 + (\nu/\gamma)t}, & \text{if} \quad a = \beta = 0,\ \gamma \neq 0. \end{cases}$$

Thus, the proposition is proved.

Now we will show that the measure $d\sigma$ in Proposition 2 is precisely $\varrho\, dt$. For this, observe first that if the polynomials $\{P_n\}$, $n = 0, 1, 2, \ldots$, $\deg P_n = n$, are orthogonal with respect to two measures with the same total mass, then all the moments of the difference measure vanish. This fact is easily obtainable by induction.

PROPOSITION 4. *With the same assumptions as in Proposition* 2, *it is* $d\sigma = \varrho\, dt$, *except for a constant factor* (*where* ϱ *is defined above and the* a_i*'s are the coefficients of* L), *on the intervals mentioned in the following, and* $\varrho = 0$ *outside them*:

If $a \neq 0$, *on* $\langle t_1, t_2 \rangle$ (*the* t_i*'s are real and different, because it is* $a \leqslant 0$);

If $a = 0$, *on* $(-\infty, t_0\rangle$ *if* $\beta < 0$; *on* $\langle t_0, +\infty)$ *if* $\beta > 0$; *and on* $(-\infty, +\infty)$ *if* $\beta = 0$.

Proof. By Proposition 1, the polynomials P_n that appear in Proposition 2 are uniquely determined, and by Proposition 3 they are orthogonal with respect to $\varrho\, dt$. On the other hand, they are also orthogonal with respect to $d\sigma$, by assumption in Proposition 2. Therefore, because of the remark above, all the moments of $d\sigma - \varrho\, dt$ vanish (we can suppose $\int d\sigma = \int \varrho\, dt$, multiplying ϱ by a suitable constant).

Therefore, we only need to prove that the problem of moments $M_k = \int t^k d\sigma$, where $M_k = \int t^k \varrho\, dt$, has a unique solution, which will be then $\varrho\, dt$. A sufficient condition for this to be true is ([4], p. 19, Th. 1.10)

$$\sum_{k=1}^{\infty} (M_{2k})^{-1/2k} = +\infty .$$

Let us see that this actually happens in our case $M_k = \int t^k \varrho\, dt$. In case $a \neq 0$, in which ϱ has compact support, the condition is obviously satisfied, and it is also satisfied in case $a = \beta = 0$, because in this case it is

$$\varrho(t) = \exp(-t^2/2) \quad \text{and} \quad M_{2k} = \int_{-\infty}^{\infty} t^{2k} e^{-t^2/2} dt = \sqrt{2\pi} \cdot 1 \cdot 3 \cdot 5 \ldots (2k-1) .$$

It remains only to consider the case $a = 0$, $\beta > 0$ ($\beta < 0$ is similar). In this case,

$$\varrho(t) = Ce^{-t/\beta} |\beta t + 1|^{(1/\beta^2) - 1},$$

and then, being $A > -1$,

$$M_{2k} \leqslant C \int_0^\infty (t-1)^{2k} \beta^{-2k-1} e^{-t/\beta^2} t^A \, dt .$$

Consider the two integrals $\int_0^1$ and $\int_1^\infty$. The first one is $\leqslant C\beta^{-2k-1}$, and the second one is $\leqslant C[\beta(2k+A)]^{2k+A}$. The condition is now easily verified, and the proposition is therefore proved.

Now we are able to fulfill the purpose of this section proving Theorems 1 and 2. Let us begin with the sufficient conditions of both theorems.

A direct calculation shows that the function $\varrho(t) = a_0^{-1}\exp\int(a_1/a_0)\,dt$ is precisely the $\varrho(t)$ of Theorem 1. Because of the conditons on μ and ν, $\varrho(t)$ is summable on $\langle a, b\rangle$. Since in L, $-\mu/a$ is not greater than or equal to zero (a being the coefficient of t^2 in a_0), because of Proposition 1, L admits one and only one polynomial P_n of degree n, for every n, and with eigenvalues all different. By Proposition 3, such P_n are orthogonal with respect to $\varrho\,dt$ in $\langle a, b\rangle$. This shows the sufficient conditions of Theorems 1 and 2.

Consider now the necessary conditions. By Proposition 2, L has one of the forms given in Theorem 2, and by Proposition 4 it is $d\sigma = \varrho\,dt$, except by a constant factor. This proves Theorem 2. Moreover, a direct calculation shows that $\varrho(t)$ is precisely the function of Theorem 1, and being $\varrho(t)$ summable on $\langle a, b\rangle$, μ and ν must have the mentioned properties. This proves Theorem 1.

Finally, we describe the situation when $d\sigma$ is a trivial measure, without proofs. $d\sigma$ must consists of either one or two point masses. If it is a single mass, L is equivalent either to $Lx = (at^2+\beta t)x'' - tx'$, with $1/a$ not natural, or to $Lx = -t^2x''+ptx'$, p being a non-negative integer. If $d\sigma$ is constituted by two point masses, then L is equivalent to $Lx = (t^2+\beta t-1)x''$. The actual form of the polynomials in this cases, without refering to the orthogonality properties, can be seen in [1].

§ 3. Associated polynomials and proof of Theorem 3. Suppose $Lx = a_0x''+a_1x'+a_2x$ satisfies the conditions imposed in Theorem 2. We will use the notation $(d^h/dt^h)f = f^{(h)}$.

Definition. Let n and k be non-negative integers, and $k \leqslant n$. The polynomial $P_{nk} = a_0^{k-n}\varrho^{-1}(a_0^n\varrho)^{(k)}$ is called the *associated polynomial* (with respect to L) *of order n and degree k.*

Let us see first that P_{nk} is actually a polynomial, a fact which is also true for $k > n$.

From $\varrho' = (a_1 - a_0')a_0^{-1}\varrho$ we get $\varrho^{(i)} = R_i a_0^{-i}\varrho$, R_i being a polynomial. Moreover, it is $(a_0^n)^{(j)} = S_{nj}a_0^{n-j}$ with S_{nj} a polynomial. Consequently,

$$P_{nk} = \sum_{j=0}^{k} \binom{k}{j} S_{nj} R_{k-j} ,$$

and our claim is proved.

We will prove now that $\deg P_{nk} \leqslant k$; later on it will be seen that $\deg P_{nk} = k$.

From the definition of S_{nj} we get $S_{n0} = 1$ and $S_{nj} = S'_{n,j-1}a_0 + (n-j+1)S_{n,j-1}a_0'$. Since $\deg a_0 \leqslant 2$, we obtain by induction $\deg S_{nj} \leqslant j$. Analogously, we get $R_0 = 1$ and $R_i = a_0 R'_{i-1} + (a_1 - ia_0')R_{i-1}$, which implies $\deg R_i \leqslant i$. Therefore $\deg P_{nk} \leqslant k$.

On the other hand, if $0 \leqslant h < n$,

$$(a_0^n \varrho)^{(h)} = a_0^{n-h-1} \exp\left(\int (a_1/a_0)\,dt\right) \sum_{j=0}^{h} \binom{h}{j} S_{nj} R_{h-j}$$

implies $(a_0^n \varrho)^{(h)} \equiv 0$ at a and b, in the sense that $(a_0^n\varrho)^{(h)}(t) \to 0$ when $t \to a+0$ and $t \to b-0$, more rapidly than any polynomial tends to infinity. Applying this fact we have, if $k > h$,

$$\int P_{nh}P_{km}a_0^{m-k}\varrho\,dt = \int P_{nh}(a_0^m \varrho)^{(k)}dt = (-1)^k \int (P_{nh})^{(k)} a_0^m \varrho\,dt = 0 ,$$

because $\deg P_{nh} \leqslant h < k$.

This implies in particular that for each $k \geqslant 0$ fixed, the polynomials $P_{n,n-k}$, $n = k, k+1, \ldots$, are orthogonal with respect to $a_0^k \varrho\,dt$, or what is the same, the functions $P_n^k = P_{n,n-k}a_0^{\frac{1}{2}k}$ are orthogonal with respect to $\varrho\,dt$. This fact is well known when L is the Legendre's operator; in this case the P_n^k's are usually called associated functions.

Now we are able to prove $\deg P_{nk} = k$, for $k \leqslant n$. If $k = 0$ it is $P_{n0} = 1$, so $\deg P_{n0} = 0$. Suppose now $\deg P_{nh} = h$ for $h < k$. We will suppose $\deg P_{nk} = k' < k$ and arrive to a contradiction. In fact, because of these assumptions we must have

$$P_{nk} = \sum_{h=0}^{k-1} c_h P_{nh} ,$$

which implies

$$\int (P_{nk})^2 a_0^{n-k}\varrho\,dt = \sum_{h=0}^{k-1} c_h \int P_{nk}P_{nh}a_0^{n-k}\varrho\,dt = 0 ,$$

because of the orthogonality formula above. Consequently $P_{nk} = 0$, that is $a_0^n \varrho = T_{k-1}$, T_{k-1} being a polynomial of degree at most $k-1$. This contradicts the fact that ϱ is summable. In synthesis, PROPOSITION. *The associated polynomials P_{nk} have degree k. Moreover, $k > h$ implies*

$$\int P_{nh} P_{mk} a_0^{m-k} \varrho \, dt = 0 ;$$

in particular, for each $k \geqslant 0$, the $P_{n,n-k}$'s, $n = k, k+1, \ldots$, are orthogonal with respect to $a_0^k \varrho \, dt$. Moreover, the P_{nk}'s satisfy the differential equation

$$a_0 x'' + [a_1 + (n-k) a_0'] x' - [k a_1' + \tfrac{1}{2} a_0'' k (2n-k-1)] x = 0 .$$

In particular, this proposition says that $\deg P_{nn} = n$ and the P_{nn}'s are orthogonal with respect to $\varrho \, dt$. By uniqueness we conlcude that $P_{nn} = P_n$, except for a constant factor depending on n, and Theorem 3 is therefore proved.

In spite of the fact that the P_{nk}'s are polynomials even in the case $k > n$, the proposition above does not hold in general in this case. So, $\deg P_{nk} = k$ is false for Chebychev polynomials, and true for Legendre's polynomials if $k \leqslant 2n$.

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