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HETEROGENEOUS QUEUEING SYSTEMS $M/M_i/2$ WITH BALKING

1. Introduction. Singh studied in [2] a two-channel queueing system $M/M_i/2$ with input intensity λ and different service rates μ_1 and μ_2 in two channels. The following queueing discipline was assumed:

(a) If both channels are busy when a new unit applies to service it joins the queue with probability β and abandons the system with probability $1-\beta$.

(b) If only one channel is occupied, the new unit with probability 1 enters the empty channel and service starts immediately.

(c) If both channels are empty, the new unit with probability 1 chooses the faster channel and occupies it immediately.

This system has been compared with a homogeneous system $M/M/2$ having equal service rates $\frac{1}{2}(\mu_1+\mu_2)$ in both channels. A necessary and sufficient condition has been found under which the steady-state probabilities p_k $(k = 1, 2, \ldots)$ of having k units in a heterogeneous system are smaller than those for the corresponding homogeneous case. Singh has also shown that in a class of heterogeneous systems with a given total of service rates $\mu = \mu_1+\mu_2$ the minimum of the expected value (as well as that of the variance) of the queue length is achieved for $\mu_2 = \lambda\left(\sqrt{1+\mu/\lambda}-1\right)$.

Here a more general case of the heterogeneous system $M/M_i/2$ is considered. We assume that the balking probability now depends on the length of queue. If there are k $(k \geqslant 0)$ units in the system, the new unit decides to enter with probability q_k and abandons the system with probability $1-q_k$. Furthermore, if both channels are empty, the entering unit chooses the first channel with probability π_1, and the second channel — with probability $\pi_2 = 1-\pi_1$.

2. Steady-state distribution of the length of queue. Assuming they exist, let us define the following steady-state probabilities:

p_k — probability of k units present in the described system $(k = 0, 1, \ldots)$;

$p_{1,0}$ — probability of one unit present in the system and being served in the first channel;

$p_{0,1}$ — probability of one unit present in the system and being served in the second channel.

They have to satisfy the following system of equations:

$$
\begin{aligned}
-\lambda q_0 p_0 + \mu_1 p_{1,0} + \mu_2 p_{0,1} &= 0, \\
-(\lambda q_1 + \mu_1) p_{1,0} + \mu_2 p_2 + \lambda q_0 \pi_1 p_0 &= 0, \\
-(\lambda q_1 + \mu_2) p_{0,1} + \mu_1 p_2 + \lambda q_0 \pi_2 p_0 &= 0, \\
-(\lambda q_n + \mu) p_n + \lambda q_{n-1} p_{n-1} + \mu p_{n+1} &= 0, \quad n = 2, 3, \ldots, \\
p_1 = p_{1,0} + p_{0,1}, \quad p_0 + p_1 + \ldots &= 1.
\end{aligned}
$$

The solution of this system is of the form

(1) $$p_1 = A p_0, \quad p_n = \varrho^{n-1} a_{n-1} p_1 = A \varrho^{n-1} a_{n-1} p_0, \; n = 2, 3, \ldots,$$

where

(2) $$A = \frac{\lambda q_0 [\lambda q_1 + \mu_1 + (\mu - 2\mu_1)\pi_1]}{\mu_1(\mu - \mu_1)(2\varrho q_1 + 1)},$$
$$a_n = \prod_{k=1}^{n} q_k, \quad \varrho = \frac{\lambda}{\mu_1 + \mu_2} = \frac{\lambda}{\mu}$$

and

(3) $$p_0 = 1 \Big/ \Big(1 + A + A \sum_{k=1}^{\infty} \varrho^k a_k\Big).$$

It follows from these formulas that the convergence of the series in (3) is a necessary condition for the existence of the steady-state distribution (1). Let us notice that the convergence depends only on λ, μ and q_k.

Knowing the solution (1) we can evaluate the following parameters:

the expected value $\mathrm{E}(Q)$ of the number of units in the system,

(4) $$\mathrm{E}(Q) = \sum_{n=1}^{\infty} n p_n = p_1 \Big\{1 + \sum_{n=2}^{\infty} n \varrho^{n-1} a_{n-1}\Big\};$$

the expected queue length

(5) $$\mathrm{E}(L) = \sum_{n=3}^{\infty} (n-2) p_n = p_1 \sum_{n=3}^{\infty} (n-2) \varrho^{n-1} a_{n-1};$$

the expected virtual waiting time

(6) $$\mathrm{E}(V) = \sum_{n=2}^{\infty} \frac{n-1}{\mu} p_n = \frac{p_1}{\mu} \sum_{n=2}^{\infty} (n-1) \varrho^{n-1} a_{n-1};$$

the expected waiting time $\mathrm{E}(W)$ for the units which do not resign,

$$\mathrm{E}(W) = \sum_{n=2}^{\infty} \frac{n-1}{\mu} q_n p_n \Big/ \sum_{n=0}^{\infty} q_n p_n \tag{7}$$

$$= \frac{A}{\mu} \sum_{n=2}^{\infty} (n-1) \varrho^{n-1} a_n \Big/ \Big\{ q_0 + A \sum_{n=1}^{\infty} \varrho^{n-1} a_n \Big\};$$

the expected number of resigning units $\mathrm{E}(R)$ in the time interval of length one,

$$\mathrm{E}(R) = \lambda \sum_{n=0}^{\infty} (1-q_n) p_n = \lambda p_1 \sum_{n=0}^{\infty} (1-q_n) \varrho^{n-1} a_{n-1}. \tag{8}$$

3. Efficiency of the system. From now on the values of parameters λ, μ and q_k $(k = 0, 1, \ldots)$ are fixed and we assume that they are chosen so that the steady-state distribution exists. A system is then completely specified by giving the values of μ_1 and π_1. By $S(\mu_1, \pi_1)$ we denote the system with given values of μ_1 and π_1; by $S(\cdot, \pi_1)$ — the class of all systems with $\mu_1 \in [0, \mu]$ and fixed π_1; by $S(\mu_1, \cdot)$ — the class of all systems with fixed μ_1 and $\pi_1 \in [0, 1]$; and by $S(\cdot, \cdot)$ — the class of all systems with $\mu_1 \in [0, \mu]$ and $\pi_1 \in [0, 1]$.

We compare now two systems

$$S^{(1)} = S(\mu_1^{(1)}, \pi_1^{(1)}) \quad \text{and} \quad S^{(2)} = S(\mu_1^{(2)}, \pi_1^{(2)})$$

belonging to $S(\cdot, \cdot)$. Let us denote by $\mathrm{E}(Q^{(i)})$, $\mathrm{E}(L^{(i)})$, $\mathrm{E}(V^{(i)})$, $\mathrm{E}(W^{(i)})$ and $\mathrm{E}(R^{(i)})$ the corresponding expected values (4)-(8) for the system $S^{(i)}$ $(i = 1, 2)$.

Definition 1. The system $S^{(1)}$ is called *more efficient than* $S^{(2)}$ if the inequalities

$$\begin{gathered} \mathrm{E}(Q^{(1)}) < \mathrm{E}(Q^{(2)}), \quad \mathrm{E}(L^{(1)}) < \mathrm{E}(L^{(2)}), \\ \mathrm{E}(V^{(1)}) < \mathrm{E}(V^{(2)}), \quad \mathrm{E}(W^{(1)}) < \mathrm{E}(W^{(2)}), \quad \mathrm{E}(R^{(1)}) < \mathrm{E}(R^{(2)}) \end{gathered} \tag{9}$$

hold.

Definition 2. The system $S^{(1)}$ is called *equally efficient as* $S^{(2)}$ if

$$\begin{gathered} \mathrm{E}(Q^{(1)}) = \mathrm{E}(Q^{(2)}), \quad \mathrm{E}(L^{(1)}) = \mathrm{E}(L^{(2)}), \\ \mathrm{E}(V^{(1)}) = \mathrm{E}(V^{(2)}), \quad \mathrm{E}(W^{(1)}) = \mathrm{E}(W^{(2)}), \quad \mathrm{E}(R^{(1)}) = \mathrm{E}(R^{(2)}). \end{gathered} \tag{10}$$

Now we prove

THEOREM 1. *The set of inequalities* (9) *is equivalent to the single inequality*

$$\frac{\lambda q_1+\mu_1^{(1)}+(\mu-2\mu_1^{(1)})\pi_1^{(1)}}{\mu_1^{(1)}(\mu-\mu_1^{(1)})} < \frac{\lambda q_1+\mu_1^{(2)}+(\mu-2\mu_1^{(2)})\pi_1^{(2)}}{\mu_1^{(2)}(\mu-\mu_1^{(2)})}, \tag{11}$$

and the set of equations (10) *is equivalent to*

$$\frac{\lambda q_1+\mu_1^{(1)}+(\mu-2\mu_1^{(1)})\pi_1^{(1)}}{\mu_1^{(1)}(\mu-\mu_1^{(1)})} = \frac{\lambda q_1+\mu_1^{(2)}+(\mu-2\mu_1^{(2)})\pi_1^{(2)}}{\mu_1^{(2)}(\mu-\mu_1^{(2)})}.$$

Proof. It follows from the definition of A (formula (2)) that (11) is equivalent to

$$A^{(1)} < A^{(2)} \tag{12}$$

which is of course equivalent to

$$\frac{1}{A^{(1)}}+1+\sum_{k=1}^{\infty}\varrho^k a_k > \frac{1}{A^{(2)}}+1+\sum_{k=1}^{\infty}\varrho^k a_k.$$

In view of (1) and (3), the last inequality is equivalent to

$$p_1^{(1)} < p_1^{(2)}. \tag{13}$$

Using formulas (4)-(8) we now easily deduce that (12) and (13) are sufficient and necessary conditions for inequalities (9) to hold. This completes the proof of the first part of the theorem. Replacing in what was said above all inequality signs by equality signs we prove the second part of the theorem.

Three corollaries easily follow from Theorem 1 and its proof.

COROLLARY 1. *Any two systems in* $S(\cdot,\cdot)$ *are comparable, i.e. either one of them is more efficient than the other or both are equally efficient.*

COROLLARY 2. *Symmetrical systems* $S(\mu_1, \pi_1)$ *and* $S(\mu-\mu_1, 1-\pi_1)$ *are equally efficient.*

COROLLARY 3. $S^{(1)}$ *is more efficient than* $S^{(2)}$ *if and only if inequality* (12) *or the equivalent inequality* (13) *holds.*

4. The most efficient systems. Three theorems of this section indicate the most effective systems in different classes.

THEOREM 2. *There exists the most effective system in* $S(\cdot, \pi_1)$. *It is the system*

$$S\left(\frac{\mu\pi_1+\lambda q_1-\lambda\sqrt{q_1^2+q_1/\varrho+\pi_1(1-\pi_1)/\varrho^2}}{2\pi_1-1}, \pi_1\right) \quad \text{if } \pi_1 \neq \frac{1}{2},$$

$$S\left(\frac{\mu}{2}, \frac{1}{2}\right) \quad \text{if } \pi_1 = \frac{1}{2}.$$

Proof. According to Corollary 3 it is enough to prove that the quantity A (defined in (2)) achieves its minimum at

$$\mu_1^0 = \begin{cases} \dfrac{\mu\pi_1 + \lambda q_1 - \lambda\sqrt{q_1^2 + q_1/\varrho + \pi_1(1-\pi_1)/\varrho^2}}{2\pi_1 - 1} & \text{if } \pi_1 \neq \dfrac{1}{2}, \\ \dfrac{\mu}{2} & \text{if } \pi_1 = \dfrac{1}{2}. \end{cases}$$

This can be easily checked by calculating the derivatives.

It may be interesting here to mention paper [1] by Gumbel who studied heterogeneous systems $M/M_i/n$ without balking, assuming random choice of an empty channel by a unit which enters the system while two or more channels are empty. His results for $n = 2$ coincide with the statement of Theorem 2 for $\pi_1 = 1/2$.

From Theorem 2 and Corollary 3 we have

COROLLARY 4. *The most efficient in* $S(\cdot, 0)$ *is the system*

$$S\left(\lambda\sqrt{q_1^2 + q_1/\varrho} - \lambda q_1, 0\right)$$

which is equally efficient as the most efficient system

$$S\left(\mu - \lambda\sqrt{q_1^2 + q_1/\varrho} + \lambda q_1, 1\right)$$

in $S(\cdot, 1)$.

A reasoning similar to that in the proof of Theorem 2 allows for the formulation of

THEOREM 3. *The most efficient in* $S(\mu_1, \cdot)$ *is the system* $S(\mu_1, 0)$ *if* $\mu_1 < \mu/2$, *and the system* $S(\mu_1, 1)$ *if* $\mu_1 > \mu/2$. *In* $S(\mu/2, \cdot)$ *all systems are equally efficient.*

Theorems 2, 3 and Corollary 2 bring us to the final result:

THEOREM 4. *In* $S(\cdot, \cdot)$ *two most efficient systems are*

$$S\left(\lambda\sqrt{q_1^2 + q_1/\varrho} - \lambda q_1, 0\right) \quad \textit{and} \quad S\left(\mu - \lambda\sqrt{q_1^2 + q_1/\varrho} + \lambda q_1, 1\right).$$

Let us notice that in the case $q_0 = q_1 = 1$ and $q_k = \beta$ for $k \geqslant 2$ our systems reduce to those studied by Singh in [2]. The most efficient are then the systems

$$S\left(\lambda\sqrt{1 + 1/\varrho} - \lambda, 0\right) \quad \text{and} \quad S\left(\mu - \lambda\sqrt{1 + 1/\varrho} + \lambda, 1\right).$$

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References

[1] H. Gumbel, *Waiting lines with heterogeneous servers*, Opns. Res. 8 (1960), p. 504-511.

[2] V. P. Singh, *Two-server Markovian queues with balking: heterogeneous servers*, ibidem 18 (1970), p. 145-159.

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NIEJEDNORODNE SYSTEMY OBSŁUGI MASOWEJ TYPU $M/M_i/2$ Z REZYGNACJĄ

STRESZCZENIE

W pracy rozważa się dwukanałowy niejednorodny system obsługi masowej $M/M_i/2$ z poissonowskim strumieniem wejścia o intensywności λ i z wykładniczym rozkładem czasu obsługi o intensywnościach μ_1 i μ_2 w dwu kanałach. Zakłada się, że zgłaszająca się do systemu jednostka może zrezygnować z obsługi z prawdopodobieństwem $1-q_k$, zależnym od liczby jednostek w systemie oraz że jednostka zastająca oba kanały wolne wybiera pierwszy z nich z prawdopodobieństwem π_1 i drugi z prawdopodobieństwem $\pi_2 = 1-\pi_1$.

Dla tak zdefiniowanego systemu znaleziono stacjonarny rozkład liczby jednostek w systemie oraz obliczono następujące średnie charakterystyki (wzory (4)-(8)): średnią liczbę jednostek w systemie $E(Q)$, średnią długość kolejki $E(L)$, średni wirtualny czas czekania na obsługę $E(V)$, średni czas czekania na obsługę dla jednostek nie rezygnujących z obsługi $E(W)$ oraz średnią liczbę jednostek rezygnujących z obsługi w jednostce czasu $E(R)$.

Przy założeniu, że systemy $S^{(1)} = S(\mu_1^{(1)}, \pi_1^{(1)})$ i $S^{(2)} = S(\mu_1^{(2)}, \pi_1^{(2)})$, należące do klasy $S(\cdot,\cdot)$, są równie efektywne wtedy i tylko wtedy, gdy spełniony jest układ równań (10), oraz że system $S^{(1)}$ jest efektywniejszy niż system $S^{(2)}$ wtedy i tylko wtedy, gdy spełniony jest układ nierówności (9), pokazano, że systemy symetryczne $S(\mu_1, \pi_1)$ i $S(\mu-\mu_1, 1-\pi_1)$ są równie efektywne, i znaleziono warunek konieczny i wystarczający na to, żeby system $S^{(1)}$ był efektywniejszy niż system $S^{(2)}$ (wniosek 3 z twierdzenia 1).

W twierdzeniach 2, 3 i 4 wskazano najefektywniejsze systemy w następujących klasach systemów:

$S(\cdot, \pi_1)$ — klasa wszystkich systemów z $\mu_1 \in [0, \mu]$ $(\mu = \mu_1+\mu_2)$ i z ustaloną wartością parametru π_1;

$S(\mu_1, \cdot)$ — klasa wszystkich systemów z ustaloną wartością parametru μ_1 i $\pi_1 \in [0, 1]$;

$S(\cdot, \cdot)$ — klasa wszystkich systemów z $\mu_1 \in [0, \mu]$ i $\pi_1 \in [0, 1]$.
