FOURIER ANALYSIS OF THE BANACH INDICATRIX

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Let \( \varphi(t) \) be a continuous function of bounded variation on \( \langle a, b \rangle \), and let \( N_{\varphi}(x) \) denote the number of solutions of the equation \( \varphi(t) = x \), \( t \in \langle a, b \rangle \); \( N_{\varphi}(x) \) may be infinite. The function \( N_{\varphi} \) is known as the Banach indicatrix (cf. [4]). It was proved by Banach [1] that

\[
\int_{-\infty}^{\infty} N_{\varphi}(y) \, dy = \int_{a}^{b} |d\varphi(t)| = \varphi \text{ var} \varphi.
\]

We shall call the function \( \varphi \) piece-wise monotonic if there exists a finite partition \( a = t_0 < t_1 < \ldots < t_n = b \) such that \( \varphi \) is monotonic (non-decreasing or non-increasing) in each interval \( \langle t_{i-1}, t_i \rangle \).

Let \( \varphi \) be continuous and piece-wise monotonic and let \( m_{\varphi}(x) \) denote the number of components of \( \varphi^{-1}(x) \). Obviously, \( 0 \leq m_{\varphi}(x) < \infty \). Now define

\[
N^*_{\varphi}(x) = \begin{cases} 
    m_{\varphi}(x) & \text{if } x \neq \varphi(a), x \neq \varphi(b), \\
    m_{\varphi}(x) - \frac{1}{2} & \text{if } x = \varphi(a) \neq \varphi(b) \text{ or } x = \varphi(b) \neq \varphi(a), \\
    m_{\varphi}(x) - 1 & \text{if } x = \varphi(a) = \varphi(b).
\end{cases}
\]

It is easy to see that \( N^*_{\varphi}(x) \leq N_{\varphi}(x) \) for all \( x \in (-\infty, \infty) \), and the equality holds except for at most a countable set of points. It was proved by Kac [3] that for piece-wise monotonic \( \varphi \) with continuous derivative the formula

\[
N^*_{\varphi}(x) = \frac{1}{\pi} \int_{0}^{\infty} du \left[ \int_{a}^{b} \cos u(\varphi(t) - x) |d\varphi(t)| \right]
\]

holds for all real \( x \). Kac established this formula without referring to the Banach indicatrix. It turns out, however, that by the methods of Fourier analysis a somewhat stronger result than (2) can be deduced from formula (1). We are also able to derive a similar formula for \( N_{\varphi} \) under the assumption that \( \varphi \) is continuous and of bounded variation on \( \langle a, b \rangle \).
In order to state our results we need the definition of \((C, k)\) summability for integrals. Let \(a\) be a continuous function on \((0, \infty)\). Then we put \((k > -1)\) (cf. \([2]\), p. 111)

\[
(C, k) \int_0^\infty a(u) \, du = \lim_{T \to \infty} \int_0^T \left(1 - \frac{u}{T}\right)^k a(u) \, du.
\]

It is easy to see that

\[
(C, 1) \int_0^\infty a(u) \, du = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ \int_0^T a(u) \, du \right] \, d\lambda.
\]

**Theorem 1.** Let \(\varphi\) be a continuous function of bounded variation on \((a, b)\). Then

\[
N_\varphi(x) = (C, 1) \frac{1}{\pi} \int_a^b du \left[ \int_a^b \cos u (\varphi(t) - x) \, d\varphi(t) \right]
\]

for almost all real \(x\). Moreover, the right-hand side of this equality converges to \(\frac{1}{2} [N_\varphi(x_+) + N_\varphi(x_-)]\) at each point \(x\), where the limits \(N_\varphi(x_\pm)\) exist. The convergence is uniform over each finite and closed interval of continuity of \(N_\varphi\).

**Proof.** Notice that the Banach result implies \(N_\varphi \in L^1(-\infty, \infty)\). Let \(J = \langle c, d \rangle, \; -\infty < c < d < \infty,\) and let \(I_E(x)\) be 1 if \(x \in E\) and 0 if \(x \notin E\). Applying (1) to the function

\[
\varphi_J(t) = \max \{c, \min(d, \varphi(t))\}
\]

we obtain

\[
\int_{-\infty}^{\infty} N_{\varphi_J}(y) \, dy = \int_{-\infty}^{\infty} I_J(y) N_\varphi(y) \, dy
\]

\[
= \int_{\varphi^{-1}(J)} |d\varphi(t)| = \int_{a}^{b} I_J(\varphi(t)) |d\varphi(t)|.
\]

Now let \(E\) be a Borel subset of \((-\infty, \infty)\). Then using (3) we can show that

\[
\int_{-\infty}^{\infty} I_E(y) N_\varphi(y) \, dy = \int_{a}^{b} I_E(\varphi(t)) |d\varphi(t)|,
\]

where the left-hand side is the Lebesgue integral and the right-hand side is the Lebesgue-Stieltjes integral. Formula (4) implies that

\[
\int_{-\infty}^{\infty} f(y) N_\varphi(y) \, dy = \int_{a}^{b} f(\varphi(t)) |d\varphi(t)|
\]
holds for any bounded, real valued Borel function. In particular, if $f(y) = \cos u(x - y)$, where $x$ and $u$ are fixed real parameters, equation (5) gives

$$\int_{-\infty}^{\infty} N_\varphi(y) \cos u(y - x) \, dy = \int_{a}^{b} \cos u(\varphi(t) - x) \, d\varphi(t).$$

The partial integral of the Fourier repeated integral of $N_\varphi$ is

$$S_\omega(x) = \frac{1}{\pi} \int_{0}^{\omega} du \left[ \int_{-\infty}^{\infty} N_\varphi(y) \cos u(y - x) \, dy \right],$$

hence by (6)

$$S_\omega(x) = \frac{1}{\pi} \int_{0}^{\omega} du \left[ \int_{a}^{b} \cos u(\varphi(t) - x) \, d\varphi(t) \right].$$

To complete the proof it is sufficient to apply the (1.21) Theorem from [5], p. 246.

The suggestion was made to me by Lee Lorch to employ a Tauberian theorem to improve the summability in Theorem 1. This method leads to a slightly stronger result than that obtained by Kac.

**Theorem 2.** Let $\varphi$ be a piece-wise monotonic and continuous function on $\langle a, b \rangle$. Then for each real $x$ and for any $k > -1$ we have

$$N^*_\varphi(x) = (C, k) \frac{1}{\pi} \int_{a}^{b} du \left[ \int_{a}^{b} \cos u(\varphi(t) - x) \, d\varphi(t) \right].$$

In particular, the integral converges to $N^*_\varphi(x)$.

**Proof.** Let $I_x(y)$ be 0 if $y \neq x$ and 1 if $y = x$. Then by the very definition of $N^*_\varphi$ we have

$$N^*_\varphi(x) = \frac{1}{2} \sum_{a \leq t \leq \varphi(y)} \text{var} I_x(\varphi(t)).$$

The total variation of a given function of bounded variation is an additive function of intervals. Therefore

$$N^*_\varphi(x) = \frac{1}{2} \sum_{j=1}^{n} \text{var} I_x(\varphi(t)), \quad \langle t_{j-1}, t_j \rangle$$

where $\langle t_{j-1}, t_j \rangle$, $j = 1, \ldots, n$, are the intervals of monotonicity of $\varphi$. Let

$$\psi_{\langle t_{j-1}, t_j \rangle}(x) = \frac{\text{var} I_x(\varphi(t))}{\text{var} I_x(\varphi(t))}, \quad j = 1, \ldots, n.$$
One checks that

\[ \psi_j(x) \equiv 0 \quad \text{if} \quad \varphi(t_{j-1}) = \varphi(t_j), \]

and if \( \varphi(t_j) \neq \varphi(t_{j-1}) \), \( \alpha_j = \min[\varphi(t_{j-1}), \varphi(t_j)] \), \( \beta_j = \max[\varphi(t_{j-1}), \varphi(t_j)] \), then

\[
\psi_j(x) = \begin{cases} 
2 & \text{for } x \in (\alpha_j, \beta_j), \\
1 & \text{for } x = \alpha_j \text{ and } x = \beta_j, \\
0 & \text{for } x \notin (\alpha_j, \beta_j).
\end{cases}
\]

Notice that

\[
(8) \quad \psi_j(x) = \frac{\psi_j(x_+) + \psi_j(x_-)}{2} \quad \text{for } x \in (-\infty, \infty),
\]

and that

\[
(9) \quad \var_{(\infty, \infty)} \psi_j(x) \leq 4 \quad \text{for } j = 1, \ldots, n.
\]

Combining (7), (8) and (9) we obtain

\[
(10) \quad \var_{(\infty, \infty)} N^*_\varphi(x) \leq 2n < \infty,
\]

and

\[
(11) \quad N^*_\varphi(x) = \frac{N^*_\varphi(x_+) + N^*_\varphi(x_-)}{2} \quad \text{for } x \in (-\infty, \infty).
\]

We remember that \( N^*_\varphi(y) = N^*_\varphi(y) \) for almost all \( y \). Following step by step the proof of Theorem 1 it is not hard to see that the function \( N^*_\varphi \), in Theorem 1, can be replaced by \( N^*_\varphi \). This and (11) give

\[
(12) \quad N^*_\varphi(x) = (C, 1) \frac{1}{\pi} \int_\infty^b \int_a^b d\varphi \left[ \int_{a}^{b} \cos u (\varphi(t) - a) |d\varphi(t)| \right].
\]

Equation (6) implies

\[
\int_a^b \cos u (\varphi(t) - a) |d\varphi(t)| = \int_{-\infty}^{\infty} N^*_\varphi(y) \cos u (y - x) dy
\]

\[
= \int_{-\infty}^{\infty} N^*_\varphi(y) \cos u (y - x) dy = \frac{1}{u} \int_{-\infty}^{\infty} N^*(y) d\sin u (y - x)
\]

\[
= -\frac{1}{u} \int_{-\infty}^{\infty} \sin u (y - x) dN^*_\varphi(y),
\]
hence, by (10), for large $u$ we get

\begin{equation}
\left| \int_{a}^{b} \cos u(\varphi(t) - x) |d\varphi(t)| \right| \leq \frac{2n}{u} = O(u^{-1}).
\end{equation}

However, (12) and (13) are the hypotheses of the Tauberian theorem for integrals stated in §6.8 on p. 135 of [2]. Applying this theorem we get the required result.

**REFERENCES**


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