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On generalizations of harmonic and killing tensors

by RAMESH CHANDRA SRIVASTAVA (Lucknow, India)

Summary. Harmonic and killing tensors in a compact Riemannian manifold were studied by Yano and Bochner in [2]. In an earlier paper [1] we defined generalized harmonic and killing vectors. In the present paper, harmonic and killing tensors of type q (which, when $q = 1$, reduce to harmonic and killing tensors as given in [2]) have been defined and the corresponding properties are obtained. The last section deals with conformal killing tensors of type q.

1. Introduction. Let V_n be a compact Riemannian manifold with a positive definite metric $g_{ij}dx^i dx^j$, the curvature tensor R^h_{ijk}, the Ricci tensor R_{ij} and the scalar curvature R. We assume that the manifold is of class C^{q+2} and g_{ij} and all the tensors considered in this paper are of class C^{q+1}. For brevity we shall denote

$$\xi_{i_1\ldots i_p} \quad \text{by} \quad \xi_{i_1/i_p},$$

$$\xi_{i_1\ldots i_p;i_{p+1}\ldots i_{p+q}} \quad \text{by} \quad \xi_{i_1/i_p;i_{p+1}/i_{p+q}},$$

and

$$g^{i_1j_1}\ldots g^{i_pj_p} \quad \text{by} \quad g^{i_1j_1/g^{i_pj_p}}.$$

For the convenience of the reader, we state the following three results which we shall use later on:

GREEN'S THEOREM. *In a compact orientable Riemannian manifold V_n, we have*

$$\int_{V_n} \lambda^i;i\,dv = 0,$$

for an arbitrary vector field $\lambda^i(x)$ ([2], p. 31).

BOCHNER'S LEMMA 1. *In a compact Riemannian manifold V_n with positive definite metric, if a function $\varphi(x)$ satisfies*

$$\Delta\varphi \geqslant 0,$$

everywhere in the manifold, then we have $\varphi =$ constant,

$$\Delta\varphi = 0,$$

everywhere in the manifold ([2], p. 30).

BOCHNER'S LEMMA 2. *On a compact Riemannian manifold V_n, if for an arbitrary tensor (or scalar), the r-th successive covariant derivative vanishes,*

$$\xi^{i_1/i_p}{}_{j_1/j_q;k_1/k_r} = 0 \,,$$

then the first derivative already vanishes, that is.

$$\xi^{i_1/i_p}{}_{j_1/j_q;k} = 0$$

([2], p. 170).

2. Some applications of Green's Theorem and Bochner's Lemmas. Let ξ_{i_1/i_p} be a tensor field of order p and

$$\varphi = \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}$$

so that the Laplacian of φ is given by

$$\Delta\varphi = 2(\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}} g^{jk} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}jk} + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q}} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q}}) \,.$$

Now

$$\xi^{i_1/i_p;\, i_{p+1}/i_{p+q}} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} = g^{i_1 j_1}/g^{i_{p+q} j_{p+q}} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} \xi_{j_1/j_p;\, j_{p+1}/j_{p+q}}$$

is a positive definite form in

$$\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}}$$

and therefore if ξ_{i_1/i_p} satisfies an equation of the form

$$g^{kl} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}kl} = \overline{T}_{i_1/i_{p+q-1}\, j_1/j_{p+q-1}} \xi^{j_1/j_p;\, j_{p+1}/j_{p+q-1}} \tag{2.1}$$

and

$$\overline{T} = \overline{T}_{i_1/i_{p+q-1}\, j_1/j_{p+q-1}} \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}} \xi^{j_1/j_p;\, j_{p+1}/j_{p+q-1}} \geqslant 0 \,,$$

then

$$\Delta\varphi \geqslant 0 \,.$$

Consequently from Bochner's Lemma 1

$$\Delta\varphi = 0$$

and in view of the above discussion

$$\begin{cases} \xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} = 0 \,, \\ \overline{T} \equiv \overline{T}_{i_1/i_{p+q-1}\, j_1/j_{p+q-1}} \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}} \xi^{j_1/j_p;\, j_{p+1}/j_{p+q-1}} = 0 \,. \end{cases} \tag{2.1'}$$

And then from Bochner's Lemma 2

$$\xi_{i_1/i_p;\, i_{p+1}} = 0 \,.$$

Thus we have proved

THEOREM 1. *In a compact Riemannian manifold V_n, there exists no tensor field ξ_{i_1/i_p} which satisfies*

$$g^{kl}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}kl} = \overline{T}_{i_1/i_{p+q-1}\,j_1/j_{p+q-1}}\xi^{j_1/j_p;\,j_{p+1}/j_{p+q-1}}$$

and

$$\overline{T} \equiv \overline{T}_{i_1/i_{p+q-1}\,j_1/j_{p+q-1}}\xi^{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\xi^{j_1/j_p;\,j_{p+1}/j_{p+q-1}} \geqslant 0$$

unless we have

$$\xi_{i_1/i_p;\,i_{p+1}/i_{p+q}} = 0\,,$$

or equivalently

$$\xi_{i_1/i_p;\,i_{p+1}} = 0$$

and consequently

$$\overline{T} = 0\,.$$

Using the identity suitably, we have, for an antisymmetric tensor field ξ_{i_1/i_p},

$$\begin{aligned}(2.2)\qquad & \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-1}i_1k} + \ldots + \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-1}i_pk} - \\ & \qquad - \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-1}ki_1} - \ldots - \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-1}ki_p} \\ & = [-\xi_{ai_2/i_p;\,i_{p+1}/i_{p+q-1}}R^a{}_{ji_1k} - \ldots - \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-2}a}R^a{}_{i_{p+q-1}i_1k}] + \\ & \quad + [-\xi_{aji_3/i_p;\,i_{p+1}/i_{p+q-1}}R^a{}_{i_1i_2k} - \ldots - \xi_{i_1ji_3/i_p;\,i_{p+1}/i_{p+q-2}a}R^a{}_{i_{p+q-1}i_2k}] + \\ & \quad + \ldots + [-\xi_{ai_2/i_{p-1}j;\,i_{p+1}/i_{p+q-1}}R^a{}_{i_1i_pk} - \ldots - \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-2}a}R^a{}_{i_{p+q-1}i_pk}]\,.\end{aligned}$$

Adding $\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} - \xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk}$ to (2.2) and then contracting with g^{jk} gives

$$\begin{aligned} & g^{jk}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} - g^{jk}(\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}j} - \\ & \qquad - \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-1}i_1} - \ldots - \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-1}i_p})_{;k} - \\ & \qquad - (\xi^k{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}ki_1} - \ldots - \xi^k{}_{i_2/i_{p-1}i_1;\,i_{p+1}/i_{p+q-1}ki_p}) \\ & = \sum_{s=1}^{p}\xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{p+q-1}}R^a{}_{i_s} + \\ & \quad + \sum_{\substack{s,t=1\\ s<t}}^{p}\xi_{i_1/i_{s-1}ai_{s+1}/i_{t-1}bi_{t+1}/i_p;\,i_{p+1}/i_{p+q-1}}R^{ab}{}_{i_si_t} - \\ & \quad - \sum_{s=1}^{p}\sum_{t=p+1}^{p+q-1}\xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{t-1}bi_{t+1}/i_{p+q-1}}R^a{}_{i_si_t}{}^b\,.\end{aligned}$$

Thus if an antisymmetric tensor field ξ_{i_1/i_p} of order p satisfies

$$\begin{aligned}(2.3)\qquad & g^{jk}(\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}j} - \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-1}i_1} - \ldots - \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-1}i_p})_{;k} - \\ & \qquad - (\xi^k{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}ki_1} - \ldots - \xi^k{}_{i_2/i_{p-1}i_1;\,i_{p+1}/i_{p+q-1}ki_p}) = 0\,,\end{aligned}$$

then it also satisfies

$$(2.4)\qquad g^{jk}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} = \sum_{s=1}^{p} \xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{p+q-1}} R^{a}{}_{i_s} + \sum_{\substack{s,t=1\\ s<t}}^{p} \xi_{i_1/i_{s-1}ai_{s+1}/i_{t-1}bi_{t+1}/i_p;\,i_{p+1}/i_{p+q-1}} R^{ab}{}_{i_si_t} - \sum_{s=1}^{p}\sum_{t=p+1}^{p+q-1} \xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{t-1}bi_{t+1}/i_{p+q-1}} R^{a}{}_{i_si_t}{}^{b}$$

and consequently

$$g^{jk}\xi^{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} = pR_{ij}\xi^{ii_2/i_p;\,i_{p+1}/i_{p+q-1}}\xi^{j}{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}} + \frac{p(p-1)}{2}R_{ijkl}\xi^{iji_3/i_p;\,i_{p+1}/i_{p+q-1}}\xi^{kl}{}_{i_3/i_p;\,i_{p+1}/i_{p+q-1}} - p\sum_{t=p+1}^{p+q-1} R^{k}{}_{lij}\xi^{ii_2/i_p;\,i_{p+1}/i_{t-1}li_{t+1}/i_{p+q-1}}\xi^{j}{}_{i_2/i_p;\,i_{p+1}/i_{t-1}ki_{t+1}/i_{p+q-1}}\,.$$

Thus if we introduce the quadratic form

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} = R_{ij}\xi^{ii_2/i_p;\,i_{p+1}/i_{p+q-1}}\xi^{j}{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}} + \frac{p-1}{2}R_{ijkl}\xi^{iji_3/i_p;\,i_{p+1}/i_{p+q-1}}\xi^{kl}{}_{i_3/i_p;\,i_{p+1}/i_{p+q-1}} - \sum_{t=p+1}^{p+q-1} R^{k}{}_{lij}\xi^{ii_2/i_p;\,i_{p+1}/i_{t-1}li_{t+1}/i_{p+q-1}}\xi^{j}{}_{i_2/i_p;\,i_{p+1}/i_{t-1}ki_{t+1}/i_{p+q-1}}\,.$$

Theorem 1 yields:

THEOREM 2. *In a compact Riemannian manifold V_n, there exists no antisymmetric tensor field ξ_{i_1/i_p} of order p which satisfies* (2.3) *and*

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} \geqslant 0$$

unless we have

$$\xi_{i_1/i_p;\,i_{p+1}/i_{p+q}} = 0$$

or equivalently

$$\xi_{i_1/i_p;\,i_{p+1}} = 0$$

and consequently

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} = 0\,.$$

Next the addition of

$$p\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} - p\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk}$$

to (2.2) and contraction with g^{jk} gives

$$\begin{aligned}
&-pg^{jk}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} + g^{jk}(p\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}j} +\\
&\quad + \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-1}i_1} + \dots + \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-1}i_p})_{;k} -\\
&\quad - (\xi^{k}{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}ki_1} - \dots - \xi^{k}{}_{i_2/i_{p-1}i_1;\,i_{p+1}/i_{p+q-1}ki_p})\\
&= \sum_{s=1}^{p} \xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{p+q-1}} R^{a}{}_{i_s} +\\
&\quad + \sum_{\substack{s,t=1\\ s<t}}^{p} \xi_{i_1/i_{s-1}ai_{s+1}/i_{t-1}bi_{t+1}/i_p;\,i_{p+1}/i_{p+q-1}} R^{ab}{}_{i_si_t} -\\
&\quad - \sum_{s=1}^{p}\sum_{t=p+1}^{p+q-1} \xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{t-1}bi_{t+1}/i_{p+q-1}} R^{a}{}_{i_si_t}{}^{b}\,.
\end{aligned}$$

Thus if an antisymmetric tensor ξ_{i_1/i_p} satisfies

$$\begin{aligned}
(2.5)\qquad & g^{jk}(p\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}j} + \xi_{ji_2/i_p;\,i_{p+1}/i_{p+q-1}i_1} + \dots + \xi_{i_1/i_{p-1}j;\,i_{p+1}/i_{p+q-1}i_p})_{;k} -\\
&\quad - (\xi^{k}{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}ki_1} - \dots - \xi^{k}{}_{i_2/i_{p-1}i_1;\,i_{p+1}/i_{p+q-1}ki_p}) = 0\,,
\end{aligned}$$

then it also satisfies

$$\begin{aligned}
(2.6)\qquad & g^{jk}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} + \frac{1}{p}\sum_{s=1}^{p} \xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{p+q-1}} R^{a}{}_{i_s} +\\
&\quad + \frac{1}{p}\sum_{\substack{s,t=1\\ s<t}}^{p} \xi_{i_1/i_{s-1}ai_{s+1}/i_{t-1}bi_{t+1}/i_p;\,i_{p+1}/i_{p+q-1}} R^{ab}{}_{i_si_t} -\\
&\quad - \frac{1}{p}\sum_{s=1}^{p}\sum_{t=p+1}^{p+q-1} \xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{t-1}bi_{t+1}/i_{p+q-1}} R^{a}{}_{i_si_t}{}^{b} = 0
\end{aligned}$$

and consequently

$$g^{jk}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk}\,\xi^{i_1/i_p;\,i_{p+1}/i_{p+q-1}} = -F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\}$$

and hence Theorem 1 yields:

THEOREM 3. *In a compact Riemannian manifold V_n, there exists no antisymmetric tensor field ξ_{i_1/i_p} of order p which satisfies* (2.5) *and*

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} \leqslant 0$$

unless we have

$$\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} = 0$$

or equivalently

$$\xi_{i_1/i_p;\, i_{p+1}} = 0$$

and consequently

$$F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0 \,.$$

3. Harmonic and killing tensors of type q. A tensor ξ_{i_1/i_p} of order p will be called *harmonic of type q* if it satisfies the conditions

(3.1) ξ_{i_1/i_p} is antisymmetric in all its indices,

$$\xi_{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|\,i_{p+q}]} = 0 \tag{3.2}$$

or explicitly

$$\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} = \xi_{i_{p+q}i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} + \dots + \xi_{i_1/i_{p-1}i_{p+q};\, i_{p+1}/i_{p+q-1}i_p} \tag{3.3}$$

and furthermore

$$g^{ij}\xi_{ii_2/i_p;\, i_{p+1}/i_{p+q-1}j} = 0 \,. \tag{3.4}$$

Now if ξ_{i_1/i_p} is a harmonic tensor of order p and type q, then it satisfies (2.3) and consequently we have (2.4). Thus, as a special case of Theorem 2, we state:

THEOREM 4. *In a compact Riemannian manifold V_n, there exists no harmonic tensor field of order p and type q which satisfies*

$$F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} \geqslant 0$$

unless we have

$$\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} = 0$$

or equivalently

$$\xi_{i_1/i_p;\, i_{p+1}} = 0$$

and consequently

$$F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0 \,.$$

A tensor ξ_{i_1/i_p} will be called *killing of type q* if it satisfies the conditions

(3.5) ξ_{i_1/i_p} is antisymmetric in all the indices,

$$\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} + \xi_{i_{p+q}i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} = 0 \,. \tag{3.6}$$

Consequently a killing tensor of order p and type q satisfies

$$\xi_{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|\,i_{p+q}]} = \xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} \tag{3.7}$$

or explicitly

$$p\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}i_{p+q}} + \xi_{i_{p+q}i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} + \dots + \xi_{i_1/i_{p-1}i_{p+q};\, i_{p+1}/i_{p+q-1}i_p} = 0 \tag{3.8}$$

and such a tensor automatically satisfies

$$\xi^{k}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}k} = 0 \,. \tag{3.9}$$

Now if ξ_{i_1/i_p} is a killing tensor of order p and type q, then it satisfies (2.5) and consequently we have (2.6). Thus, as a special case of Theorem 3, we have:

THEOREM 5. *In a compact Riemannian manifold V_n, there exists no killing tensor of order p and type q which satisfies*

$$F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} \leqslant 0$$

unless we have

$$\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} = 0$$

or equivalently

$$\xi_{i_1/i_p;\, i_{p+1}} = 0 \,,$$

and consequently

$$F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0 \,.$$

4. A fundamental formula. In this section we establish a fundamental formula for an orientable manifold and use it to obtain alternative proofs of Theorems 4 and 5.

In a compact orientable Riemannian manifold V_n, with an antisymmetric tensor ξ_{i_1/i_p} we form

$$\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\, \xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}}$$

and the divergence

$$\begin{aligned}(\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\, \xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}})_{;i_1} &= \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}i_1}\, \xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}} + \\ &+ \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\, \xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} \,.\end{aligned} \tag{4.1}$$

From the Ricci identity

$$\begin{aligned}\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}j} - \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{j i_{p+q}} \\ = \xi^{a i_2/i_p;\, i_{p+1}/i_{p+q-1}} R^{i_1}{}_{a i_{p+q} j} + \ldots + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-2}a} R^{i_{p+q-1}}{}_{a i_{p+q} j}\end{aligned}$$

we have, by contracting with respect to i_1 and j,

$$\begin{aligned}\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}i_1} &= \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1 i_{p+q}} + \xi^{a i_2/i_p;\, i_{p+1}/i_{p+q-1}} R_{a i_{p+q}} + \\ &+ \xi^{i_1 a i_3/i_p;\, i_{p+1}/i_{p+q-1}} R^{i_2}{}_{a i_{p+q} i_1} + \ldots + \xi^{i_1/i_{p-1}a;\, i_{p+1}/i_{p+q-1}} R^{i_p}{}_{a i_{p+q} i_1} + \\ &+ \xi^{i_1/i_p;\, a i_{p+2}/i_{p+q-1}} R^{i_{p+1}}{}_{a i_{p+q} i_1} + \ldots + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-2}a} R^{i_{p+q-1}}{}_{a i_{p+q} i_1}\end{aligned}$$

and consequently, on substituting this into (4.1), we obtain

$$
\begin{aligned}
&(\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}})_{;i_1} \\
&\quad = \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1 i_{p+q}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}} + \\
&\quad + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1} i_1} + \\
&\quad + R_{i_1 i_{p+q}}\,\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}} + \\
&\quad + (p-1)\,R_{ijkl}\,\xi^{iki_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{jl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}} + \\
&\quad + \sum_{t=p+1}^{p+q-1} R^{i_t}{}_{a i_{p+q} i_1}\,\xi^{i_1/i_p;\, i_{p+1}/i_{t-1} a i_{t+1}/i_{p+q-1}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}}
\end{aligned}
$$

by virtue of

$$R_{ijkl} = R_{lkji}\,.$$

But according to the identity

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

the term

$$(p-1)\,R_{ijkl}\,\xi^{iki_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{jl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}}$$

appearing in the right-hand side of above equation can also be written as

$$-(p-1)[R_{iklj} + R_{iljk}]\,\xi^{iki_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{jl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}}\,,$$

i.e., as

$$
\begin{aligned}
&(p-1)[-R_{ijkl}\,\xi^{iji_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{lk}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}} - \\
&\qquad - R_{ijlk}\,\xi^{iki_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{lj}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}}]\,,
\end{aligned}
$$

i.e., as

$$
\begin{aligned}
&(p-1)\,R_{ijkl}\,\xi^{iji_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{kl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}} - \\
&\qquad - (p-1)\,R_{ijkl}\,\xi^{iki_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{jl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}}
\end{aligned}
$$

so that

$$
\begin{aligned}
&(p-1)\,R_{ijkl}\,\xi^{iki_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{jl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}} \\
&\qquad = \frac{p-1}{2}\,R_{ijkl}\,\xi^{iji_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{kl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}}
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
(4.2)\quad &(\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}})_{;i_1} \\
&\quad = \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1 i_{p+q}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}} + \\
&\quad + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\,\xi_{i_{p+q} i_2/i_p;\, i_{p+1}/i_{p+q-1} i_1} + \\
&\quad + R_{ij}\,\xi^{ii_2/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{j}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}} + \\
&\quad + \frac{p-1}{2}\,R_{ijkl}\,\xi^{iji_3/i_p;\, i_{p+1}/i_{p+q-1}}\,\xi^{kl}{}_{i_3/i_p;\, i_{p+1}/i_{p+q-1}} + \\
&\quad + \sum_{t=p+1}^{p+q-1} R^{l}{}_{k i_{p+q} i_1}\,\xi^{i_1/i_p;\, i_{p+1}/i_{t-1} k i_{t+1}/i_{p+q-1}}\,\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{t-1} l i_{t+1}/i_{p+q-1}}\,.
\end{aligned}
$$

Next we consider

$$\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}}$$

and form the divergence

$$(4.3)\qquad (\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}})_{;i_{p+q}}$$
$$= \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}} +$$
$$+ \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1 i_{p+q}}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}\,,$$

and from (4.2) and (4.3) we obtain

$$(4.4)\qquad (\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_{p+q}}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}})_{;i_1} -$$
$$- (\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}})_{;i_{p+q}}$$
$$= F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_{p+q} i_2/i_p;\, i_{p+1}/i_{p+q-1} i_1} -$$
$$- \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}\,.$$

Integrating both members of (4.4) over the whole manifold and applying Green's Theorem we get

$$(4.5)\qquad \int_{v_n} [F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_{p+q} i_2/i_p;\, i_{p+1}/i_{p+q-1} i_1} -$$
$$- \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}]\, dv = 0\,.$$

Now

$$\xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_{p+q} i_2/i_p;\, i_{p+1}/i_{p+q-1} i_1}$$
$$= \frac{1}{p}\xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} -$$
$$- \frac{p+1}{p}\xi^{[i_1/i_p;\, |i_{p+1}/i_{p+q-1}|\, i_{p+q}]}\xi_{[i_1/i_p;\, |i_{p+1}/i_{p+q-1}|\, i_{p+q}]}$$

where $\xi_{[i_1/i_p;|i_{p+1}/i_{p+q-1}|i_{p+q}]}$ denotes the antisymmetric part of the tensor $\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}}$ w.r.t. the indices $i_1, i_2, \ldots, i_p$ and i_{p+q} only and on introducing this into (4.5) we get the fundamental formula

$$(4.6)\qquad \int_{v_n} \Big[F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} + \frac{1}{p}\xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} -$$
$$- \frac{p+1}{p}\xi^{[i_1/i_p;\, |i_{p+1}/i_{p+q-1}|\, i_{p+q}]}\xi_{[i_1/i_p;\, |i_{p+1}/i_{p+q-1}|\, i_{p+q}]} -$$
$$- \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}\Big]\, dv = 0$$

which will often be used.

If ξ_{i_1/i_p} is a harmonic tensor of order p and type q, then the substitution of

$$\xi_{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|\,i_{p+q}]} = 0$$

and

$$\xi^{i_1}{}_{i_2/i_p;\,i_{p+1}/i_{p+q-1}i_1} = 0$$

into (4.6) gives

$$\int_{V_n}\left[F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} + \frac{1}{p}\,\xi^{i_1/i_p;\,i_{p+1}/i_{p+q}}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q}}\right]dv = 0 \tag{4.7}$$

and thus

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} \geqslant 0$$

implies

$$\xi_{i_1/i_p;\,i_{p+1}/i_{p+q}} = 0$$

or equivalently

$$\xi_{i_1/i_p;\,i_{p+1}} = 0$$

and consequently

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} = 0\,.$$

This is an alternative proof of Theorem 4 for an orientable manifold.

Similarly, for a killing tensor, we obtain

$$\int_{V_n}[F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} - \xi^{i_1/i_p;\,i_{p+1}/i_{p+q}}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q}}]dv = 0 \tag{4.8}$$

and thus

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} \leqslant 0$$

implies

$$\xi_{i_1/i_p;\,i_{p+1}/i_{p+q}} = 0$$

or equivalently

$$\xi_{i_1/i_p;\,i_{p+1}} = 0$$

and consequently

$$F\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} = 0\,.$$

This is an alternative proof of Theorem 5 for an orientable manifold.

5. A necessary and sufficient condition for a tensor field to be harmonic of type q. If we introduce the symbol

$$\begin{aligned} S\{\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}}\} = {} & g^{jk}\xi_{i_1/i_p;\,i_{p+1}/i_{p+q-1}jk} - \\ & \sum_{s=1}^{p}\xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{p+q-1}}R^{a}{}_{i_s} - \\ & \sum_{\substack{s,t=1\\ s<t}}^{p}\xi_{i_1/i_{s-1}ai_{s+1}/i_{t-1}bi_{t+1}/i_p;\,i_{p+1}/i_{p+q-1}}R^{ab}{}_{i_si_t} + \\ & + \sum_{s=1}^{p}\sum_{t=p+1}^{p+q-1}\xi_{i_1/i_{s-1}ai_{s+1}/i_p;\,i_{p+1}/i_{t-1}bi_{t+1}/i_{p+q-1}}R^{a}{}_{i_si_t}{}^{b}\,, \end{aligned} \tag{5.1}$$

then for a harmonic tensor we have

$$S\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0$$

and we are going to prove the converse for a compact orientable Riemannian manifold.

For

$$\varphi = \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}$$

we have

$$\Delta\varphi = 2[\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}jk}g^{jk} + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}}]$$

and therefore integrating over the whole manifold and applying Green's Theorem, we find

$$(5.2)\qquad \int_{v_n}[g^{jk}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}jk}\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}} + \xi^{i_1/i_p;\, i_{p+1}/i_{p+q}}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}}]dv = 0 .$$

Subtracting p times (4.6) from (5.2), we have

$$(5.3)\qquad \int_{v_n}[\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}g^{jk}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}jk} - pF\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} + (p+1)\xi^{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|i_{p+q}]}\xi_{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|i_{p+q}]} + p\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}]dv = 0 ,$$

which may also be written in the form

$$(5.4)\qquad \int_{v_n}[\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}S\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} + (p+1)\xi^{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|i_{p+q}]}\xi_{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|i_{p+q}]} + p\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}]dv = 0 ,$$

and thus, if

$$S\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0 ,$$

we must have

$$(5.5)\qquad \xi_{[i_1/i_p;\,|i_{p+1}/i_{p+q-1}|i_{p+q}]} = 0$$

and

$$(5.6)\qquad \xi^{i_1}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} = 0 .$$

Thus we have proved:

THEOREM 6. *In a compact orientable Riemannian manifold V_n, a necessary and sufficient condition that an antisymmetric tensor field ξ_{i_1/i_p} be harmonic of order p and type q is that it satisfies*

$$S\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0 \; .$$

6. A necessary and sufficient condition for a tensor field to be killing of order p and type q. For a killing tensor the analogue to $S\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\}$ is the form

$$(6.1)\qquad T\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = g^{jk}\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}jk} + \\ +\frac{1}{p}\sum_{s=1}^{p}\xi_{i_1/i_{s-1}a i_{s+1}/i_p;\, i_{p+1}/i_{p+q-1}}R^a{}_{i_s} + \\ +\frac{1}{p}\sum_{\substack{s,t=1\\ s<t}}^{p}\xi_{i_1/i_{s-1}a i_{s+1}/i_{t-1}b i_{t+1}/i_p;\, i_{p+1}/i_{p+q-1}}R^{ab}{}_{i_s i_t} - \\ -\frac{1}{p}\sum_{s=1}^{p}\sum_{t=p+1}^{p+q-1}\xi_{i_1/i_{s-1}a i_{s+1}/i_p;\, i_{p+1}/i_{t-1}b i_{t+1}/i_{p+q-1}}R^a{}_{i_s i_t}{}^b$$

and we will again prove that, in a compact orientable Riemannian manifold,

$$T\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0$$

and

$$\xi^{i_1}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} = 0$$

imply killing property of type q.

In fact, the analogue to (5.4) is

$$(6.2)\qquad \int_{V_n}[\xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}T\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} + \\ +\frac{p+1}{p}(\xi^{i_1/i_p;\, i_{p+1}/i_{p+q}} - \xi^{[i_1/i_p;\, |i_{p+1}/i_{p+q-1}|i_{p+q}]})\times \\ \times(\xi_{i_1/i_p;\, i_{p+1}/i_{p+q}} - \xi_{[i_1/i_p;\, |i_{p+1}/i_{p+q-1}|i_{p+q}]}) - \\ - \xi^{i_1/i_p;\, i_{p+1}/i_{p+q-1}}{}_{i_1}\xi^{i_{p+q}}{}_{i_2/i_p;\, i_{p+1}/i_{p+q}}]dv = 0$$

and our conclusion follows:

THEOREM 7. *In a compact orientable Riemannian manifold V_n, a necessary and sufficient condition that an antisymmetric tensor field of order p be killing of type q is that it satisfies*

$$T\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0$$

and

$$\xi^{i_1}{}_{i_2/i_p;\, i_{p+1}/i_{p+q-1}i_1} = 0\,.$$

7. Conformal killing tensors of type q. We define a vector ξ_i to be conformal killing of type q if it satisfies

$$(7.1)\qquad \xi_{i;\, i_2/i_q j} + \xi_{j;\, i_2/i_q i} = 2\varphi_{i_2/i_q} g_{ij}\,.$$

Here

$$\xi^i{}_{;i_2/i_q i} = n\varphi_{i_2/i_q}\,.$$

For such a vector,

$$\frac{\delta}{\delta s}\left(\xi_{i;\, i_2/i_q}\frac{dx^i}{ds}\right) = \frac{1}{2}(\xi_{i;\, i_2/i_q j} + \xi_{j;\, i_2/i_q i})\frac{dx^i}{ds}\frac{dx^j}{ds}\,,$$

i.e.,

$$\frac{\delta}{\delta s}\left(\xi_{i;\, i_2/i_q}\frac{dx^i}{ds}\right) = \varphi_{i_2/i_q}$$

along any geodesic $x^i(s)$ and thus

$$\frac{\delta}{\delta s}\left(\xi_{i;\, i_2/i_q}\frac{dx^i}{ds}\right)$$

depends only on the point and not the direction of the geodesic through that point.

In order to obtain a corresponding property for an antisymmetric tensor ξ_{i_1/i_p}, we assume that

$$(7.2)\qquad \frac{\delta}{\delta s}\left(\xi_{ii_2/i_p;\, i_{p+1}/i_{p+q-1}}\frac{dx^i}{ds}\right) = \frac{1}{2}(\xi_{ii_2/i_p;\, i_{p+1}/i_{p+q-1}j} + \xi_{ji_2/i_p;\, i_{p+1}/i_{p+q-1}i})\frac{dx^i}{ds}\frac{dx^j}{ds}$$

depends only on the point and not on the direction of the geodesic $x^i(s)$ through this point, and from (7.2), we find

$$(7.3)\qquad \xi_{ii_2/i_p;\, i_{p+1}/i_{p+q-1}j} + \xi_{ji_2/i_p;\, i_{p+1}/i_{p+q-1}i} = 2\varphi_{i_2/i_{p+q-1}} g_{ij}$$

where

$$\varphi_{i_2/i_{p+q-1}} = \frac{1}{n} g^{ij}\xi_{ii_2/i_p;\, i_{p+1}/i_{p+q-1}j}\,.$$

An antisymmetric tensor field ξ_{i_1/i_p} which satisfies (7.3) will be called a *conformal killing tensor* of order p and type q.

From (7.1), we have

$$\xi_{j;\, i_2/i_p i} = -\xi_{i;\, i_2/i_q j} + 2\varphi_{i_2/i_q} g_{ij}$$

and hence

$$\xi^{i_1;\, i_2/i_{q+1}}\xi_{i_{q+1};\, i_2/i_q i_1} = -\xi^{i_1;\, i_2/i_{q+1}}\xi_{i_1;\, i_2/i_{q+1}} + 2n\varphi_{i_2/i_q}\varphi^{i_2/i_q}$$

and therefore by the fundamental formula ([1], (5.2)), we obtain

$$\int_{v_n}\Big[\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_qj_q}-\sum_{r=2}^{q}R_{i_1j_1j_ri_r}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_qj_q}\Big)\xi^{i_1;i_2/i_q}\xi^{j_1;j_2/j_q}-\\ -\xi^{i_1;i_2/i_{q+1}}\xi_{i_1;i_2/i_{q+1}}+2n\varphi_{i_2/i_q}\varphi^{i_2/i_q}-n^2\varphi_{i_2/i_q}\varphi^{i_2/i_q}\Big]dv=0\,.$$

i.e.,

$$\int_{v_n}\Big[\Big(R_{i_1j_1}g_{i_2j_2}/g_{i_qj_q}-\sum_{r=2}^{q}R_{i_1j_1j_ri_r}g_{i_2j_2}/g_{i_{r-1}j_{r-1}}g_{i_{r+1}j_{r+1}}/g_{i_qj_q}\Big)\xi^{i_1;i_2/i_q}\xi^{j_1;j_2/j_q}-\\ -\xi^{i_1;i_2/i_{q+1}}\xi_{i_1;i_2/i_{q+1}}-n(n-2)\varphi_{i_2/i_q}\varphi^{i_2/i_q}\Big]dv=0\,.$$

Consequently we have:

Theorem 8. *In a compact orientable Riemannian manifold V_n, there exists no conformal killing vector of type q which satisfies*

$$T'\leqslant 0\qquad([1])$$

unless we have

$$\xi_{i_1;i_2/i_{q+1}}=0$$

or equivalently

$$\xi_{i_1;i_2}=0\,,$$

$$T'=0\,,$$

and

$$\varphi_{i_2/i_q}=0\,.$$

For a conformal killing tensor of order p and type q, we have

$$\xi^{i_1/i_p;i_{p+1}/i_{p+q}}\xi_{i_{p+q}i_2/i_p;i_{p+1}/i_{p+q-1}i_1}\\ =-\xi^{i_1/i_p;i_{p+1}/i_{p+q}}\xi_{i_1/i_p;i_{p+1}/i_{p+q}}+2n\varphi_{i_2/i_{p+q-1}}\varphi^{i_2/i_{p+q-1}}$$

and therefore by (5.4) we obtain

$$\text{(7.4)}\qquad \int_{v_n}[F\{\xi_{i_1/i_p;i_{p+1}/i_{p+q-1}}\}-\xi^{i_1/i_p;i_{p+1}/i_{p+q}}\xi_{i_1/i_p;i_{p+1}/i_{p+q}}-\\ -n(n-2)\varphi_{i_2/i_{p+q-1}}\varphi^{i_2/i_{p+q-1}}]\,dv=0\,.$$

Consequently we have:

Theorem 9. *In a compact orientable Riemannian manifold V_n, there exists no conformal killing tensor of order p and type q which satisfies*

$$F\{\xi_{i_1/i_p;i_{p+1}/i_{p+q-1}}\}\leqslant 0$$

unless we have

$$\xi_{i_1/i_p;i_{p+1}/i_{p+q}}=0$$

or equivalently

$$\xi_{i_1/i_p;\, i_{p+1}} = 0$$

and

$$\psi_{i_2/i_{p+q-1}} = 0$$

and then automatically

$$F\{\xi_{i_1/i_p;\, i_{p+1}/i_{p+q-1}}\} = 0 \, .$$

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