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# ON TRANSFINITE DIMENSION

BY

L. POLKOWSKI (WARSZAWA)

We shall discuss here the two transfinite dimensions tr Ind and tr ind.

First, motivated by the fact that the existence of the transfinite dimension tr Ind is an invariant of closed mappings with fibres of cardinality less than $k$, where $k$ is a positive integer (see Theorem A, below), we give an estimation of $\operatorname{tr}\operatorname{Ind} Y$ in terms of $\operatorname{tr}\operatorname{Ind} X$ and $k$. This estimation, however, depends heavily upon the method of proof; it can be slightly improved in the case of open-and-closed mappings for some values of $\operatorname{tr}\operatorname{Ind} X$.

Next, we introduce the sets $S_I^\alpha(X)$, where $X$ is a normal space and $\alpha$ is an ordinal number, and by means of these sets we define the class of small spaces. We investigate the behaviour of such spaces under closed mappings.

Finally, we prove the sum theorem and the Cartesian product theorem for the sets $S_I^\alpha(X)$. Employing these theorems along with a localization lemma for the sets $S_I^\alpha(X)$ (see Lemma 2.4, below) we give an account of the other dimensional properties of the class of small spaces; in particular, our results on the sets $S_I^\alpha(X)$ imply some known facts about the transfinite dimension $D$ introduced by Henderson ([6]).

Our terminology and notation follow [3] and [4]. We shall quote definitions and theorems from [5] when transfinite dimension is concerned. By "dimension" we always mean the large inductive dimension Ind.

**1.** We begin with recalling the definitions of the transfinite dimensions tr Ind and tr ind.

Definition 1.1. For a normal space $X$ we let

1) $\operatorname{tr}\operatorname{Ind} X = -1$ if the space $X$ is empty;

2) $\operatorname{tr}\operatorname{Ind} X \leqslant \alpha$ if for every pair $A$, $B$ of disjoint closed subsets of $X$ there exists a partition $L$ between $A$ and $B$ such that $\operatorname{tr}\operatorname{Ind} L < \alpha$;

3) $\operatorname{tr}\operatorname{Ind} X = \alpha$ if $\operatorname{tr}\operatorname{Ind} X \leqslant \alpha$ and there exists no ordinal number $\beta < \alpha$ such that $\operatorname{tr}\operatorname{Ind} X \leqslant \beta$.

A normal space $X$ is said *to have* tr Ind if $\operatorname{tr}\operatorname{Ind} X = \alpha$ for an ordinal number $\alpha$.

Definition 1.2. For a regular space $X$ we let

1) $\operatorname{tr}\operatorname{ind} X = -1$ if the space $X$ is empty;

2) $\operatorname{tr}\operatorname{ind} X \leqslant \alpha$ if for every point $x \in X$ and each closed subset $F$ of $X$ that does not contain $x$ there exists a partition $L$ between $\{x\}$ and $F$ such that $\operatorname{tr}\operatorname{ind} L < \alpha$;

3) $\operatorname{tr}\operatorname{ind} X = \alpha$ if $\operatorname{tr}\operatorname{ind} X \leqslant \alpha$ and there exists no ordinal number $\beta < \alpha$ such that $\operatorname{tr}\operatorname{ind} X \leqslant \beta$.

We state, now, two theorems on invariance and inverse invariance of the transfinite dimension tr Ind (see [16], Theorems 4.9 and 4.10).

THEOREM A. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and the space $X$ has* tr Ind, *then the space $Y$ has* tr Ind.

THEOREM B. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $\operatorname{Ind} f^{-1}(y) \leqslant k$ for every $y \in Y$ and the space $Y$ has* tr Ind, *then the space $X$ has* tr Ind.

Motivated by Theorem A, we give an estimation for $\operatorname{tr}\operatorname{Ind} Y$. Let us recall that every ordinal number $\alpha$ can be expressed in a unique way in the form $\omega_0 \cdot \lambda(\alpha) + n(\alpha)$, where $n(\alpha) < \omega_0$ (see [10], Ch. VII, § 5, Theorem 4).

We begin with a lemma (cf. Theorem 3.16 in [5] and its proof given therein).

LEMMA C. *If a hereditarily normal space $X$ contains a closed subspace $C$ such that* $\operatorname{tr}\operatorname{Ind} C \leqslant \alpha$ *with the property that each closed subspace $T$ of $X$ contained in $X \setminus C$ satisfies the inequality* $\operatorname{tr}\operatorname{Ind} T \leqslant \beta$, *then* $\operatorname{tr}\operatorname{Ind} X$ *is defined and*

$$\operatorname{tr}\operatorname{Ind} X \leqslant \begin{cases} \beta+\alpha+1 & \text{if } \alpha < \omega_0, \\ \beta+\alpha & \text{otherwise.} \end{cases}$$

THEOREM 1.3. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and* $\operatorname{tr}\operatorname{Ind} X \leqslant \alpha$, *then*

$$\operatorname{tr}\operatorname{Ind} Y \leqslant \begin{cases} n(\alpha)+k-1 & \text{if } \lambda(\alpha) = 0, \\ \omega_0^{\lambda(\alpha)} \cdot m(\alpha, k) + n(\alpha, k) + k - 1 & \\ & \text{if } \lambda(\alpha) > 0 \text{ and } \alpha \text{ is a non-limit number}, \\ \omega_0^{\lambda(\alpha)} \cdot m(\alpha, k) & \text{if } \lambda(\alpha) > 0 \text{ and } \alpha \text{ is a limit number}, \end{cases}$$

*where, for every ordinal number* $\alpha$ *and each positive integer* $k$,

$$m(\alpha, 1) = 1,$$
$$m(\alpha, k+1) = m(\alpha, k) + m(\beta, k+1) \quad \text{if } \alpha = \beta+1,$$
$$m(\alpha, k+1) = m(\alpha, k) + 1 \quad \text{if } \alpha \text{ is a limit number}$$

*and*

$$n(\alpha, 1) = n(\alpha),$$
$$n(\alpha, k+1) = n(\alpha, k) + 1 \quad \text{if } \alpha \text{ is a non-limit number},$$
$$n(\alpha, k) = 0 \quad \text{if } \alpha \text{ is a limit number}.$$

Proof. It follows from the assumptions that

(1) $$m(\alpha, k) \geqslant m(\beta, k) \quad \text{whenever} \quad n(\alpha) \geqslant n(\beta).$$

As the case when $\lambda(\alpha) = 0$ is the theorem on dimension-raising mappings (see [4], Theorem 4.3.3), it suffices to prove the theorem in the case when $\lambda(\alpha) > 0$. This will be done by induction with respect to the number $\alpha$.

Let us assume that the theorem holds for all spaces with $\operatorname{tr Ind}$ less than $\alpha$ and consider a space $X$ such that $\operatorname{tr Ind} X = \alpha$. Now, we apply induction with respect to the integer $k$. Since $\omega_0^\beta \cdot \beta \leqslant \omega_0^\beta$ for every ordinal number $\beta$ and $n(\beta, k) \geqslant n(\beta)$ for every ordinal number $\beta$ and $k = 1, 2, \ldots$, our theorem holds for $k = 1$. Let us assume, now, that the theorem is true for $k = 1, 2, \ldots, l-1$ and consider the case when $k = l$.

Consider a pair of disjoint closed subsets of $Y$. There exists a partition $L$ between $f^{-1}(A)$ and $f^{-1}(B)$ in $X$ such that $\operatorname{tr Ind} L = \beta < \alpha$. There exist closed subsets $M$, $N$ of $X$ such that $X = M \cup N$, $L = M \cap N$, $f^{-1}(A) \subset M \setminus N$ and $f^{-1}(B) \subset N \setminus M$. It is easy to see that the subset $K = f(M) \cap f(N)$ of $Y$ is a partition between $A$ and $B$ in $Y$.

There are two cases to consider.

Case 1. $\alpha$ is a non-limit number, $\alpha = \gamma + 1$. It follows from the first inductive assumption applied to the restriction $f|L\colon L \to f(L)$ that

$$\operatorname{tr Ind} f(L) \leqslant \begin{cases} n(\beta) + l - 1 & \text{if } \lambda(\beta) = 0 \\ \omega_0^{\lambda(\beta)} \cdot m(\beta, l) + n(\beta, l) + l - 1 & \\ & \text{if } \lambda(\beta) > 0 \text{ and } \beta \text{ is a non-limit number}, \\ \omega_0^{\lambda(\beta)} \cdot m(\beta, l) & \text{if } \lambda(\beta) > 0 \text{ and } \beta \text{ is a limit number}. \end{cases}$$

Consider, now, a closed subset $T$ of $K$ contained in $K \setminus f(L)$ and the closed subset $C = M \cap f^{-1}(T)$ of $X$. Since the restriction $f|C\colon C \to T$ has the property that $|(f|C)^{-1}(y)| \leqslant l-1$ for each $y \in T$, it follows from the second inductive assumption applied to the restriction $f|C\colon C \to T$ that

$$\operatorname{tr Ind} T \leqslant \omega_0^{\lambda(\alpha)} \cdot m(\alpha, l-1) + n(\alpha, l-1) + l - 2.$$

It follows from Lemma C that

$$\operatorname{tr}\operatorname{Ind} K \leqslant \begin{cases} \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1)+n(\alpha, l-1)+l-2+n(\beta)+l \\ \qquad\qquad \text{if } \lambda(\beta)=0, \\ \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1)+n(\alpha, l-1)+l-2+\omega_0^{\lambda(\beta)}\cdot m(\beta, l)+n(\beta, l)+l-1 \\ \qquad\qquad \text{if } \lambda(\beta)>0 \text{ and } \beta \text{ is a non-limit number,} \\ \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1)+n(\alpha, l-1)+l-2+\omega_0^{\lambda(\beta)}\cdot m(\beta, l) \\ \qquad\qquad \text{if } \lambda(\beta)>0 \text{ and } \beta \text{ is a limit number.} \end{cases}$$

It is easy to see, by (1), that in either of the two cases $\lambda(\beta)<\lambda(\alpha)$, $\lambda(\beta)=\lambda(\alpha)$

$$\operatorname{tr}\operatorname{Ind} Y \leqslant \begin{cases} \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l)+n(\alpha, l)+l-1 & \text{if } \lambda(\beta)=0, \\ \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l)+n(\alpha, l)+l-1 & \text{if } \lambda(\beta)>0 \text{ and } \beta \text{ is a non-limit number,} \\ \omega_0^{\lambda(\alpha)}\cdot [m(\alpha, l-1)+m(\gamma, l)]+n(\alpha, l)+l-1 & \\ & \text{if } \lambda(\beta)>0 \text{ and } \beta \text{ is a limit number} \end{cases}$$

and thus in Case 1 the theorem is proved.

Case 2. $\alpha$ is a limit number. By the second inductive assumption we have, in the notation of Case 1,

$$\operatorname{tr}\operatorname{Ind} T \leqslant \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1).$$

Thus,

$$\operatorname{tr}\operatorname{Ind} K \leqslant \begin{cases} \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1)+n(\beta)+l \quad \text{if } \lambda(\beta)=0, \\ \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1)+\omega_0^{\lambda(\beta)}\cdot m(\beta, l)+n(\beta, l)+l-1 \\ \qquad\qquad \text{if } \lambda(\beta)>0 \text{ and } \beta \text{ is a non-limit number,} \\ \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l-1)+\omega_0^{\lambda(\beta)}\cdot m(\beta, l) \\ \qquad\qquad \text{if } \lambda(\beta)>0 \text{ and } \beta \text{ is a limit number.} \end{cases}$$

Since $\lambda(\beta)<\lambda(\alpha)$, in all three cases

$$\operatorname{tr}\operatorname{Ind} K < \omega_0^{\lambda(\alpha)}\cdot [m(\alpha, l-1)+1]$$

and thus

$$\operatorname{tr}\operatorname{Ind} Y \leqslant \omega_0^{\lambda(\alpha)}\cdot m(\alpha, l).$$

This completes the proof of the theorem. □

We begin our discussion of the case of open-and-closed mappings with a lemma.

LEMMA 1.4. *If $f\colon X\to Y$ is an open-and-closed mapping of a space $X$ onto a space $Y$, two closed subsets $A$, $B$ of $Y$ are disjoint and a closed subset $L$ of $X$ is a partition between $f^{-1}(A)$ and $f^{-1}(B)$ in $X$, then the closed subset $f(L)$ of $Y$ is a partition between $A$ and $B$ in $Y$.*

Proof. Consider disjoint open subsets $G$, $H$ of $X$ such that $X\setminus L = G\cup H$, $f^{-1}(A)\subset G$ and $f^{-1}(B)\subset H$. The open sets $U = Y\setminus f(X\setminus G)$ and $V = Y\setminus f(X\setminus H)$ are disjoint, $A\subset U$ and $B\subset V$. Consider also the open subsets $W_1 = f(G)\cap f(H)$ and $W_2 = Y\setminus f(L)$ of $Y$. Since

$$U\cap W_1 = f(G)\cap f(H)\setminus f(X\setminus G) = \emptyset,$$
$$V\cap W_1 = f(G)\cap f(H)\setminus f(X\setminus H) = \emptyset$$

and

$$\begin{aligned} Y\setminus[U\cup V\cup(W_1\cap W_2)] &= [Y\setminus(U\cup V)]\cap[(Y\setminus W_1)\cup(Y\setminus W_2)] \\ &= f(X\setminus G)\cap f(X\setminus H)\cap[(Y\setminus f(G)\cap f(H))\cup f(L)] \\ &= f(L)\cup[f(X\setminus G)\cap f(X\setminus H)\setminus(f(G)\cap f(H))] = f(L), \end{aligned}$$

the set $f(L)$ is a partition between $A$ and $B$. Thus, the lemma is proved. □

THEOREM 1.5. *If $f\colon X\to Y$ is an open-and-closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $|f^{-1}(y)|\leqslant k$ for every $y\in Y$ and* $\operatorname{tr Ind} X\leqslant\alpha = \omega_0\cdot\lambda(\alpha)$, *then* $\operatorname{tr Ind} Y\leqslant\omega_0^{\lambda(\alpha)}$.

Proof. Let $A$, $B$ be a pair of disjoint closed subsets of $Y$. There exists a partition $L$ between $f^{-1}(A)$ and $f^{-1}(B)$ in $X$ such that $\operatorname{tr Ind} L = \beta < \alpha$. It follows from Theorem 1.3 that

$$\operatorname{tr Ind} f(L)\leqslant\omega_0^{\lambda(\beta)}\cdot m(\beta, k)+n(\beta, k)+k-1 < \omega_0^{\lambda(\alpha)}.$$

By Lemma 1.4 the set $f(L)$ is a partition between $A$ and $B$ and thus $\operatorname{tr Ind} Y\leqslant\omega_0^{\lambda(\alpha)}$ so that the proof is concluded. □

2. In this section we shall investigate the behaviour of small spaces under closed mappings and we shall show that in this case more precise results can be obtained.

We begin with some definitions.

Definition 2.1. For a normal space $X$ we let

$$S_I(X) = X\setminus\bigcup\{U\subset X\colon\ U \text{ is open in } X \text{ and } \operatorname{Ind}\bar{U}<\infty\}.$$

Definition 2.2. For a normal space $X$ and each ordinal number $\alpha$ of cardinality less than $w(X)^+$ we let

1) $S_I^0(X) = X$,
2) $S_I^\alpha(X) = S_I(S_I^\beta(X))$ if $\alpha = \beta+1$,
3) $S_I^\alpha(X) = \bigcap_{\beta<\alpha} S_I^\beta(X)$ if $\alpha$ is a limit number.

Clearly, for every normal space $X$, the sequence $S_I^0(X)$, $S_I^1(X)$, $\ldots, S_I^\alpha(X),\ldots$ is eventually constant. We denote by $\alpha(X)$ the first ordinal number $\alpha_0$ with the property that $S_I^\alpha(X) = S_I^{\alpha_0}(X)$ whenever $\alpha\geqslant\alpha_0$.

Definition 2.3. A normal space $X$ is said to be *small* if $S_I^{\alpha(X)}(X) = \emptyset$.

It should be observed that a more general approach was described by Stone ([18]), who — for any property $\mathscr{P}$ hereditary with respect to closed subsets — defined the sets $F^{\alpha}(X)$ and $K(\mathscr{P}, X)$ of which the sets $S_I^{\alpha}(X)$ and $S_I^{\alpha(X)}(X)$ defined here are particular cases; Stone's idea goes back to the classical idea of localization (see [9], Ch. 1, §7, IV). Later, Nagami defined independently — for any property $\mathscr{P}$ hereditary with respect to closed subsets — the class of $\mathscr{P}$-dispersed spaces ([14]); it turns out that the class of $\mathscr{P}$-dispersed spaces coincides with the class of spaces $X$ with the property that $K(\mathscr{P}, X) = \emptyset$ and thus the class of small spaces coincides with the class of normal $\mathscr{P}$-dispersed spaces with $\mathscr{P}$ being the property of being locally finite-dimensional space. The class of small spaces coincides also — in the realm of strongly hereditarily normal spaces (see [4], Definition 2.1.2) — with the class of spaces $X$ satisfying the inequality $D(X) < \Delta$, where $D(X)$ denotes the transfinite dimension of $X$ in the sense of Henderson [6], defined originally for metrizable spaces only (cf. Proposition 3.3, below and Theorem 1 in [7]), which can be generalized without difficulty to strongly hereditarily normal spaces. A couple of results on $D$-dimension was announced in [12] and [11]. Kozlovskiĭ ([8]), without mentioning the more general constructions of [18] and [14], defined in the realm of metrizable spaces the class of small spaces (under the name of $d$-spaces) and announced a couple of results on dimensional properties of these spaces. It should be added that some subclasses of the class of small spaces were also discussed by Shmuely ([17]) and Baković ([1]).

Let us recall that if a normal space $X$ has $\operatorname{tr}\operatorname{Ind}$, then the subspace $S_I(X)$ of $X$ is countably compact (see [5], Lemma 3.14; the argument applied in [5] in the case of metrizable spaces holds in the more general case) so that if the space $X$ is, moreover, weakly paracompact, then $S_I(X)$ is compact (see [3], Theorem 5.3.2). This implies that if a weakly paracompact small space $X$ has $\operatorname{tr}\operatorname{Ind}$, then the ordinal number $\alpha(X)$ has a predecessor, i.e. $\alpha(X) = \varkappa(X) + 1$; one easily checks that $0 \leqslant \operatorname{Ind} S_I^{\varkappa(X)}(X) < \infty$ (see [5], Lemma 3.13).

We shall now state two properties of the operation $S_I^{\alpha}(X)$ to be frequently used in the sequel. It should be observed that it follows from the monotonicity of the dimension Ind in the class of strongly hereditarily normal spaces that if $X$ is a strongly hereditarily normal space, then

$$S_I(X) = X \setminus \bigcup \{U \subset X\colon\ U \text{ is open in } X \text{ and } \operatorname{Ind} U < \infty\}.$$

We denote from now the sets $S_I^{\alpha}(X)$ by $S^{\alpha}(X)$.

LEMMA 2.4. *If $X$ is a strongly hereditarily normal space, then*

(i) *$S^{\alpha}(A) \subset S^{\alpha}(X)$ for every ordinal number $\alpha$ and each subset $A$ of $X$;*

(ii) *if $U$ is an open subset of $X$, then $S^{\alpha}(A \cap U) = S^{\alpha}(A) \cap U$ for every ordinal number $\alpha$ and each subset $A$ of $X$.*

Proof. Since the family of open finite-dimensional subsets of the space

$X$ is an ideal of sets (see [4], Theorems 2.2.5 and 2.3.6), (i) holds for $\alpha = 1$ by [9], Ch. 1, § 7, IV, property (1), the general case follows by obvious inductive argument.

We shall prove (ii) by induction with respect to the ordinal number $\alpha$. We consider first the case when $\alpha = 1$. It follows from (i) that $S(A \cap U) \subset S(A) \cap U$. To prove the reverse inclusion, consider a point $x \in X \setminus S(A \cap U)$. There are two possibilities: either $x \in X \setminus (A \cap U)$, then of course $x \in X \setminus (S(A) \cap U)$ or $x \in (A \cap U)$. In the latter case, since $x \in A \cap U \setminus S(A \cap U)$, there exists an open subset $V$ of $A \cap U$ such that $x \in V$ and $\operatorname{Ind} V < \infty$. Consider an open subset $W$ of $X$ such that $W \cap (A \cap U) = V$. Thus, $A \cap (U \cap W)$ is a neighbourhood of $x$ in $A$ and $\operatorname{Ind} A \cap (U \cap W) < \infty$ so that $x \in X \setminus S(A) \subset X \setminus (S(A) \cap U)$. The proof in the case $\alpha = 1$ is concluded.

Assume, now, that (ii) holds for all ordinal numbers $\beta < \alpha < 1$. There are two possibilities: either $\alpha$ is a limit number, then it follows from the inductive assumption that

$$S^{\alpha}(A \cap U) = \bigcap_{\beta < \alpha} S^{\beta}(A \cap U) = \bigcap_{\beta < \alpha} (S^{\beta}(A) \cap U) = S^{\alpha}(A) \cap U$$

or $\alpha$ has a predecessor $\beta$, then it follows from the inductive assumption for $\beta$ and the already proved case $\alpha = 1$, that

$$S^{\alpha}(A \cap U) = S(S^{\beta}(A \cap U)) = S(S^{\beta}(A) \cap U) = S^{\alpha}(A) \cap U.$$

Thus, the lemma is proved. □

It should be noted that for $A = X$ (ii) was proved in [18] (property (2.4)).

We shall now define for each small space $X$ an ordinal invariant $\sigma(X)$ and for some small spaces we define an ordinal invariant $\iota$.

Definition 2.5. For each small space $X$ we let $\sigma(X) = \omega_0 \cdot \alpha(X)$ and for each small space $X$ such that $\alpha(X)$ has a predecessor $\varkappa(X)$ and $0 \leqslant \operatorname{Ind} S^{\varkappa(X)}(X) < \infty$ we let

$$\iota(X) = \omega_0 \cdot \varkappa(X) + \operatorname{Ind} S^{\varkappa(X)}(X).$$

Let us observe that if for a strongly hereditarily normal space $X$ the invariant $\iota(X)$ is defined, then $\iota(X) \leqslant D(X)$. Indeed, let $D(X) = \alpha$ and consider an $(\alpha - D)$-representation $\{A_\delta\}_{\delta \leqslant \gamma}$ for the space $X$ (see [7], Definitions); we prove by induction with respect to the ordinal number $\beta$, that $S^{\beta}(X) \subset \bigcup_{\delta \geqslant \omega_0 \cdot \beta} A_\delta$. This inclusion obviously holds for $\beta = 0$. We assume that the inclusion holds for each ordinal $\gamma < \beta$. To prove that $S^{\beta}(X) \subset \bigcup_{\delta \geqslant \omega_0 \cdot \beta} A_\delta$ we consider two cases.

Case 1. $\beta$ has a predecessor $\gamma$. By the inductive assumption

$S^\gamma(X) \subset \bigcup_{\delta \geqslant \omega_0 \cdot \gamma} A_\delta$. Consider a point $x \in S^\gamma(X) \setminus \bigcup_{\delta \geqslant \omega_0 \cdot \beta} A_\delta$. There exists a positive integer $n$ such that $x \in A_{\omega_0 \cdot \gamma + n} \setminus \bigcup_{m > n} A_{\omega_0 \cdot \gamma + m}$ (see [7], l. c., condition (d)) and thus the open subspace $U = S^\gamma(X) \setminus \bigcup_{\delta \geqslant \omega_0 \cdot \gamma + n} A_\delta$ of $S^\gamma(X)$ is a finite-dimensional neighbourhood of $x$ in $S^\gamma(X)$, so that $x \in S^\gamma(X) \setminus S^{\gamma+1}(X)$. Thus, $S^\beta(X) \subset \bigcup_{\delta \geqslant \omega_0 \cdot \beta} A_\delta$.

Case 2. $\beta$ is a limit number. Consider a point $x \in S^\beta(X)$. Since for each $\gamma < \beta$, by the inductive assumption, $x \in \bigcup_{\delta \geqslant \omega_0 \cdot \gamma} A_\delta$, it follows that $\delta(x) \geqslant \omega_0 \cdot \beta$, $\delta(x)$ being the largest ordinal $\delta$ such that $x \in A_\delta$. Thus, $S^\beta(X) \subset \bigcup_{\delta \geqslant \omega_0 \cdot \beta} A_\delta$.

This implies that $\varkappa(X) \leqslant \lambda(\gamma)$, hence $\iota(X) \leqslant D(X)$. It should be also noted that if $A_\gamma = \emptyset$, then $\alpha(X) \leqslant \lambda(\gamma)$, so that $\sigma(X) \leqslant D(X)$. Let us also observe that if the space $X$ is, moreover, hereditarily paracompact and the invariant $\iota(X)$ is defined, then $D(X) \leqslant \iota(X)$. Indeed, for each $\alpha < \varkappa(X)$, consider the subspace $X_\alpha = S^\alpha(X) \setminus S^{\alpha+1}(X)$. Since $X_\alpha$ is locally finite-dimensional, by the hereditary paracompactness of $X$, there exists a locally finite in $X_\alpha$ cover $\mathscr{F}_\alpha = \{F_s\}_{s \in S_\alpha}$ of $X_\alpha$ by closed in $X_\alpha$ finite-dimensional sets. For $n = 1, 2, \ldots$, we let

$$F_{\alpha,n} = \bigcup \{F_s\colon \operatorname{Ind} F_s = n \text{ and } s \in S_\alpha\};$$

by the locally finite sum theorem for Ind (see [4], Theorem 2.3.10), $\operatorname{Ind} F_{\alpha,n} \leqslant n$ for $n = 1, 2, \ldots$ and $\alpha < \varkappa(X)$. Moreover, since $\bigcup_{m \geqslant n} F_{\alpha,m}$ is closed in $X_\alpha$, there exists a closed subset $F_\alpha$ of $X$ such that

$$F_\alpha \cap X_\alpha = F_\alpha \cap S^\alpha(X) \cap X_\alpha = \bigcup_{m \geqslant n} F_{\alpha,m}$$

and thus

$$S^{\alpha+1}(X) \cup (F_\alpha \cap S^\alpha(X)) = S^{\alpha+1}(X) \cup \bigcup_{m \geqslant n} F_{\alpha,m},$$

so that the last set is closed in $X$ for $n = 1, 2, \ldots$ and $\alpha < \varkappa(X)$. Since the countable family $\{F_{\alpha,n}\}_{n=1}^{\infty}$ is locally finite, for each $x \in X$, there exists a positive integer $n$ such that $x \in F_{\alpha,n} \setminus \bigcup_{m > n} F_{\alpha,m}$ for $\alpha < \varkappa(X)$. The family

$$\mathscr{F} = \{F_{\alpha,n}\colon \alpha < \varkappa(X);\ n = 1, 2, \ldots\} \cup \{S^{\varkappa(X)}(X)\},$$

where the subfamily $\{F_{\alpha,n}\colon \alpha < \varkappa(X);\ n = 1, 2, \ldots\}$ is ordered lexicographically, with the last element $S^{\varkappa(X)}(X)$ is a well-ordered family of type $\gamma + 1$, where $\gamma = \omega_0 \cdot \varkappa(X)$. We let $A_\delta = F_{\tau,n}$ whenever $\delta = \omega_0 \cdot \tau + n$ and $\tau < \varkappa(X)$ and $A_\gamma = S^{\varkappa(X)}(X)$. It follows that the family $\mathscr{F}$ is a $((\gamma + \operatorname{Ind} A_\gamma) - D)$-representation and thus

$$D(X) \leqslant \gamma + \operatorname{Ind} A_\gamma = \omega_0 \cdot \varkappa(X) + \operatorname{Ind} S^{\varkappa(X)}(X) = \iota(X).$$

Let us finally emphasize that we distinguish between the two cases: $\alpha(X)$ is a limit number and $\alpha(X)$ is a non-limit number to avoid the difficulty caused by the empty set, whose dimension, $-1$, is no ordinal number.

The following two theorems give some evaluations of the transfinite dimensions tr ind and tr Ind in the realm of small spaces. The existence of $\operatorname{tr\,ind} X$ for every small metrizable space $X$ was announced by Kozlovskiĭ ([8]). The second evaluation was established by Henderson ([7]) for metrizable $X$.

THEOREM 2.6. *If a strongly hereditarily normal space $X$ is small, then $X$ has* tr ind *and*

$$\operatorname{tr\,ind} X \leqslant \sigma(X) = \omega_0 \cdot \alpha(X).$$

Proof. We proceed by induction with respect to the ordinal number $\alpha(X)$. If $\alpha(X) = 1$, then $X$ is locally finite-dimensional and thus $\operatorname{tr\,ind} X \leqslant \omega_0$ (see [4], Theorem 1.6.3). Let us assume now that the theorem is true for all spaces $Y$ with $\alpha(Y) < \alpha$ and consider a space $X$ with $\alpha(X) = \alpha$. There are two cases to consider.

Case 1. $\alpha$ is a limit number. For each $x \in X$, choose an open neighbourhood $U(x)$ and an ordinal number $\alpha(x) < \alpha$ such that $U(X) \cap S^{\alpha(x)}(X) = \emptyset$. By Lemma 2.4 (ii) and the inductive assumption,

$$\operatorname{tr\,ind} U(x) \leqslant \sigma(U(x)) = \omega_0 \cdot \alpha(U(x)) < \omega_0 \cdot \alpha \quad \text{for each } x \in X.$$

Thus, $\operatorname{tr\,ind} X \leqslant \omega_0 \cdot \alpha$.

Case 2. $\alpha$ has a predecessor $\beta$. By Lemma 2.4 (ii) and the inductive assumption, $\operatorname{tr\,ind}(X \setminus S^{\beta}(X)) \leqslant \omega_0 \cdot \beta < \omega_0 \cdot \alpha$. Consider a point $x \in S^{\beta}(X)$. There exists a neighbourhood $W$ of $x$ in $X$ such that $\operatorname{Ind} W \cap S^{\beta}(X) = n < \infty$. It is easy to prove by induction with respect to $n$, that

$$\operatorname{tr\,ind} W \leqslant \omega_0 \cdot \beta + n.$$

This implies that $\operatorname{tr\,ind} X \leqslant \omega_0 \cdot \alpha$ and the theorem is proved. □

A similar inductive argument will be applied to obtain an evaluation for tr Ind.

THEOREM 2.7. *If a strongly hereditarily normal weakly paracompact small space $X$ has* tr Ind, *then*

$$\operatorname{tr\,Ind} X \leqslant \iota(X) = \omega_0 \cdot \varkappa(X) + \operatorname{Ind} S^{\varkappa(X)}(X).$$

Proof. We prove the theorem by induction with respect to $\varkappa(X)$. If $\varkappa(X) = 0$, then $S(X) = \emptyset$, so that $\iota(X) = \operatorname{Ind} X$. Now, we assume that if $\varkappa(Y) < \alpha$, then $\operatorname{tr\,Ind} Y \leqslant \iota(Y)$ and consider a space $X$ such that $\varkappa(X) = \alpha$. We prove by induction with respect to the non-negative integer $n = \operatorname{Ind} S^{\varkappa(X)}(X)$ that $\operatorname{tr\,Ind} X \leqslant \omega_0 \cdot \alpha + n$. Let $A$, $B$ be a pair of disjoint closed subsets of $X$.

First, consider the case where $n = 0$. There exists a partition $L$ between $A$ and $B$ in $X$ such that $L \subset X \setminus S^{\varkappa}(X)$ (see [4], Theorem 2.2.4) and Lemma 2.4 (i) implies that $\varkappa(L) < \alpha$, so that by the inductive assumption

$$\operatorname{tr\,Ind} L \leqslant \omega_0 \cdot \varkappa(L) + \operatorname{Ind} S^{\varkappa(L)}(L) < \omega_0 \cdot [\varkappa(L)+1] \leqslant \omega_0 \cdot \alpha.$$

We assume, now, that if $\operatorname{Ind} S^{\varkappa(X)}(X) = m$, then $\operatorname{tr\,Ind} X \leqslant \omega_0 \cdot \varkappa(X) + m$ for $m = 0, 1, \ldots, n-1$ and consider the case where $\operatorname{Ind} S^{\varkappa(X)}(X) = n$. There exists a partition $L$ between $A$ and $B$ in $X$ such that $\operatorname{Ind} L \cap S^{\varkappa(X)}(X) \leqslant n-1$ and Lemma 2.4(i) implies that either $\varkappa(L) < \alpha$ or $\varkappa(L) = \alpha$ and $\operatorname{Ind} S^{\varkappa(L)}(L) \leqslant n-1$. In either case by the inductive assumption $\operatorname{tr\,Ind} L \leqslant \omega_0 \cdot \alpha + n - 1$, so that $\operatorname{tr\,Ind} X \leqslant \omega_0 \cdot \alpha + n$. Thus, the theorem is proved. □

We pass now to a discussion of closed mappings. We begin with a lemma.

LEMMA 2.8. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$ and there exists a positive integer $k$ such that $\operatorname{Ind} f^{-1}(y) \leqslant k$ for each $y \in Y$, then $f(S^{\alpha}(X)) \subset S^{\alpha}(Y)$ for every ordinal number $\alpha$.*

Proof. We prove the lemma by induction with respect to the ordinal number $\alpha$. First, we prove that $f(S(X)) \subset S(Y)$. Consider a point $y \in Y \setminus S(Y)$; there exists an open subset $U$ of $Y$ such that $y \in U$ and $\operatorname{Ind} \bar{U} < \infty$. The restriction $f_{\bar{U}}\colon f^{-1}(\bar{U}) \to \bar{U}$ being closed, it follows from the theorem on dimension-lowering mappings (see [4], Theorem 4.3.6), that $\operatorname{Ind} f^{-1}(\bar{U}) < \infty$, so that $\operatorname{Ind} f^{-1}(U) < \infty$. Thus, $f^{-1}(y) \subset X \setminus S(X)$. This implies that $f(S(X)) \subset S(Y)$. We assume now that $f(S^{\beta}(X)) \subset S^{\beta}(Y)$ whenever $\beta < \alpha$. To prove that $f(S^{\alpha}(X)) \subset S^{\alpha}(Y)$ we have to consider two cases.

Case 1. $\alpha$ has a predecessor $\beta$. The inductive assumption implies that

$$\begin{aligned} f(S^{\alpha}(X)) &= f\big(S(S^{\beta}(X))\big) = (f \mid S^{\beta}(X))\big(S(S^{\beta}(X))\big) \\ &\subset S\big(f(S^{\beta}(X))\big) \subset S(S^{\beta}(Y)) = S^{\alpha}(Y). \end{aligned}$$

Case 2. $\alpha$ is a limit number. The inductive assumption implies that

$$f(S^{\alpha}(X)) = f\Big(\bigcap_{\beta<\alpha} S^{\beta}(X)\Big) \subset \bigcap_{\beta<\alpha} f(S^{\beta}(X)) \subset \bigcap_{\beta<\alpha} S^{\beta}(Y) = S^{\alpha}(Y).$$

Thus, the lemma is proved. □

Remark 2.9. Let us observe that the above argument, employing the theorem on dimension-lowering mappings due to Zarelua [20], can be applied to prove the lemma in the case when $X$ is an $L$-space (see Nagami [15]) and the space $Y$ is normal.

THEOREM 2.10. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable small space $Y$, there exists a positive integer $k$ such that $\operatorname{Ind} f^{-1}(y) \leqslant k$ for each $y \in Y$ and the space $Y$ has* tr Ind, *then $X$ has* tr Ind *and $\iota(X) \leqslant \iota(Y) + k$, so that* $\operatorname{tr\,Ind} X \leqslant \iota(Y) + k$.

Proof. It follows from Theorem A that $\operatorname{tr} \operatorname{Ind} X$ is defined. By Lemma 2.8, $X$ is a small space, $\varkappa(X) \leqslant \varkappa(Y)$ and if $\varkappa(X) = \varkappa(Y)$, then $f(S^{\varkappa(X)}(X)) \subset S^{\varkappa(Y)}(Y)$; hence

$$\operatorname{Ind} S^{\varkappa(X)}(X) \leqslant \operatorname{Ind} S^{\varkappa(Y)}(Y) + k.$$

Thus,

$$\iota(X) = \omega_0 \cdot \varkappa(X) + \operatorname{Ind} S^{\varkappa(X)}(X) \leqslant \omega_0 \cdot \varkappa(Y) + \operatorname{Ind} S^{\varkappa(Y)}(Y) + k = \iota(Y) + k.$$

To conclude the proof it suffices to apply Theorem 2.7. □

Applying Theorem 2.6 and Lemma 2.8 in a similar way we obtain a counterpart of Theorem 2.10 for the dimension $\operatorname{tr} \operatorname{ind}$.

THEOREM 2.11. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $\operatorname{Ind} f^{-1}(y) \leqslant k$ for each $y \in Y$ and $Y$ is a small space, then $X$ has* $\operatorname{tr} \operatorname{ind}$ *and $\sigma(X) \leqslant \sigma(Y)$, so that* $\operatorname{tr} \operatorname{ind} X \leqslant \sigma(Y)$.

It should be noted that Theorem 2.11 holds in the case when $X$ is an $L$-space and $Y$ is normal (cf. Remark 2.9).

Lemma 2.8 can be strenghtened in the case of closed mappings with finite fibres.

LEMMA 2.12. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$ and there exists a positive integer $k$ such that $|f^{-1}(y)| \leqslant k$ for each $y \in Y$, then $f(S^\alpha(X)) = S^\alpha(Y)$ for every ordinal number $\alpha$.*

Proof. By Lemma 2.8, $f(S^\alpha(X)) \subset S^\alpha(Y)$ for every $\alpha$; to prove the reverse inclusion we apply induction with respect to $\alpha$. We start with $\alpha = 1$. Consider a point $y \in Y \setminus f(S(X))$. Since $f^{-1}(y) \subset X \setminus S(X)$, the finiteness of $f^{-1}(y)$ implies that there exists an open subset $W$ of $X$ such that $f^{-1}(y) \subset W \subset X \setminus S(X)$ and $\operatorname{Ind} W < \infty$. We let $V = Y \setminus f(X \setminus W)$. The restriction $f_V\colon f^{-1}(V) \to V$ being closed, it follows from the theorem on dimension-raising mappings (see [4], Theorem 4.3.3) that $\operatorname{Ind} V < \infty$. Thus, $y \in Y \setminus S(Y)$. This implies that $S(Y) \subset f(S(X))$. We assume now that $S^\beta(Y) = f(S^\beta(X))$ for each $\beta < \alpha$ and consider two cases.

Case 1. $\alpha$ has a predecessor $\beta$. It follows from the inductive assumption that

$$S^\alpha(Y) = S(S^\beta(Y)) = (f \mid S^\beta(X))(S(S^\beta(X))) = f(S^\alpha(X)).$$

Case 2. $\alpha$ is a limit number. It follows from the inductive assumption and the finiteness of the fibres of the mapping $f$ that

$$S^\alpha(Y) = \bigcap_{\beta<\alpha} S^\beta(X) = \bigcap_{\beta<\alpha} f(S^\beta(X)) = f\Big(\bigcap_{\beta<\alpha} S^\beta(X)\Big) = f(S^\alpha(X)).$$

Thus, the lemma is proved. □

Remark 2.13. Let us observe that the above argument, employing the theorem on dimension-raising mappings due to Morita [13] can be applied to prove the lemma in the case when $X$ is an $L$-space and $Y$ is a normal space (cf. Remark 2.9).

THEOREM 2.14. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $|f^{-1}(y)| \leqslant k$ and $X$ is a small space which has* tr Ind, *then $Y$ has* tr Ind *and $\iota(Y) \leqslant \iota(X)+k-1$, so that* $\operatorname{tr\,Ind} Y \leqslant \iota(X)+k-1$.

Proof. By Theorem A $\operatorname{tr\,Ind} Y$ is defined and it follows from Lemma 2.12 that $Y$ is a small space, $\varkappa(X) = \varkappa(Y)$ and $S^{\varkappa(Y)}(Y) = f(S^{\varkappa(X)}(X))$; hence $\operatorname{Ind} S^{\varkappa(Y)}(Y) \leqslant \operatorname{Ind} S^{\varkappa(X)}(X)+k-1$. Thus,

$$\iota(Y) = \omega_0 \cdot \varkappa(Y)+\operatorname{Ind} S^{\varkappa(Y)}(Y) \leqslant \omega_0 \cdot \varkappa(X)+\operatorname{Ind} S^{\varkappa(X)}(X)+k-1 = \iota(X)+k-1.$$

To conclude the proof it suffices to apply Theorem 2.7. □

Applying Theorem 2.6 and Lemma 2.12 in a similar way we obtain a counterpart of Theorem 2.14 for the dimension tr ind.

THEOREM 2.15. *If $f\colon X \to Y$ is a closed mapping of a metrizable space $X$ onto a metrizable space $Y$, there exists a positive integer $k$ such that $|f^{-1}(y)| \leqslant k$ for every $y \in Y$ and $X$ is a small space, then $Y$ has* tr ind *and $\sigma(X) = \sigma(Y)$, so that* $\operatorname{tr\,ind} Y \leqslant \sigma(X)$.

It should be noted that Theorem 2.15 holds in the case when $X$ is an $L$-space and $Y$ is a normal space (cf. Remark 2.13).

Let us add that in Baković [1] one can find some special cases of Lemmas 2.8 and 2.12.

**3.** In this section we establish some new facts about small spaces. We begin with a definition.

Definition 3.1. A normal space $X$ is said to be *strongly countable-dimensional* in the sense of Ind (abbreviation s.c.d.I.) if it can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces each of which is finite-dimensional in the sense of the dimension Ind.

We establish now a relation between the class of small spaces and the class of s.c.d.I. spaces.

THEOREM 3.2. *Every perfectly normal weakly paracompact small space $X$ is a* s.c.d.I. *space.*

Proof. For each $\alpha < \alpha(X)$, we let $X_\alpha = S^\alpha(X) \setminus S^{\alpha+1}(X)$. It follows from the assumptions that $X_\alpha$ is a $G_\delta$-set in $X$ for each $\alpha < \alpha(X)$. Now, as proved by Chaber ([2]), under the weaker assumption that $X$ is $\theta$-refinable rather than weakly paracompact, for the cover $\{X_\alpha\}_{\alpha<\alpha(X)}$ of $X$ there exists a closed refinement $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$, where the family $\mathscr{F}_n$ is discrete in $X$ for $n$

$= 1, 2, \ldots$ Consider a subspace $F \in \mathcal{F}$. Since $F$ is locally finite-dimensional, there exists an open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of $F$ such that $\operatorname{Ind} U_s < \infty$ for each $s \in S$. Since, for $k = 1, 2, \ldots$, the subspace $U_k(F) = \bigcup \{U \in \mathcal{U}: \operatorname{Ind} U \leqslant k\}$ is an $F_\sigma$-set in $X$, $U_k(F)$ is weakly paracompact (see [3], Exercise 5.3. C(b)) and thus $\operatorname{Ind} U_k(F) \leqslant k$ (see [4], Theorem 2.3.14); we let $U_k(F) = \bigcup_{j=1}^{\infty} K_{k,j}(F)$, where $K_{k,j}(F)$ is closed in $X$ for $k, j = 1, 2, \ldots$ For $n = 1, 2, \ldots$, $\operatorname{Ind} \bigcup \{K_{k,j}(F): F \in F_n\} \leqslant k$ for $k, j = 1, 2, \ldots$ and this implies that $X$ is a s.c.d.I. space. □

Theorem 3.2 was proved by Henderson ([7]) for separable metric spaces and announced by Kozlowskiĭ for metrizable spaces ([8]). In the case when the space $X$ is perfectly paracompact, Theorem 3.2 follows also from Theorem 4′ in Stone [18].

Let us note the following characterization of small spaces.

PROPOSITION 3.3. *For every strongly hereditarily normal space $X$ the following conditions are equivalent*:

a) *the space $X$ is small*;

b) *each non-empty subset $A$ of $X$ contains a non-empty relatively open subset $U$ such that* $\operatorname{Ind} U < \infty$;

c) *each non-empty closed subset $A$ of $X$ contains a non-empty relatively open subset $U$ such that* $\operatorname{Ind} U < \infty$.

Condition b), above, was introduced by Stone (cf. Theorem 3 in [18]); condition c), above, was discussed by Stone (l. c.), Nagami ([14]) and, in an equivalent form, by Henderson (cf. Theorem 1 in [7]).

From Proposition 3.3, applying the Baire category theorem which holds in Čech-complete spaces, we obtain

COROLLARY 3.4. *Every strongly hereditarily normal Čech-complete* s.c.d.I. *space $X$ is a small space.*

From the already mentioned fact that if a normal space $X$ has $\operatorname{tr} \operatorname{Ind}$, then $S(X)$ is countably compact and from Corollary 3.4 we obtain

COROLLARY 3.5. *Every strongly hereditarily normal weakly paracompact space $X$ that is a* s.c.d.I. *space and has* $\operatorname{tr} \operatorname{Ind}$ *is a small space.*

We now prove a sum theorem.

THEOREM 3.6. *If a strongly hereditarily normal space $X$ can be represented as the union of a locally finite family $\{F_s\}_{s\in S}$ of closed subsets, then $S^\alpha(X) = \bigcup_{s\in S} S^\alpha(F_s)$ for every ordinal number $\alpha$.*

Proof. We prove the theorem by induction with respect to the ordinal number $\alpha$. In the case when $\alpha = 1$ it suffices to prove that $S(X) \subset \bigcup_{s\in S} S(F_s)$; the reverse inclusion follows from Lemma 2.4 (i). Consider a point

$x \in X \setminus \bigcup_{s \in S} S(F_s)$; let $S_0 = \{s \in S\colon x \in F_s\}$. For each $s \in S_0$, we choose an open subset $U_s$ of $X$ such that $x \in U_s$ and $\operatorname{Ind}(U_s \cap F_s) < \infty$. The open subset $W = (X \setminus \bigcup_{s \in S \setminus S_0} F_s) \cap \bigcap_{s \in S_0} U_s$ of $X$ is a neighbourhood of $X$ such that $\operatorname{Ind} W < \infty$ (see [4], Theorem 2.2.5) and thus $x \in X \setminus S(X)$; the proof in this case is concluded. We assume now that $S^\beta(X) = \bigcup_{s \in S} S^\beta(F_s)$ for each $\beta < \alpha \geqslant 2$ and consider the two cases.

Case 1. $\alpha$ has a predecessor $\beta$. It follows from the inductive assumption applied to the locally finite cover $\{S^\beta(F_s)\}_{s \in S}$ of $S^\beta(X)$ and the already established case of $\alpha = 1$ that

$$S^\alpha(X) = S(S^\beta(X)) = \bigcup_{s \in S} S(S^\beta(F_s)) = \bigcup_{s \in S} S^\alpha(F_s).$$

Case 2. $\alpha$ is a limit number. Since

$$S^\alpha(X) = \bigcap_{\beta < \alpha} S^\beta(X) = \bigcap_{\beta < \alpha} \bigcup_{s \in S} S^\beta(F_s),$$

it suffices to show that $\bigcap_{\beta < \alpha} \bigcup_{s \in S} S^\beta(F_s) \subset \bigcup_{s \in S} S^\alpha(F_s)$. But, if a point $x \in X \setminus \bigcup_{s \in S} S^\alpha(F_s)$, then for each $s \in S$ such that $x \in F_s$ there exists an ordinal number $\alpha(s) < \alpha$ such that $x \in X \setminus S^{\alpha(s)}(F_s)$ and thus $x \in X \setminus \bigcup_{s \in S} S^{\tilde{\alpha}}(F_s)$, where $\tilde{\alpha} = \max\{\alpha(s)\colon x \in F_s\} < \alpha$.

This completes the proof. $\square$

COROLLARY 3.7. *If a strongly hereditarily normal space $X$ can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed subsets and $F_s$ is a small space for each $s \in S$, then $X$ is a small space and $\sigma(X) = \sup\{\sigma(F_s)\colon s \in S\}$. If, moreover $X$ is weakly paracompact and* $\operatorname{tr}\operatorname{Ind} X$ *is defined, then there exists an $s_0 \in S$ such that $\iota(F_{s_0}) = \iota(X)$.*

The inequality $D(X) \leqslant \sup\{D(F_s)\colon s \in S\}$ was proved by Henderson in the case of metrizable $X$ ([6]).

The next theorem is a counterpart for small spaces of the enlargement theorem for Ind (see [4], Theorem 4.1.19).

THEOREM 3.8. *If a metrizable small space $X$ is a subspace of a metrizable space $Z$, then there exists a subspace $\tilde{X}$ of $Z$ such that $X \subset \tilde{X}$, $\tilde{X}$ is a $G_\delta$-set in $Z$, $\tilde{X}$ is a small space and $\alpha(\tilde{X}) = \alpha(X)$.*

Proof. We prove the theorem by induction with respect to the number $\alpha(X)$. We can suppose that $X$ is dense in $Z$. We consider first the case when $\alpha(X) = 1$. Let $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$ be an open cover of $Z$, where $\mathscr{B}_m = \{V_s\}_{s \in S_m}$ is a discrete family of open subsets of $Z$ for $m = 1, 2, \ldots$, such that $\mathscr{B} | X$ contains a refinement of a cover of $X$ by open in $X$ finite-dimensional sets.

Space $X$ is locally finite-dimensional, so the closed subset $Y = \{z \in Z\colon \operatorname{Ind}(V \cap X) = \infty$ whenever $z \in V \in \mathscr{B}\}$ of $Z$ is disjoint from $X$. We let $\Omega_m = \{s \in S_m\colon \operatorname{Ind}(V_s \cap X) < \infty\}$ and $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$. For each $s \in \Omega$ there exists a $G_\delta$-set $Q_s$ in $Z$ with the following properties:

(1) $$V_s \cap X \subset Q_s \subset V_s;$$

(2) $$\operatorname{Ind} Q_s = \operatorname{Ind}(V_s \cap X).$$

We let $T = \bigcup_{s \in \Omega} (V_s \setminus Q_s)$ and $\tilde{X} = Z \setminus (Y \cup T)$. By virtue of (1) and (2) $\operatorname{Ind}(V_s \cap \tilde{X}) < \infty$ for each $s \in \Omega$ and thus the equality $\tilde{X} = \bigcup_{s \in \Omega} (V_s \cap \tilde{X})$ implies that $\alpha(\tilde{X}) = 1$. Clearly, $X \subset \tilde{X}$ and $\tilde{X}$ is a $G_\delta$-set in $Z$ so that the proof in case $\alpha(X) = 1$ is concluded.

We assume now that the theorem holds for all spaces $Y$ such that $\alpha(Y) < \alpha$ and consider a space $X$ with $\alpha(X) = \alpha$. There are two cases to consider.

Case 1. $\alpha$ has a predecessor $\beta$. Consider an open subset $U$ of $Z$ such that $U \cap X = X \setminus S^\beta(X)$. By Lemma 2.4 (ii) $\alpha(U \cap X) < \beta < \alpha$ and thus, by the inductive assumption, there exists a $G_\delta$-set $Q$ in $Z$ such that

(3) $$U \cap X \subset Q \subset U \quad \text{and} \quad \alpha(Q) = \alpha(U \cap X).$$

Consider also a closed subset $K$ of $Z$ such that $K \cap X = S^\beta(X)$. Since $\alpha(S^\beta(X)) = 1$, it follows from the already established special case of $\alpha(X) = 1$ that there exists a $G_\delta$-set $P$ in $Z$ such that

(4) $$S^\beta(X) \subset P \subset K \quad \text{and} \quad \alpha(P) = 1.$$

We let $T = U \setminus Q$ and $R = K \setminus P$; consider the subset $\tilde{X} = (U \cup K) \setminus (T \cup R)$ of $Z$. By virtue of (3) and (4) $X \subset \tilde{X}$; clearly $\tilde{X}$ is a $G_\delta$-set in $Z$. Since $U \cap \tilde{X} \subset Q$, it follows from Lemma 2.4 (ii) that $U \cap \tilde{X} \cap S^\beta(\tilde{X}) = \emptyset$, so that $S^\beta(\tilde{X}) \subset K \setminus (T \cup R) \subset P$. Thus, by virtue of (4), $S^{\beta+1}(\tilde{X}) = \emptyset$, so that $\tilde{X}$ is a small space and $\alpha(\tilde{X}) \leqslant \beta + 1 = \alpha$. By Lemma 2.4 (i) $\alpha(\tilde{X}) \geqslant \alpha$ and this concludes the proof in Case 1.

Case 2. $\alpha$ is a limit number. For each $x \in X$ there exists an ordinal number $\alpha(x) < \alpha$ and a neighbourhood $U(x)$ of $x$ in $X$ such that $U(x) \cap S^{\alpha(x)}(X) = \emptyset$ so that, by Lemma 2.4(ii), $\alpha(U(x)) < \alpha(x) < \alpha$. For each $x \in X$, consider an open subset $V(x)$ of $Z$ such that $V(x) \cap X = U(x)$. Consider the open subset $V = \bigcup_{x \in X} V(x)$ of $Z$ and a locally finite in $V$ open refinement $\{W_s\}_{s \in S}$ of the open cover $\{V(x)\}_{x \in X}$ of $V$. For each $s \in S$, we choose, by the inductive assumption, a $G_\delta$-set $Q_s$ in $Z$ that is a small space such that

(5) $$W_s \cap X \subset Q_s \subset W_s \quad \text{and} \quad \alpha(Q_s) = \alpha(W_s \cap X).$$

We let $T = \bigcup_{s \in S} (W_s \setminus Q_s)$ and consider the subset $\tilde{X} = V \setminus T$ of $Z$. It follows from (5) and Lemma 2.4 (i) that $W_s \cap \tilde{X} = Q_s \cap \tilde{X}$, hence $\alpha(W_s \cap \tilde{X}) \leqslant \alpha(Q_s) < \alpha$ for each $s \in S$. The paracompactness of $\tilde{X}$ implies that there exists a locally finite closed shrinking $\{F_s\}_{s \in S}$ of the open cover $\{W_s \cap \tilde{X}\}_{s \in S}$ of $\tilde{X}$. By Lemma 2.4 (i), $\alpha(F_s) \leqslant \alpha(W_s \cap X) < \alpha$ for each $s \in S$ and thus Corollary 3.7 implies that $\tilde{X}$ is a small space and $\alpha(\tilde{X}) \leqslant \alpha$. Clearly $X \subset \tilde{X}$ so that, by Lemma 2.4 (i), $\alpha(\tilde{X}) \geqslant \alpha$; $\tilde{X}$ being a $G_\delta$-set in $Z$, the proof in Case 2 is concluded. □

Let us add that Kozlovskiĭ announced ([8]) that for every small metrizable space $X$ there exists a completion $\tilde{X}$ such that $\overline{S^\alpha(X)}^{\tilde{X}} = S^\alpha(\tilde{X})$ whenever $\alpha < \alpha(X)$. This condition, however, can be easily satisfied by introducing minor modifications in the proof of Theorem 3.8 (cf. also [19], for the case of locally finite-dimensional metric separable $X$); a special case of Theorem 3.8 was discussed by Shmuely ([17]).

We now pass to Cartesian products of small spaces. We restrict ourselves to metrizable spaces, one easily notes, however, that the corresponding theorems hold for strongly hereditarily normal spaces $X_0$, $X_1$ with the property that $\operatorname{Ind}(U \times V) \leqslant \operatorname{Ind} U + \operatorname{Ind} V$ for each pair $U$, $V$ of open subsets, where $U \subset X_0$ and $V \subset X_1$.

The following lemma is obvious.

LEMMA 3.9. *For every pair of metric spaces $X_0$, $X_1$ we have*

$$S(X_0 \times X_1) = (S(X_0) \times X_1) \cup (X_0 \times S(X_1)).$$

We state now a more general result. For every ordinal number $\alpha$, we denote by $T(\alpha)$ the set of all pairs $(\beta, \gamma)$ or ordinal numbers such that $\beta \mp \gamma = \alpha$, where $\beta \mp \gamma$ denotes the natural sum of $\beta$ and $\gamma$ (see [10], Ch. VII, § 7). Clearly, $T(\alpha)$ is finite for every ordinal number $\alpha$.

THEOREM 3.10. *For every pair of metric spaces $X_0$, $X_1$ we have*

$$(*)\quad S^\alpha(X_0 \times X_1) = \bigcup \{(S^\beta(X_0) \times S^\gamma(X_1)) \cup (S^\gamma(X_0) \times S^\beta(X_1)) \colon (\beta, \gamma) \in T(\alpha)\}.$$

Proof. We prove the theorem by induction with respect to the ordinal number $\alpha$. From Lemma 3.9 it follows that the theorem is true for $\alpha = 1$. We obtain by induction that

$$(1)\qquad (S^\gamma(X_0) \times X_1) \cup (X_0 \times S^\gamma(X_1)) \subset S^\gamma(X_0 \times X_1)$$

for every ordinal number $\gamma$. We assume now that the theorem is true for $\beta < \alpha \geqslant 2$. We denote the right-hand side of $(*)$ by $\bigcup(\alpha)$. We prove first that $\bigcup(\alpha) \subset S^\alpha(X_0 \times X_1)$. Suppose, on the contrary, that $\bigcup(\alpha) \not\subset S^\alpha(X_0 \times X_1)$ and consider a point $(x_0, x_1) \in \bigcup(\alpha) \setminus S^\alpha(X_0 \times X_1)$. There exists an ordinal number $\alpha_0 < \alpha$ such that

(2) $$(x_0, x_1) \in S^{\alpha_0}(X_0 \times X_1) \setminus S^{\alpha_0+1}(X_0 \times X_1).$$

Thus, for some $(\beta_0, \gamma_0) \in T(\alpha_0)$, we have

(3) $$(x_0, x_1) \in S^{\beta_0}(X_0) \times S^{\gamma_0}(X_1).$$

This, along with (1) and (2), implies that

(4) $$x_0 \in S^{\beta_0}(X_0) \setminus S^{\beta_0+1}(X_0) \quad \text{and} \quad x_1 \in S^{\gamma_0}(X_1) \setminus S^{\gamma_0+1}(X_1).$$

Since $(x_0, x_1) \in \bigcup(\alpha)$, for some $(\beta_1, \gamma_1) \in T(\alpha)$, we have

(5) $$(x_0, x_1) \in S^{\beta_1}(X_0) \times S^{\gamma_1}(X_1).$$

Thus, by virtue of (4) and (5), $\beta_1 \leqslant \beta_0$ and $\gamma_1 \leqslant \gamma_0$, so that $\alpha = \beta_1 \mp \gamma_1 \leqslant \beta_0 \mp \gamma_0 = \alpha_0 < \alpha$, a contradiction. Thus, $\bigcup(\alpha) \subset S^\alpha(X_0 \times X_1)$.

To prove that $S^\alpha(X_0 \times X_1) \subset \bigcup(\alpha)$, we consider two cases.

Case 1. $\alpha$ has a predecessor $\beta$. By the inductive assumption $S^\beta(X_0 \times \times X_1) = \bigcup(\beta)$ and the equality $S^\alpha(X_0 \times X_1) = \bigcup(\alpha)$ follows from Lemma 3.9 and Theorem 3.6.

Case 2. $\alpha$ is a limit number. We may write $\alpha = \delta + \omega_0^\gamma$, where $\gamma > 0$. By the inductive assumption, $S^\delta(X_0 \times X_1) = \bigcup(\delta)$ and thus, since $S^\alpha(X_0 \times X_1) = S^{\omega_0^\gamma}(S^\delta(X_0 \times X_1))$, we can assume that $\alpha = \omega_0^\gamma$. We have to prove that $S^{\omega_0^\gamma}(X_0 \times X_1) \subset \bigcup(\omega_0^\gamma)$. But were it that $S^{\omega_0^\gamma}(X_0 \times X_1) \not\subset \bigcup(\omega_0^\gamma)$, we could choose a point $(x_0, x_1) \in S^{\omega_0^\gamma}(X_0 \times X_1)$ and an ordinal number $\tau < \omega_0^\gamma$ in such a way that

(6) $$x_0 \in S^\tau(X_0) \setminus S^{\tau+1}(X_0) \quad \text{and} \quad x_1 \in S^\tau(X_1) \setminus S^{\tau+1}(X_1).$$

Since $\tau \mp \tau + 1 < \omega_0^\gamma$, it follows from the inductive assumption that, by virtue of (6),

$$(x_0, x_1) \in X_0 \times X_1 \setminus S^{\tau \mp \tau + 1}(X_0 \times X_1) \subset X_0 \times X_1 \setminus S^{\omega_0^\gamma}(X_0 \times X_1),$$

a contradiction. Thus, the theorem is proved. □

For a pair $\alpha, \beta$ of ordinal numbers, we let

$$\gamma(\alpha, \beta) = \begin{cases} \text{predecessor of } \alpha \mp \beta & \text{if either } \alpha \text{ or } \beta \text{ is a non-limit number,} \\ \sup\{\sigma \mp \tau\colon \sigma < \alpha \text{ and } \tau < \beta\} & \text{otherwise.} \end{cases}$$

The following corollary follows from Theorem 3.10.

COROLLARY 3.11. *The Cartesian product $X_0 \times X_1$ of small metrizable spaces $X_0$, $X_1$ is a small space and*

$$\alpha(X_0 \times X_1) \leqslant \gamma(\alpha(X_0), \alpha(X_1)).$$

As a special case we obtain

COROLLARY 3.12. *If for small metric spaces $X_0$, $X_1$ the invariants $\iota(X_0)$ and $\iota(X_1)$ are defined, then $\iota(X_0 \times X_1)$ is defined and $\iota(X_0 \times X_1) \leqslant \iota(X_0) \mp \iota(X_1)$ so that, if* $\operatorname{tr}\operatorname{Ind}(X_0 \times X_1)$ *is defined, then*

$$\operatorname{tr}\operatorname{Ind}(X_0 \times X_1) \leqslant \iota(X_0) \mp \iota(X_1).$$

It should be noted that Henderson established ([6]) the inequality $D(X_0 \times X_1) \leqslant D(X_0) \mp D(X_1)$ for metrizable $X_0$ and $X_1$.

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