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Some properties of the class of arithmetic functions $T_r(N)$

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0. Introduction. For r, N positive integers, we define $T_r(N)$ as the number of integers k with $1 \leqslant k \leqslant N$ such that (k, N), the greatest common divisor of k and N, has no factor which is the r-th power of any prime. Some results for these functions have been obtained in [1]. In this note we show properties of these functions analogous to the ones enjoyed by Euler's totient function $\varphi(\equiv T_1)$. We show also that the T_r's, for $r \geqslant 2$, do not admit a restricted Busche-Ramanujan identity [2]. We recall from [1] that the T_r's are multiplicative.

1. Theorem A.

$$V_r(N) = \sum_{d^r|N} T_r\left(\frac{N}{d^r}\right) = N .$$

Proof. Let $N = \prod p^c$, where p's are distinct primes. Suffixes are omitted and the notation followed here is as in [3, p. 54]. Hence, for each d such that $d^r|N$, N/d^r is of the form $\prod p^a$ where $a = c, c-r, c-2r, \ldots, c-kr$; here k is the unique positive integer satisfying $0 \leqslant c-kr < r$.

$$\begin{aligned} V_r(N) &= \sum_{p,a} T_r(\prod p^a) \\ &= \prod_p \{T_r(p^c) + T_r(p^{c-r}) + \ldots + T_r(p^{c-kr})\} \\ &= \prod_p \{p^{c-kr} + p^{c-kr+r}(1-p^{-r}) + \ldots + p^c(1-p^{-r})\} \\ &= \prod_p \{p^{c-kr} + p^{c-kr}(p^r-1) + \ldots + p^{c-r}(p^r-1)\} \\ &= \prod_p p^c = N . \end{aligned}$$

Theorem B. *Let*

$$W_r(N) = \sum_{d^r|N} (-1)^{d-1} T_r\left(\frac{N}{d^r}\right);$$

then

$$W_r(N) = \begin{cases} N, & \text{if } \quad N = 2^n M, \quad \text{where } M \text{ is odd and } 0 \leqslant n \leqslant r-1, \\ N(1-2^{-(r-1)}), & \text{if } \quad N = 2^n M, \quad \text{where } M \text{ is odd and } n \geqslant r \end{cases}$$

Proof. If N is odd, all the divisors d are odd and then

$$W_r(N) = V_r(N) = N \quad \text{by Theorem A}.$$

Now let N be even, $= 2^n M$ where M is odd. Let $n = kr + m$ where k, m are integers such that $0 \leqslant m \leqslant r-1$, $0 \leqslant k$ and at least one of them is not equal to zero. Denote by D_1 the collection of all the odd divisors of N; by D_2 the even ones. Write

$$a_N = \sum_{\substack{g \in D_1 \\ g^r | N}} T_r\left(\frac{N}{g^r}\right), \quad b_N = \sum_{\substack{d \in D_2 \\ d^r | N}} T_r\left(\frac{N}{d^r}\right).$$

Hence

$$W_r(N) = a_N - b_N.$$

Now

$$a_N = \sum_{g^r | M} T_r\left(\frac{N}{g^r}\right) = \sum_{g^r | M} T_r(2^n)\, T_r\left(\frac{M}{g^r}\right) = T_r(2^n)\, M, \text{ by Theorem A},$$

$$b_N = \sum_{\substack{j=1 \\ g^r | M}}^{k} T_r\left(\frac{2^n M}{2^{rj} g^r}\right) = \sum_{g^r | M} T_r\left(\frac{M}{g^r}\right) \left\{ \sum_{j=0}^{k} T_r\left(\frac{2^n}{2^{rj}}\right) - T_r(2^n) \right\}$$

$$= M\{2^n - T_r(2^n)\}, \text{ again by Theorem } A.$$

Therefore $W_r(N) = 2M\,T_r(2^n) - N$. If $n \leqslant r-1$, this equals $2(2^n M) - N = N$. If $n \geqslant r$, then this equals $2M\,2^n(1-2^{-r}) - N = N(1-2^{-(r-1)})$.

COROLLARY (cf. [4], p. 214, exercise 31). *If N is even, then*

$$\sum_{d | N} (-1)^{d-1} \varphi\left(\frac{N}{d}\right) = 0.$$

2. We are using in this paragraph the terminology of [2]. The generating function of T_r for base p is $(1-x^r)/(1-px)$. Thus for $r \geqslant 2$, T_r is not an integral quadratic function, a totient function, the second convolute of a totient function or crosses between these types. Hence, by theorem XXXVIII of [2] we clonclude that T_r, when $r \geqslant 2$, does not admit a restricted Busche-Ramanujan identity.

References

[1] Paul J. McCarthy, *On a certain family of arithmetic functions*, Amer. Math Monthly 65 (1958), pp. 586-90.

[2] R. Vaidyanathaswamy, *The theory of Multiplicative Arithmetic Functions*, Trans. Amer. Math. Soc. 33 (1931), pp. 579-662.

[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 3rd ed., Oxford University Press, 1954.

[4] W. Sierpiński, *Teoria Liczb*, Cz. II, Warszawa 1959.

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