Approximation and interpolation of entire functions
and generalized orders

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Abstract. The problem of generalized orders of an entire function \( f \), defined by a series of polynomials and coinciding with the partial sum of series at a finite number of points, is studied. The same problem is considered when (i) the given function \( f \) is continuous on a compact set such that its complement with respect to the complex plane is connected and the approximation error tends to zero rapidly, (ii) \( f \) is holomorphic in the unit disc \( A \) and \( f \in L^2(A) \), (iii) \( f \) is entire and defined by a Newton series.

1. Introduction. In the first part of this paper, we consider the expansion of an entire function \( f(z) \) in a series of polynomials such that the \( n \)-th partial sum of the series coincides with \( f(z) \) at a given set of points. Let

\[
p(z) = \prod_{i=1}^{N} (z - z_i),
\]

and let \( f(z) \) be entire. Then \( f(z) \) can be expanded in a series

\[
f(z) = \sum_{k=1}^{\infty} q_k(z) (p(z))^{k-1},
\]

where \( q_k(z) \) is a uniquely determined polynomial of degree \( N - 1 \) or less ([14], p. 56). Further, \( S_n(z) = \sum_{k=1}^{n} q_k(z) (p(z))^{k-1} \) coincides with \( f(z) \) at the points \( z_i, i = 1, 2, \ldots, N \).

Rice [8] and Juneja and Kapoor [4] (see also Winiarski [15]) have studied entire functions \( f(z) \) defined by (1.2) and have obtained expressions for the order [8], Lemma 3, type [8], Lemma 5, and the \( q \)-order and the lower \( q \)-order [4] of \( f(z) \), analogous to those for entire functions defined by a power series (see [1], p. 9–12, [10], [11]). In Theorem 1 of this paper we obtain expressions for the generalized order \( g(\alpha, \beta, f) \) (defined below) for the function \( f(z) \) defined by (1.2). These results extend some of the theorems proved in [4] and [8].

Let \( \mu \) be a non-negative number and let the functions \( \alpha \) and \( \beta \) satisfy the following:
(H.i) \( a(x) \) and \( \beta(x) \) are positive, strictly increasing and differentiable on \([a, \infty) \) \((a > 0)\) and tend to infinity as \( x \to \infty \).

(H.ii) \( a(x) \) is slowly oscillating, that is, \( \lim_{x \to \infty} a(tx)/a(x) = 1 \) for every positive constant \( t \).

(H.iii) \( \beta(x)/x^\mu \) is slowly oscillating for some \( \mu \geq 0 \). If \( \mu = 0 \), we further assume that \( \beta(x^\psi(x))/\beta(x^\sigma) \) tends to zero as \( x \to \infty \) for some function \( \psi \) tending to \( \infty \) (however, slowly) as \( x \to \infty \).

(This implies that the growth of \( \beta \) is not "too slow".)

(H.iv) \( F(x, t) = \beta^{-1}(t a(x)) \) satisfies, for every positive constant \( t \),
\[
dF(x, t) \quad d(\log x) = O(1), \quad x \to \infty.
\]

Let \( f(z) \) be any entire function and write
\[
\varphi(a, \beta, f) = \lim_{r \to \infty} \sup_{r \to \infty} a(\log M(r, f)) \lambda(a, \beta, f) = \inf \frac{a(\log M(r, f))}{\beta(\log r)}.
\]
Then \( \varphi(a, \beta, f) \) is called the generalized order, and \( \lambda(a, \beta, f) \) the generalized lower order, of \( f(z) \) (cf. [9], [11]). Let \( I_R \) be the lemniscate
\[
I_R = \{ z \mid |p(z)| = R \} \quad \text{and} \quad ||I_R|| = \text{length of } I_R.
\]
Then \( ||I_R|| = 2\pi R^{1+\epsilon} (1 + o(1)) \), as \( R \to \infty \). Further there exists [8] a polynomial \( Q(z) \) of degree \( N - 1 \), independent of \( n \) and \( R \), such that for \( R > c > 0 \),
\[
\|q_n(z)\|_{r_c} \leq \left\{ \|I_R\| M(I_R, f) \|Q(z)\|_{r_c} \right\}/(2\pi R^n),
\]
where \( f(z) \) is defined by (1.2), \( M(I_R, f) = \max_{z \in I_R} |f(z)| \), and \( ||Q(z)||_{r_c} = \max_{z \in I_R} |Q(z)| \) and \( ||q_n(z)||_{r_c} = \max_{z \in I_R} |q_n(z)| \). In what follows we suppose that \( c > 1 \) is a fixed constant and write \( ||q_n|| \) for \( ||q_n(z)||_{r_c} \). We prove

**Theorem 1.** Let \( f(z) \) be an entire function defined by (1.2). Then we have:

\[
\varphi(a, \beta, f) = N^n \limsup_{n \to \infty} a(n)/\beta \left( -\frac{1}{n} \log ||q_n|| \right).
\]

\[
\lambda(a, \beta, f) \geq N^n \liminf_{n \to \infty} a(n)/\beta \left( -\frac{1}{n} \log ||q_n|| \right).
\]

(1.6) If \( \{m_k\} \) is any strictly increasing sequence of natural numbers, then
\[
\lambda(a, \beta, f) = N^n \sup_{m_k} \liminf_{k \to \infty} \alpha(m_{k-1})/\beta \left( -1/m_k \log ||q_{m_k}|| \right).
\]

(1.7) \( \lambda(a, \beta, f) = N^n \sup_{m_k} \liminf_{k \to \infty} \alpha(m_{k-1})/\beta \left( -1/(m_k - m_{k-1}) \log \left( ||q_{m_{k-1}}||/||q_{m_k}|| \right) \right) \),
where supremum, in (1.6) and (1.7), is taken over all sequences \( \{m_k\} \) and we define \( \|q_{m_{k-1}}\|/\|q_{m_k}\| \) equal to \( \infty \) when \( \|q_{m_k}\|=0 \) and \( \|q_{m_{k-1}}\| \) zero or otherwise.

Remarks. (a) Let \( \alpha(x) = \log x, \mu = 1, \beta(x) = x \). Then conditions (H,i)–(H,iv) are satisfied and we get an extension of Lemma 4 of [8]. Note that if the right-hand side of (1.4) is infinite, then \( g(\alpha, \beta, f) \) is infinite and conversely.

(b) Let \( \beta(x) = x \) and \( \alpha(x) = l_p x \) (\( p \)-th iterate of the logarithm, \( l_p x = \log x \)). Then we get Theorems 1, 3 and 4 of [4].

2. Approximation and interpolation. Next we consider, in Theorems 2 and 3, two approximation problems and, in Theorem 4, a function defined by Newton interpolation series.

(a) Let \( E \) be a compact set of points in the complex plane \( C \) and let \( K = C \setminus E \). We assume that \( K \) is connected and the transfinite diameter \( d(E) \), of the set \( E \), is positive. Let \( f(z) \) be continuous on \( E \) and write

\[
\|f\|_E = \sup_{z \in E} |f(z)|.
\]

Let \( P_n \) denote the set of all polynomials in \( z \) of degree not exceeding \( n \). Then for every \( f(z) \) continuous on \( E \), there is exactly one polynomial \( \pi_n \in P_n \) such that the approximation error

\[
E_n(f, E) = \inf_{p \in P_n} \|f - p\|_E = \|f - \pi_n\|_E.
\]

We prove

**Theorem 2.** Let \( f(z) \) be continuous on \( E \) and suppose that \( \{E_n(f, E)\}^{1/n} \to 0 \) as \( n \to \infty \). Then \( f(z) \) has an analytic extension \( \tilde{f}(z) \) which is an entire function. Furthermore, the function \( g(z) \) defined by

\[
g(z) = \sum_{n=0}^{\infty} E_n(f, E) z^n
\]

is entire and we have

\[
\epsilon(\alpha, \beta, g) = \epsilon(\alpha, \beta, \tilde{f}) = \epsilon(\alpha, \beta, f),
\]

\[
\lambda(\alpha, \beta, g) = \lambda(\alpha, \beta, \tilde{f}) = \lambda(\alpha, \beta, f);
\]

and formulae similar to (1.4) through (1.7), with \( N \) replaced by 1 and \( \|q_p\| \) by \( E_p(f, E), \) hold.

**Remark.** In (2.4) and the next theorem, we identify \( f(z) \) with its analytic extension \( \tilde{f}(z) \).

(b) Let \( L^2(\Delta) \) denote the class of functions \( f(z) \) which are holomorphic in the unit disc \( \Delta \) and for which \( \int_{\Delta} |f(z)|^2 \, dx \, dy < \infty, \, z = x + iy \). Let

\[
D_n(f) = \left\{ \min_{a_j} \int_{\Delta} |f(z) - \sum_{j=0}^{n} a_j z^j|^2 \, dx \, dy \right\}^{1/2}.
\]
It is known that if \( f(z) \in L^2(\Delta) \) and \( \{D_n(f)\}^{1/n} \to 0 \), then \( f(z) \) has an analytic extension \( \hat{f}(z) \) which is an entire function. (See [7] and the references given there.)

**Theorem 3.** Let \( f(z) \in L^2(\Delta) \) and \( \{D_n(f)\}^{1/n} \to 0 \) as \( n \to \infty \). Then \( f(z) \) has an analytic extension \( \hat{f}(z) \) which is an entire function. Furthermore,

\[
G(z) = \sum_{n=0}^{\infty} D_n(f) z^n
\]

is entire and we have

\[
\varrho(a, \beta, G) = \varrho(a, \beta, \hat{f}), \quad \lambda(a, \beta, G) = \lambda(a, \beta, \hat{f});
\]

and formulae similar to (1.4) through (1.7), with \( N \) replaced by 1 and \( \|q_p\| \) by \( D_p(f) \), hold.

(c) In Theorem 1, we considered the expansion (1.2), where \( f(z) \) is assumed to be entire and the number of points \( z_i \) is finite. We now let \( \{z_n\}_{0}^{\infty} \) be a bounded sequence of points and consider the series

\[
f(z) = \sum_{n=0}^{\infty} a_n w_{n-1}(z),
\]

where \( |a_n|^{1/n} \to 0 \) as \( n \to \infty \), and

\[
w_n(z) = (z-z_0)(z-z_1) \ldots (z-z_n) \quad (n = 0, 1, \ldots), \quad w_{-1}(z) = 1.
\]

Then \( f(z) \) defined by (2.8) is entire. The order and type of \( f(z) \), when \( \log M(r, f) = O(r^\nu) \), \( 0 < \nu < \infty \), have been studied by Winiarski [15] and Neidleman [5]. We consider here the generalized orders of \( f(z) \).

**Theorem 4.** Let \( f(z) \) defined by (2.8) be an entire function and let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \). Then \( h(z) \) is entire and

\[
\varrho(a, \beta, f) = \varrho(a, \beta, h), \quad \lambda(a, \beta, f) = \lambda(a, \beta, h).
\]

Further, the formulae similar to (1.4) through (1.7), with \( N \) replaced by 1 and \( \|q_p\| \) by \( |a_p| \), hold.

Remarks. (i) For the Newton series, with integer points \( z_n = n \), see [1], Chapter 9.

(ii) For the interpolation problem associated with an entire function and its derivatives, see [12].

3. **Proof of Theorem 1.** If \( f(z) \) is a polynomial, then (H,i) through (H,iv) show that \( \varrho(a, \beta, f) = \lambda(a, \beta, f) = 0 \). Further (1.3) shows that the right-hand side expressions of (1.4) through (1.7) are all zero. We suppose therefore that \( f(z) \) is not a polynomial. Let \( c \) be a positive constant not less than one and
(3.1) \[ F(z) = \sum_{n=1}^{\infty} \|q_n\| z^n, \]

where \( \|q_n\| = \|q_n(z)\|_{r_n} \). From (1.3) we get for all large \( R, R > R_0 \) say,
\[ \|q_n(z)\| \leq c \cdot M(2R^{1/N}, f)/R^{n-1}, \]
where \( c \) is a constant independent of \( n \) and \( R \). Consequently for \( |z| \leq R/2, R > R_0 \),
\[ \sum_{n=1}^{\infty} \|q_n\| |z|^n \leq c \cdot R \cdot M(2R^{1/N}, f) \sum_{n=1}^{\infty} |z|^n/R^n \leq c \cdot R \cdot M(2R^{1/N}, f). \]
This implies
\[ (3.2) \quad F(R/2) \leq c \cdot R \cdot M(2R^{1/N}, f). \]

Further
\[ M(G_R, f) \leq \sum_{k=1}^{\infty} \|q_k\| r_k R^{k-1} < \sum_{k=1}^{\infty} \|q_k\| r_n R^{k+N-1} \]
wherein we have used Walsh inequality ([14], Lemma, p. 77; [8]). Hence for all sufficiently large \( R \)
\[ (3.3) \quad M(R^{1/N}/2, f) < R^{N-1} F(R); \]
and from (3.2) and (3.3) we get
\[ (3.4) \quad \varphi(a, \beta, f) = N^n \varphi(a, \beta, F), \quad \lambda(a, \beta, f) = N^n \lambda(a, \beta, F). \]

By (1.9) and (1.11) of [11] applied to \( F \), we get first and second parts of Theorem 1. The remaining two parts of Theorem 1 follow from Theorem 4 of [11]. We sketch here an alternate proof of (1.6) and (1.7).

**Lemma 1.** Let \( a_n \in C \) and let \( \{m_k\}_{k=1}^{\infty} \) be any sequence of natural numbers. Write \( a(m_k) \) for \( a_{m_k} \) and suppose that \( |a_{n+1}|^{1/n} \to 0 \) as \( n \to \infty \). Write

\[ (3.5) \quad \lambda_0(\{m_k\}) = \liminf_{k \to \infty} a(m_{k-1})/\beta \left( \frac{1}{m_k} \log |a(m_k)| \right), \]

\[ (3.6) \quad \lambda_1(\{m_k\}) = \liminf_{k \to \infty} a(m_{k-1})/\beta \left( \frac{1}{m_k - m_{k-1}} \log \left| \frac{a(m_{k-1})}{a(m_k)} \right| \right). \]

Then
\[ (3.7) \quad \lambda_0(\{m_k\}) \geq \lambda_1(\{m_k\}). \]

The proof is straightforward and omitted. Note the convention, we made in the statement of Theorem 1, when \( a(m_k) = 0 \).
LEMMA 2. Let \( H(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function and let \( \{n_k\}_0^\infty \) be the range of the central index \( \nu(r, H) \) of \( H \). Let \( G(z) = \sum_{k=0}^{\infty} a(n_k) z^n/k \). Then \( G \) is entire and
\[
(3.8) \quad \lambda(a, \beta, H) = \lambda(a, \beta, G) = \lambda_1(\{n_{k+1}\}).
\]
This follows from Lemma 4 of [3] and Theorem 1 of [11].

LEMMA 3. Let \( H(z) = \sum_{n=0}^{\infty} a_n z^n \) be entire. Then for any sequence \( \{m_k\}_0^\infty \)
\[
(3.9) \quad \lambda(a, \beta, H) \geq \lambda_0(\{m_k\}).
\]
Proof. Write \( \lambda_0(\{m_k\}) = \lambda_0 \). We may suppose that \( \lambda_0 > 0 \). Given \( \varepsilon > 0 \), let \( \mu^* = \lambda_0 - \varepsilon \) if \( \lambda_0 \neq \infty \), \( \mu^* = T \) if \( \lambda_0 = \infty \), where \( T \) is an arbitrary large number. By the Cauchy inequality
\[
\log M(r, H) \geq \log |a(m_k)| + m_k \log r \\
\geq m_k \log r - m_k \beta^{-1}(a(m_{k-1})/\mu^*), \quad k > n_0.
\]
Choose \( R_k = \exp(\beta^{-1}(a(m_{k-1})/\mu^*)) \) and \( R_k \leq R < R_{k+1} \). Then
\[
a(\log M(R, H))/\beta(\log R) \geq a(m_k)/\beta(\log R_{k+1}) \quad \text{and} \quad (3.9) \text{ follows}.
\]

LEMMA 4. Let \( H(z) = \sum_{n=0}^{\infty} a_n z^n \) be entire. Then
\[
(3.10) \quad \lambda(a, \beta, H) = \sup_{\{m_k\}} \lambda(\{m_k\}) = \sup_{\{m_k\}} \lambda_1(\{m_k\})
\]
where the supremum is taken over all sequences \( \{m_k\} \).

The proof follows from Lemmas 1 through 3. The parts (1.6) and (1.7) of Theorem 1 follow if we apply (3.10) to \( F \).

4. Proof of Theorem 2. By our hypothesis, \( E \) has an infinite number of points. Following Winiarski [15] we write
\[
\xi^{(n)} = \{\xi_{n0}, \xi_{n1}, \ldots, \xi_{nn}\}
\]
for a system of \((n+1)\) points of \( E \), and
\[
V(\xi^{(n)}) = \prod_{0 \leq j < k \leq n} |\xi_{nj} - \xi_{nk}|,
\]
\[
A^{(j)}(\xi^{(n)}) = \prod_{k=0}^{n} |\xi_{nj} - \xi_{nk}|, \quad j = 0, 1, \ldots, n.
\]
Let the system of points \( \eta^{(n)} = \{\eta_{n0}, \ldots, \eta_{nn}\} \) of \( E \) satisfy the relations:
(i) \( V(\eta^{(n)}) = \sup_{\xi^{(n)} \in E} V(\xi^{(n)}) \), (ii) \( A^{(0)}(\eta^{(n)}) \leq A^{(j)}(\eta^{(n)}), j = 1, 2, \ldots, n \). Write
\[
I^{(j)}(x, \eta^{(n)}) = \prod_{k=0}^{n} (x - \eta_{nk})/(\eta_{nj} - \eta_{nk}), \quad j = 0, 1, \ldots, n.
\]
Then there exists a finite limit

\[ \lim_{n \to \infty} |L^{(n)}(z, \eta^{(n)})|^{1/n} = L(z) \geq 1 \]

for every \( z \) in \( C \setminus \mathcal{E} \) (see [15] and the references mentioned there). The convergence in (4.1) is uniform on every compact subset of \( C \setminus \mathcal{E} \), and \( L \) is a modulus of an analytic function \( \varphi \) in \( C \setminus \mathcal{E} \) which has a univalent branch

\[ \varphi(z) = \gamma z + \gamma_0 + \gamma_1/z + \ldots, \quad |\gamma| = 1/d(E) \]

in a neighborhood of infinity. Moreover, \( \log L(z) \) is the Green's function for \( K \) with singularity at \( \infty \).

By Bernstein–Walsh inequality (see [15], p. 264, [13]) and our hypothesis on \( E_n(f, E) \), the function \( \tilde{f}(z) \) defined by

\[ \tilde{f}(z) = \pi_1(z) + \sum_{n=2}^{\infty} (\pi_n(z) - \pi_{n-1}(z)) \]

is entire (see the details following (4.5) below; see also [15], p. 269). Further on \( E \), \( \tilde{f}(z) = f(z) \) and so \( \tilde{f}(z) \) provides the analytic extension of \( f(z) \) to the entire plane. By our hypothesis on \( E_n(f, E) \), \( g(z) \) is entire and we may suppose that it is not a polynomial. Hence \( \tilde{f}(z) \) is not a polynomial. Let \( r > 1 \) and

\[ E_r = \{ z \in C, \ d(E)L(z) = r \}, \quad \tilde{M}(r, \tilde{f}) = \sup_{z \in E_r} |\tilde{f}(z)|. \]

Then for all sufficiently large \( r \) [15]

\[ M(r/2, \tilde{f}) \leq \tilde{M}(r, \tilde{f}) \leq M(2r, \tilde{f}). \]

Now write \( \pi_0 = 0 \). Then

\[ |\tilde{f}(z)| \leq \sum_{n=1}^{\infty} |\pi_n(z) - \pi_{n-1}(z)|. \]

Hence by Bernstein–Walsh inequality we have for \( z \in E_r, \ r > r_0, \)

\[ |\tilde{f}(z)| \leq \sum_{n=1}^{\infty} \|\pi_n - \pi_{n-1}\|_n L^n(z) \leq 2 \sum_{n=1}^{\infty} E_{n-1}(f, E)(r/d)^n \]

wherein we have utilized (4.4), and \( \|p\|_n = \sup_{z \in E} |p(z)|. \) Since

\[ g(r/d) = \sum_{n=0}^{\infty} E_n(f, E)(r/d)^n, \]

we have for all sufficiently large \( r, \)

\[ g(r/d) \geq (d/2r)\tilde{M}(r, \tilde{f}) \geq (d/2r)M(r/2, \tilde{f}). \]
Now consider
\[ L_n(z) = \sum_{j=0}^{n} L^{(j)}(z, \eta^{(n)}) f(\eta_{nj}). \]

Given \( \epsilon > 0 \) we have for all \( r > R_0(\epsilon), n > n_1 \) [15]
\[ \|f - L_n\|_E \leq c_2 \frac{\bar{M}(r, \tilde{f})}{r^n} (de^*)^n, \]
where \( c_2 \) is a constant. Since
\[ E_n(f, E) \leq \|f - L_n\|_E, \]
we have
\[ |g(z)| \leq |p(z)| + c_2 \sum_{n=n_1+1}^{\infty} \bar{M}(r, \tilde{f}) \left( \frac{|z|de^*}{r} \right)^n, \]
where \( p(z) \) is a polynomial. Choosing \( r \) such that \( |z|de^* \leq r/2 \), we get
\[ M\left(r/(2de^*), g\right) \leq (1 + o(1)) 2c_2 \bar{M}(r, \tilde{f}). \]

This combined with (4.6) and (4.5) gives (2.4). The formulae similar to (1.4) through (1.7) follow, as in Theorem 1. The auxiliary function is now \( g(z) \).

5. Proof of Theorem 3. Let
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1, f \in L^2(\Delta). \]

We may suppose, as in Theorem 1, that \( f(z) \) is not a polynomial. Since
\[ D_n^2(f) = \sum_{k=n+1}^{\infty} \frac{\pi}{k+1} |a_k|^2 \]
and \( (D_n(f))^{1/n} \to 0 \), it follows that \( |a_n|^{1/n} \to 0 \) and \( f(z) \) is entire. From (5.2) and Cauchy’s inequality we get for \( n \geq 0, R \geq r_0, \)
\[ D_n(f) \leq M(R, f)/R^n. \]
Hence, taking \( 2r \leq R, \)
\[ M(r, G) \leq \sum_{n=0}^{\infty} \frac{M(R, f)}{R^n} r^n, \]
and so we have for all sufficiently large \( R \)
\[ M(R/2, G) \leq M(R, f). \]
Further
\[ M(r, G') = \sum_{n=1}^{\infty} nD_n r^{n-1} \geq \frac{1}{r^2} \sum_{n=1}^{\infty} n^{\frac{1}{n+2}} |a_{n+1}| r^{n+1} \]
\[ \geq \frac{1}{r^2} \sum_{n=1}^{\infty} |a_{n+1}| r^{n+1} \geq 1 + o(1)|M(r, f)/r^2; \]
and so
\[ M(r+1, G) > (1 + o(1))M(r, f)/r^2. \]

Relations in (2.7) follow from (5.4) and (5.5). The remaining statement follows, as in Theorems 1 and 2.

6. Proof of Theorem 4. We may assume that \( f(z) \) is not a polynomial.

Let
\[ h(z) = \sum_{n=0}^{\infty} a_n z^n. \]

By (2.9), \( h(z) \) is entire. Further, if \( L = \sup_{n \geq 0} |z_n|, \)
\[ M(|z|, f) \leq \sum_{n=0}^{\infty} |a_n| (|z| + L)^n \leq M(r, h) \sum_{n=0}^{\infty} \left( \frac{|z| + L}{r} \right)^n. \]

Hence
\[ M(r, f) \leq 2M(2r + 2L, h). \]

Further [15] for \( r > L, \)
\[ |a_n| \leq (r/(r - L))^{n+1} M(r, f)/r^n \]
and so
\[ M(|z|, h) \leq \sum_{n=0}^{\infty} |a_n| |z|^n \leq (rM(r, f))/(r - L - |z|). \]

Consequently
\[ M\left( \frac{r-L}{2}, h \right) \leq \frac{2rM(r, f)}{r - L}. \]

From (6.2) and (6.3) we get the required results.

Corollary. If \( |a_n|/a_{n+1} \) is a non-decreasing function of \( n \) for \( n > n_0, \)
then
\[ \lambda(\alpha, \beta, h) = \lambda(\alpha, \beta, f) = \liminf_{n \to \infty} a(n)/\beta \left( \frac{-1}{n} \log|a_n| \right). \]

This follows from Theorem 2 of [11].

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