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On injective multivalued semiflows

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Abstract. An injective multivalued semiflow is a mapping F from $\boldsymbol{R}_+ \times X$ into the set of all nonempty compact subsets of X which has properties analogous to those of negative funnel sections of local semiflows. If $x \in F(t, y)$, then the point y is uniquely appointed and we put $\pi_F(t, x) = y$. The aim of the paper is to prove that if X is a manifold, then the mapping π_F is a local semiflow if and only if all the sets $F(t, x)$ are acyclic in Alexander cohomologies over $\boldsymbol{Q}$.

1. Let X be a topological space and let D be a subset of $\boldsymbol{R}_+ \times X$ (by definition, $\boldsymbol{R}_+ = [0, \infty)$). A mapping

$$\pi\colon D \to X$$

is called a *local semiflow* if the following conditions are fulfilled (see [1]):

(P1) D is open in $\boldsymbol{R}_+ \times X$ and $\{0\} \times X \subset D$.

(P2) π is continuous.

(P3) For any $x \in X$ the set

$$D_x = \{t \in \boldsymbol{R}_+ \colon (t, x) \in D\}$$

is an interval $[0, \omega_x)$ for some $\omega_x > 0$.

(P4) $\pi(0, x) = x$ for any $x \in X$.

(P5) $s+t \in D_x$ if and only if $t \in D_x$ and $s \in D_{\pi(t,x)}$.

(P6) $\pi(s+t, x) = \pi(s, \pi(t, x))$ for any $x \in X$, $s+t \in D_x$ and $s \in D_{\pi(t,x)}$.

For any $x \in X$ and $t \in \boldsymbol{R}_+$, we define the set

$$F_\pi(t, x) = \{y \in X \colon \pi(t, y) = x\}$$

called a *funnel section*. We shall write F instead of F_π. Assume for any t and x the set $F(t, x)$ is nonempty. The multivalued mapping

$$F\colon \boldsymbol{R}_+ \times X \ni (t, x) \to F(t, x) \in \mathscr{P}(X)$$

($\mathscr{P}(X)$ denotes the set of nonempty subsets of X) has the following properties:

(F1) $$F(0, x) = \{x\} \quad \text{for any } x \in X,$$

(F2) $$F(t, F(s, x)) = F(t+s, x) \quad \text{for any } x \in X,\ t, s \in \boldsymbol{R}_+,$$

(F3) $$F(t, x) \cap F(t, y) \neq \emptyset \Rightarrow x = y.$$

Moreover, if we assume additionaly:

(F4) $F(t, x)$ is compact for any $x \in X$ and $t \in \boldsymbol{R}_+$,

then (see [3]):

(F5) F is upper semicontinuous.

2. In this section we reverse the situation considered in Section 1. Let F be a multivalued mapping

$$F\colon \boldsymbol{R}_+ \times X \to \mathscr{P}(X);$$

F is called an *injective multivalued semiflow* provided it fulfils conditions (F1)–(F5).

If $x \in F(t, y)$ for some y and t, the point y is uniquely determined and by definition we put

$$\pi_F(t, x) = y.$$

We ask whether the mapping

$$\pi_F\colon D \to X$$

(denoted in the sequel by π), where

$$D = \{(t, x) \in \boldsymbol{R}_+ \times X\colon x \in F(t, y) \text{ for some } y \in X\},$$

is a local semiflow, i.e., conditions (P1)–(P6) are fulfilled.

The following example shows the mapping $\pi = \pi_F$ need not be a local semiflow even in the case $X = \boldsymbol{R}$. Let us put for $x \in \boldsymbol{R}$ and $t \in \boldsymbol{R}_+$

$$F(t, x) = \begin{cases} \{t + x\} & \text{if } x > 0, \\ \{-t + x\} & \text{if } x < 0, \\ \{-t, t\} & \text{if } x = 0. \end{cases}$$

Condition (P1) is not fulfilled.

3. Let F and π be the same as in the previous section. We present a necessary and sufficient condition under which π is a local semiflow.

PROPOSITION. *If X is metrizable, then π is a local semiflow if and only if the set $F(t, U)$ is open for any $t \in \boldsymbol{R}_+$ and U open in X.*

Proof. *Only if* part is obvious.

If. It is easy to verify that conditions (P4), (P5) and (P6) are always fulfilled. Moreover, one can see that:

(*) If $x \in X$ and $t \in D_x$, then $[0, t] \subset D_x$ and if $\tau \in [0, t]$ then:

$$\pi(\tau, x) \in F\big(t - \tau, \pi(t, x)\big).$$

Let $x \in X$. Define $\omega_x = \sup D_x$. Assume $t \in D_x$. In order to prove (P3) it is sufficient to prove that there exists an $\varepsilon > 0$ such that $t + \varepsilon \in D_x$ since (*) is valid. By assumptions, $F(t, X)$ is open and there exists y, $x \in F(t, y)$. By (F5) there is $0 < \varepsilon \leqslant t$ such that

$$F(t-\varepsilon, y) \subset F(t, X).$$

Since $F(t, y) = F(\varepsilon, F(t-\varepsilon, y))$, there exists $w \in F(t-\varepsilon, y)$, $x \in F(\varepsilon, w)$. Thus $w \in F(t, z)$ for some $z \in X$ and $x \in F(t+\varepsilon, z)$.

Condition (P3) is proved.

Let $(t, x) \in D$. We have just proved the existence of an $\varepsilon > 0$ such that $(t+\varepsilon, x) \in D$. By (*) the set $[0, t+\varepsilon) \times F(t+\varepsilon, X)$ is an open neighbourhood of (t, x) and is contained in D. Thus (P1) is proved.

Now we will prove (P2). The proof will be divided into five steps.

Step 1. For any $x \in X$ the mapping $\pi(\cdot, x)$: $D_x \to X$ is continuous at 0.

Assume the assertion is false. There exists an open neighbourhood U of x and a sequence $\{t_n\}$, $t_n > 0$, $t_n \to 0$ if $n \to \infty$ such that for any $n \in N$

$$\pi(t_n, x) \notin U.$$

Without loss of generality we can assume $t_n < r$ for some $r < \omega_x$. Let $z = \pi(r, x)$ and $y_n = \pi(t_n, x)$. Thus $y_n \in F(r-t_n, z)$. It is easy to verify that the sequence $\{y_n\}$ has an accumulation point $y \in F(r, z)$, $y \neq x$ and we can assume $y_n \to y$. We can find V, an open neighbourhood of y, $x \notin V$. (F5) implies $F([0, \delta), W) \subset V$ for some $\delta > 0$ and W open, $y \in W$. In n is sufficiently large, $y_n \in W$ and $t_n < \delta$. Thus $x \in F(t_n, y_n) \subset V$ which is impossible.

Step 2. For any $x \in X$ the mapping $\pi(\cdot, x)$: $D_x \to X$ is continuous.

Let U be open, $\pi(t, x) \in U$. (F5) implies the existence of an $\varepsilon_1 > 0$, $F([0, \varepsilon_1), \pi(t, x)) \subset U$. From Step 1 we obtain an $\varepsilon_2 > 0$, $\pi([0, \varepsilon_2), \pi(t, x)) \subset U$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\tau \in (t-\varepsilon, t+\varepsilon)$. If $\tau - t \geqslant 0$, then $\pi(\tau, x) = \pi(t+(\tau-t), x) \subset U$. If we assume $t-\tau \geqslant 0$, then (*) implies $\pi(\tau, x) \in F(t-\tau, \pi(t, x)) \subset U$.

Step 3. Write $D^t = \{x \in X$: $(t, x) \in D\}$. Then D^t is open for any $t \geqslant 0$ and the mapping $\pi(t, \cdot)$: $D^t \to X$ is continuous.

Indeed, let U be an open neighbourhood of $\pi(t, x)$. The assumption implies that $F(t, U)$ is open. $x \in F(t, \pi(t, x)) \subset F(t, U)$. Using (F3), one can easy verify that $\pi(t, F(t, U)) \subset U$.

Step. 4. For any $x \in X$ the mapping π: $D \to X$ is continuous in $(0, x)$.

Let U be an open set, $x \in U$. There exists $\varepsilon > 0$ and V, an open neighbourhood of x, such that $F([0, \varepsilon), V) \subset U$. By Step 1 there exists $\delta \in (0, \varepsilon]$, $\pi(\delta, x) \in V$. Step 3 implies there exists an open set W, $x \in W$, $\pi(\delta, W) \subset V$. Using (*) we can prove $\pi([0, \delta), W) \subset F([0, \varepsilon), \pi(\delta, W))$ and thus $\pi([0, \delta), W) \subset U$.

Step 5. The mapping π: $D \to X$ is continuous.

It suffices to prove the continuity of π in $(t, x) \in D$, $t > 0$. Since X is metrizable we can apply results of [2] to Steps 2 and 3 and obtain the assertion. Thus Proposition is proved.

4. A topological space is called *acyclic* iff its reduced Alexander cohomology modules over $\boldsymbol{Q}$ are equal to 0. If the space is compact, one can replace the Alexander cohomologies by the Čech homologies (see [4] and [5]).

A metrizable space is called a *manifold* if any point has a neighbourhood homeomorphic with $\boldsymbol{R}^n$.

Now we can formulate the main result of the paper.

THEOREM. *If X is a manifold, F is an injective multivalued semiflow on X, then the following conditions are equivalent.*

(1) *The mapping π_F is a local semiflow.*

(2) *For any $t \in \boldsymbol{R}_+$ and U open in X the set $F(t, U)$ is open.*

(3) *For any $t \in \boldsymbol{R}_+$ and $x \in X$ the set $F(t, x)$ is acyclic.*

Proof. Proposition states the equivalence $(1) \Leftrightarrow (2)$.

$(1) \Rightarrow (3)$. This theorem is proved in [6].

$(3) \Rightarrow (2)$. If the set $F(t, U)$ is contained in a coordinate neighbourhood, the assertion follows from VII (3, 5) in [4]. This conclusion implies the following statement.

If K is compact subset of X, then there exists an $\varepsilon > 0$ such that for any $x \in K$ there exists U, an open neighbourhood of x such that if $V \subset U$, V is open and $\tau \in [0, \varepsilon]$ then $F(\tau, V)$ is open.

In order to prove (2) it suffices to assume that U is relatively compact. Let t be an arbitrary positive real number. Since F is upper semicontinuous, the set $F([0, t], \bar{U})$ is compact and thus we can choose an ε to it as in the statement. Let $t = k\delta$ for some $k \in \boldsymbol{N}$ and $0 < \delta \leqslant \varepsilon$. The statement and (F2) imply that $F(i\delta, U)$ is open for any $i = 1, \ldots, k$, thus (2) holds.

References

[1] N. P. Bhatia, O. Hajek, *Local Semi-Dynamical Systems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin–Heidelberg–New York 1969.

[2] P. Chernoff, J. Marsden, *On continuity and smothness of group actions*, Bull. Amer. Math. Soc. 76 (1970), 1044.

[3] S. Elaydi, *Semidynamical Systems with Nonunique Global Backward Extension*, Funkcialaj Ekvacioj 26 (1983), 173–187.

[4] L. Górniewicz, *Homological methods in fixed point theory of multivalued maps*, Dissert. Math. 129, Warszawa 1976.

[5] E. H. Spanier, *Algebraic topology*, New York 1966.

[6] R. Srzednicki, *On funnel sections of local semiflows*, Bull. Acad. Polon. Sci. 34 (3–4) (1986), 203–209.

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