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AN L-SUBSPACE GENERATED BY A CERTAIN MEASURE WITH COUNTABLE SPECTRUM

BY

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1. Let G be a non-discrete L.C.A. group and let $\hat{G}$ be the dual group of G. Let $L^1(G)$ and $M(G)$ be the group algebra on G and the measure algebra on G, respectively. For $\mu \in M(G)$, we write $\lambda \ll \mu$ ($\lambda \perp \mu$) if λ is absolutely continuous (singular) with respect to μ and we put

$$L^1(\mu) = \{\lambda \in M(G);\ \lambda \ll \mu\}.$$

A closed subspace (ideal, subalgebra) $N \subset M(G)$ is called an *L-subspace* (*L-ideal, L-subalgebra*) if $L^1(\mu) \subset N$ for every $\mu \in N$. $L^1(\mu)$ is an L-subspace generated by μ. For an L-subspace N, we put

$$N^{\perp} = \{\lambda \in M(G);\ \lambda \perp N\}.$$

An L-subalgebra N (L-ideal) is called *prime* if $N^{\perp}$ is an L-ideal (L-subalgebra). For $\mu \in M(G)$, we put $\mu^*(E) = \overline{\mu(-E)}$ for every Borel subset of G. By [7], there exist a compact topological abelian semigroup S and an isometric isomorphism θ of $M(G)$ into $M(S)$ such that

(i) $\theta(M(G))$ is a weak-* dense subalgebra of $M(S)$;

(ii) for $f \in \hat{S}$, $M(G) \ni \mu \to \int_S f d\theta\mu$ is a non-zero complex homomorphism of $M(G)$, where $\hat{S}$ is the set of all continuous semicharacters on S;

(iii) for a non-zero complex homomorphism F of $M(G)$, there is an $f \in \hat{S}$ such that

$$F(\mu) = \int_S f d\theta\mu \quad \text{for } \mu \in M(G).$$

Thus $\hat{S}$ is the maximal ideal space of $M(G)$ and $\hat{G} \subset \hat{S}$, and the Gelfand transform of $\mu \in M(G)$ is given by

$$\hat{\mu}(f) = \int_S f d\theta\mu \quad \text{for } f \in \hat{S}.$$

We denote by $\mathfrak{M}$ the set of all symmetric measures, that is,

$$\mathfrak{M} = \{\mu \in M(G);\ (\mu^*)\hat{}\,(f) = \overline{\hat{\mu}(f)} \text{ for each } f \in \hat{S}\}.$$

We denote by $\mathrm{Rad}L^1(G)$ the radical of $L^1(G)$ in $M(G)$, that is,

$$\mathrm{Rad}L^1(G) = \{\mu \in M(G);\ \hat{\mu}(f) = 0 \text{ for every } f \in \hat{S}\setminus\hat{G}\}.$$

Let $\mathscr{T}$ be the set of all L.C.A. group topologies on G which are stronger than the original one. We put

$$\mathscr{L}(G) = \sum_{\tau\in\mathscr{T}} \mathrm{Rad}L^1(G_\tau),$$

where G_τ is an L.C.A. group with the same underlying group G and topology τ. We denote by $\mathrm{Spec}_G(\mu)$ the spectrum of $\mu \in M(G)$, that is,

$$\mathrm{Spec}_G(\mu) = \{\hat{\mu}(f);\ f \in \hat{S}\}.$$

We put

$$\mathfrak{N}(G) = \{\mu \in M(G);\ \mathrm{Spec}_G(\mu) \text{ is a countable set}\}.$$

In [4], the author studied the properties of $\mathfrak{N}(G)$ in connection with the topological structure of G. We note that generally $\mathfrak{N}(G)$ is not norm closed and $\mathfrak{N}(G) \subset \mathfrak{M}$. For a compact subgroup H of G, we have $\mathrm{Rad}L^1(H) \subset \mathfrak{N}(G)$. It is an interesting question whatever $\mu \in \mathfrak{N}(G)$ satisfies $L^1(\mu) \subset \mathfrak{N}(G)$. In this paper, we show

THEOREM. *Assume that a measure μ on an L.C.A. group G satisfies* (a) $\mu \geqslant 0$ *and each non-zero point $x \in \mathrm{Spec}_G(\mu)$ is isolated in* $\mathrm{Spec}_G(\mu)$. *Then $L^1(\mu) \subset \mathfrak{N}(G)$.*

Remark. Some non-trivial examples of μ ($\mu \notin \mathscr{L}(G)$) satisfying condition (a) are known by [2], [3] and [6].

2. We prove our Theorem after showing 5 lemmas. For $x \in G$ we denote by δ_x and by δ_0 the unit point mass at x and at the identity of G, respectively.

LEMMA 1. *Let G be an infinite discrete abelian group. If $\mu \in M(G)$ satisfies condition* (a) *of the Theorem, then we have*

$$\mu = \sum_{j=1}^{k} a_j \delta_{x_j},$$

where $a_j > 0$ and $x_j \in G$ has finite order ($j = 1, 2, \ldots, k$).

Proof. Suppose that

$$\mu = \sum_{j=1}^{\infty} a_j \delta_{x_j}, \qquad a_j > 0 \ (j = 1, 2, \ldots)$$

and $x_i \neq x_j$ if $i \neq j$. Suppose that x_j has infinite order for some positive integer j. Let G_0 be the subgroup generated by all finite order elements of G. Then G/G_0 is an infinite group and every non-zero element of G/G_0 has infinite order, so that $(G/G_0)^\wedge$ is a connected compact group (see [5],

p. 47). Let φ be the canonical homomorphism of G onto G/G_0 and let Φ be the homomorphism of $M(G)$ onto $M(G/G_0)$ induced by φ. Then we have $\hat{\mu}(\hat{G}) \supset (\Phi\mu)^{\wedge}(G/G_0)^{\wedge}$ (see [5], p. 54). Since

$$\Phi\mu \neq \delta_0 \ (0 \in G/G_0) \quad \text{and} \quad \mathrm{Spec}_{G/G_0}(\Phi\mu) = (\Phi\mu)^{\wedge}(G/G_0)^{\wedge},$$

$\mathrm{Spec}_{G/G_0}(\Phi\mu)$ is not a countable set. Since $\mathrm{Spec}_G(\mu) = \hat{\mu}(\hat{G})$, $\mathrm{Spec}_G(\mu)$ is not a countable set. But this is a contradiction. Thus x_j has finite order $(j = 1, 2, \ldots)$. Let G_n be the group generated by $\{x_1, x_2, \ldots, x_n\}$ and let G_∞ be the group generated by $\{x_1, x_2, \ldots\}$; then G_n is a finite subgroup of G_∞ and $G_n \neq G_\infty$. Since $\|\mu\| \in \mathrm{Spec}_G(\mu)$ is isolated, there is an $\varepsilon > 0$ such that

$$\mathrm{Spec}_G(\mu) \cap \{x \in C;\ |x - \|\mu\|| < \varepsilon\} = \{\|\mu\|\}, \tag{1}$$

where C is the complex number plane. There is a positive integer n_0 such that

$$\Big\|\mu - \sum_{j=1}^{n_0} a_j \delta_{x_j}\Big\| < \frac{\varepsilon}{2}.$$

Since $\hat{G}_\infty \setminus G_{n_0}^{\perp} \neq \{0\}$, there exists a $\gamma \in \hat{G}_\infty \setminus G_{n_0}^{\perp}$ such that $\gamma(x_i) = 1$ if $1 \leqslant i \leqslant n_0$ and $\gamma(x_j) \neq 1$ for some integer j, where $G_{n_0}^{\perp}$ is the annihilator of G_{n_0} in $\hat{G}_\infty$. Consequently, $|\hat{\mu}(\gamma) - \|\mu\|| < \varepsilon$ and $\hat{\mu}(\gamma) \neq \|\mu\|$. But this contradicts (1). Thus we have

$$\mu = \sum_{j=1}^{k} a_j \delta_{x_j} \quad \text{for some } k.$$

LEMMA 2. *Let H be a compact open subgroup of an L.C.A. group G. If $\mu \in M(G)$ satisfies condition* (a) *of the Theorem, then there is a compact subgroup H' of G such that $\mu \in M(H')$, $H \subset H'$ and H'/H is a finite group.*

This is clear by Lemma 1.

For a compact subgroup H of G, there is an L.C.A. group topology on G such that H is a compact open subgroup; we denote its L.C.A. group by G_H. For $\lambda \in M(G)$, we denote by λ_H the part of λ which is contained in $M(G_H)$.

LEMMA 3. *Let G be an L.C.A. group and assume that $\mu \in M(G)$ satisfies condition* (a) *of the Theorem. Then there exist compact subgroups $H_n \subset G$ $(n = 1, 2, \ldots)$ such that, for every $\lambda \in L^1(\mu)$, λ_{H_n} coincides with the part of λ which is concentrated on H_n, and*

$$\mathrm{Spec}_G(\lambda) \subset \{0\} \cup \bigcup_{n=1}^{\infty} \hat{\lambda}_{H_n}(\hat{H}_n).$$

Proof. Let μ be a measure satisfying condition (a) of the Theorem. We put $\hat{S}^+ = \{f \in \hat{S}; f \geqslant 0\}$. For $f, g \in \hat{S}^+$ we write $f \geqslant g$ if $f(x) \geqslant g(x)$ for every $x \in S$. Since

$$\mathrm{Spec}_G(\mu) \supset \{\hat{\mu}(f); f \in \hat{S}^+\},$$

$\{\hat{\mu}(f); f \in \hat{S}^+\}$ is a countable set. We put

$$\{\hat{\mu}(f); f \in \hat{S}^+\} \setminus \{0\} = \{a_n\}_{n=1}^{\infty} \quad (a_n \neq 0).$$

Let $F_n = \{f \in \hat{S};\ \hat{\mu}(f) = a_n\}$ and $F_n^+ = \{f \in F_n; f \in \hat{S}^+\}$; then F_n is an open compact subset of $\hat{S}$ and $F_n^+ \neq \emptyset$. By Šilov's idempotent theorem, there exists an idempotent $\eta_n \in M(G)$ such that $\hat{\eta}_n = 1$ on F_n and $\hat{\eta}_n = 0$ on $\hat{S} \setminus F_n$. By Cohen's idempotent theorem, there exist compact subgroups $\{K_j\}_{j=1}^{k}$ of G such that

$$\eta_n = \sum_{i=1}^{k} b_i \gamma_i m_{K_i},$$

where $\gamma_i \in \hat{G}$, b_i is an integer and m_{K_i} is the normalized Haar measure on K_i $(i = 1, 2, \ldots, k)$. For $f_a \in F_n^+$ we have $\hat{\eta}_n(f_a) = 1$. Then we obtain

$$\{K_i;\ \hat{m}_{K_i}(f_a) = 1\} \neq \emptyset$$

and denote by P_a the compact subgroup of G which is generated by $\{K_i;\ \hat{m}_{K_i}(f_a) = 1\}$. It is clear that $\Lambda_n = \{P_a; f_a \in F_n^+\}$ is a finite set. For $P_a \in \Lambda_n$ we put

$$\chi_a(\lambda) = \int_G d\lambda_{P_a} \quad \text{for } \lambda \in M(G).$$

Then $\chi_a \in \hat{S}$, $\chi_a \geqslant 0$, $\chi_a^2 = \chi_a$ and $f_a \geqslant \chi_a$. Since $\hat{\eta}_n(\chi_a) = 1$, we have $\chi_a \in F_n^+$. We denote by F_n^0 the collection of all such $\chi_a \in F_n^+$. Since Λ_n is a finite set, so is F_n^0. We put

$$F = \bigcup_{n=1}^{\infty} F_n^0 = \{\chi_1, \chi_2, \ldots\} \quad \text{and} \quad \Lambda = \bigcup_{n=1}^{\infty} \Lambda_n.$$

For $\chi_n \in F$ we put

$$\Gamma_{\chi_n} = \{g \in \hat{S};\ |g| = \chi_n\};$$

then there is a $P_n \in \Lambda$ and we can regard that $\Gamma_{\chi_n} = \hat{G}_{P_n}$. Since $M(G_{P_n})$ is a prime L-subalgebra of $M(G)$, we have $\mathrm{Spec}_{G_{P_n}}(\mu_{P_n}) \subset \mathrm{Spec}_G(\mu)$, and $\mu_{P_n} \in M(G_{P_n})$ satisfies condition (a) of the Theorem. By Lemma 2, there is a compact subgroup H_n of G_{P_n} such that

(2) $\mu_{P_n} \in M(H_n)$, $H_n \supset P_n$ and H_n/P_n is a finite group.

Let $\lambda \in L^1(\mu)$, that is, $|\lambda| \ll \mu$. For $h \in \hat{S}$, if $\hat{\mu}(|h|) = 0$, then $\hat{\lambda}(h) = 0$, and if $\hat{\mu}(|h|) \neq 0$, then $|h| \in F_n^+$ for a positive integer n. Then there exists a $\chi_s \in F_n^0$ such that $|h| \geqslant \chi_s$. Since $\hat{\mu}(\chi_s) = \hat{\mu}(|h|)$ and $\mu \geqslant 0$, we have

$$|\lambda|^{\wedge}(\chi_s) = |\lambda|^{\wedge}(|h|) \quad \text{and} \quad \hat{\lambda}(h) = \hat{\lambda}(\chi_s h).$$

Since $\chi_s h \in \Gamma_{\chi_s}$, we have

$$\mathrm{Spec}_G(\lambda) \subset \{0\} \cup \bigcup_{\chi_s \in F} \hat{\lambda}(\Gamma_{\chi_s}) = \{0\} \cup \bigcup_{\chi_s \in F} \hat{\lambda}_{P_s}(\hat{G}_{P_s}).$$

By (2), we have

$$\lambda_{P_s} = \lambda_{H_s} \in M(H_s) \quad \text{and} \quad \hat{\lambda}_{P_s}(\hat{G}_{P_s}) = \hat{\lambda}_{H_s}(\hat{G}_{H_s}) = \hat{\lambda}_{H_s}(\hat{H}_s).$$

This completes the proof.

Lemma 4. *Let G be a metrizable L.C.A. group. If $\mu \in M(G)$ satisfies condition* (a) *of the Theorem, then $L^1(\mu) \subset \mathfrak{N}(G)$.*

Proof. We note that, for a compact subgroup H of G and $\lambda \in M(H)$, $\hat{\lambda}(\hat{H})$ is a countable set. Then we complete the proof by Lemma 3.

Lemma 5. *Let G be a compact abelian group. If $\mu \in M(G)$ satisfies condition* (a) *of the Theorem, then $\hat{\lambda}(\hat{G})$ is a countable set for each $\lambda \in L^1(\mu)$.*

Proof. Suppose that $\lambda \in L^1(\mu)$ and $\hat{\lambda}(\hat{G})$ is an uncountable set. Then there is a countable subgroup $K \subset \hat{G}$ such that $\overline{\hat{\lambda}(K)}$, the closure of $\hat{\lambda}(K)$ in a complex number plane C, is an uncountable set. Then $G/K^{\perp}$ is a compact metrizable group and $(G/K^{\perp})^{\wedge} = K$, where $K^{\perp}$ is the annihilator of K in G. Let Φ be the natural homomorphism of $M(G)$ onto $M(G/K^{\perp})$ ([5], p. 54). It is easy to show that $\mathrm{Spec}_G(\nu) \supset \mathrm{Spec}_{G/K}(\Phi\nu)$ for every $\nu \in M(G)$. Then $\Phi\mu$ satisfies condition (a) (replace G by $G/K^{\perp}$). Since $\Phi\lambda \ll \Phi\mu$, $\mathrm{Spec}_{G/K^{\perp}}(\Phi\lambda)$ is a countable set by Lemma 4. On the other hand, since $(\Phi\lambda)^{\wedge}(K) = \hat{\lambda}(K)$, we have

$$\overline{\hat{\lambda}(K)} = \overline{(\Phi\lambda)^{\wedge}(K)} \subset \mathrm{Spec}_{G/K^{\perp}}(\Phi\lambda),$$

which shows that $\mathrm{Spec}_{G/K^{\perp}}(\Phi\lambda)$ is an uncountable set. This is a contradiction which completes the proof.

Proof of the Theorem. Let $H_n \subset G$ $(n = 1, 2, \ldots)$ be compact subgroups of G which satisfy Lemma 3. Since $M(G_{H_n})$ is a prime L-subalgebra of $M(G)$ and μ_{H_n} is concentrated on H_n, we have

$$\mathrm{Spec}_{H_n}(\mu_{H_n}) = \mathrm{Spec}_{G_{H_n}}(\mu_{H_n}) \subset \mathrm{Spec}_G(\mu),$$

where $=$ is followed by the proof of Lemma 2 of [1]. By Lemma 5, $\hat{\lambda}_{H_n}(\hat{H}_n)$ is countable for $\lambda \in L^1(\mu)$. By Lemma 3, $\mathrm{Spec}_G(\lambda)$ is a countable set for $\lambda \in L^1(\mu)$. This completes the proof.

In the Theorem, the conditions that $\mu \geqslant 0$ and that every non-zero point $x \in \mathrm{Spec}_G(\mu)$ is isolated in $\mathrm{Spec}_G(\mu)$ may not be removed.

Example. Let D_∞ be the direct product of countably many copies of the group $\{-1, 1\}$. Then D_∞ is a compact metrizable abelian group.

Let $\{x_n\}_{n=1}^{\infty}$ be an independent set of D_∞. We denote by Q_n the group generated by $\{x_1, x_2, \ldots, x_n\}$; then Q_n is a finite group. Let

$$\mu = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n m_{Q_n};$$

then $\mu \in M(D_\infty)$, $\mu \geqslant 0$, $\|\mu\| = 1$ and $\operatorname{Spec}_{D_\infty}(\mu)$ is a countable set. We note that 1 is not isolated in $\operatorname{Spec}_{D_\infty}(\mu)$. Let

$$\lambda = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \delta_{x_n};$$

then $\lambda \ll \mu$. Since $\{x_n\}_{n=1}^{\infty}$ is an independent set and λ is a discrete measure we have

$$\operatorname{Spec}_{D_\infty}(\lambda) = \left\{ \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \varepsilon_n;\ \varepsilon_n = \pm 1 \right\}.$$

Then $\operatorname{Spec}_{D_\infty}(\lambda)$ is not a countable set, so that $L^1(\mu) \not\subset \mathfrak{N}(D_\infty)$.

Let $\nu = \mu - \delta_0$; then ν is not a positive measure, and every non-zero point $x \in \operatorname{Spec}_{D_\infty}(\nu)$ is isolated in $\operatorname{Spec}_{D_\infty}(\nu)$. But $\lambda \ll \nu$ and $\lambda \notin \mathfrak{N}(D_\infty)$

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