Koebe domains for univalent functions with real coefficients under Montel’s normalization

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Abstract. Let \( \overline{S}_c(r_0) \), \( \overline{S}^*(r_0) \) and \( \overline{S}^0(r_0) \) be the classes of all \( \text{analytic and univalent functions} \) \( f(z), \, \alpha \in K_1, \, K_1 = \{z: |z| < 1\} \) with real coefficients, which are convex, starlike and convex in the direction of the imaginary axis respectively and such that \( f(0) = 0, \, f(r_0) = r_0, \, 0 < r_0 < 1, \, \alpha \in K_1 \).

In this note the Koebe domains for the classes \( \overline{S}_c(r_0) \), \( \overline{S}^*(r_0) \) and \( \overline{S}^0(r_0) \) are given.

Let \( \overline{S}(r_0) \) denote the set of all functions \( f(z) \) analytic and univalent in the disc \( K_1 = \{z: |z| < 1\} \), with real coefficients, and such that

\[
1 \quad f(0) = 0, \quad f(r_0) = r_0, \quad 0 \leq r_0 < 1.
\]

The subclasses of functions, starlike functions with respect to the origin, and convex functions in the direction of the imaginary axis of the class \( \overline{S}(r_0) \) will denoted by \( \overline{S}_c(r_0) \), \( \overline{S}^*(r_0) \) and \( \overline{S}^0(r_0) \), respectively.

In this note we are going to determine the Koebe domains for the classes \( \overline{S}_c(r_0) \), \( \overline{S}^*(r_0) \), and \( \overline{S}^0(r_0) \), i.e. the sets of the form \( \bigcap f(K_1) \), where \( f \) runs through one of the classes. Let us observe that the domains \( f(K_1) \) are symmetrical with respect to the real axis.

We now prove

**Theorem 1.** The set \( \bigcap f(K_1) = \mathcal{X}[\overline{S}_c(r_0)] \) is a convex set symmetrical with respect to the real axis and bounded by the curve given by the following parametric equations:

\[
\begin{align*}
u(a) & = \frac{r_0(1-r_0)^a}{(1+r_0)^a-(1-r_0)^a} \left[ \frac{1}{\pi} \left( \ln \frac{1+r_0}{1-r_0} \right) \frac{(1+r_0)^a \sin \alpha \pi}{(1+r_0)^a-(1-r_0)^a} - 1 \right], \\
u_0(a) & = \pm \frac{1}{\pi} (1-\cos \alpha \pi) \frac{r_0(1-r_0)^a}{[(1+r_0)^a-(1-r_0)^a]^2} \ln \frac{1+r_0}{1-r_0}, \quad \alpha \in [0, 1],
\end{align*}
\]

\( (2) \)
and
\[
\begin{align*}
  u(a) &= \frac{r_0(1+r_0)^a}{(1+r_0)^a-(1-r_0)^a} \left[ -\frac{1}{\pi} \ln \frac{1+r_0}{1-r_0} (1-r_0)^\alpha \sin \alpha \pi + 1 \right], \\
  v(a) &= \pm \frac{1}{\pi} (1-\cos \alpha \pi) \frac{r_0(1-r_0)^a}{[(1+r_0)^a-(1-r_0)^a]^2} \ln \frac{1+r_0}{1-r_0}, \quad a \in [0,1],
\end{align*}
\]

together with the straight line segments
\[
\begin{align*}
  u &= \frac{1-r_0}{2}, \quad -\frac{1-r_0^2}{\pi} \ln \frac{1+r_0}{1-r_0} \leq v \leq \frac{1-r_0^2}{\pi} \ln \frac{1+r_0}{1-r_0}
\end{align*}
\]
and
\[
\begin{align*}
  u &= \frac{1+r_0}{2}, \quad -\frac{1-r_0^2}{\pi} \ln \frac{1+r_0}{1-r_0} \leq v \leq \frac{1-r_0^2}{\pi} \ln \frac{1+r_0}{1-r_0}.
\end{align*}
\]

The above-mentioned domain is also symmetrical with respect to the line \( u = r_0/2 \). Moreover, we have
\[
\begin{align*}
  u(0) &= r_0/2, \quad v(0) = -\frac{\pm \pi r_0}{2 \ln \frac{1+r_0}{1-r_0}}.
\end{align*}
\]

**Corollary 1.** With \( r_0 \to 0 \), the class \( \tilde{S}_c(r_0) \) reduces to the class \( \tilde{S}_c(0) \) of convex functions with real coefficients, such that \( f(0) = 0 \) and \( f'(0) = 1 \). whereas the domain \( \mathcal{X} [\tilde{S}_c(r_0)] \) reduces to the one \(^{(1)}\).

**Proof.** First, we observe that any domain \( f(K_1) \) convex and symmetrical with respect to the real axis, where \( f \in \tilde{S}_c(r_0) \), can be represented as the intersection of a collection of convex "angle-domains" which are also symmetrical with respect to the real axis and are such that each of them contains the points 0 and \( r_0 \), i.e. domains which are convex angles containing the points 0, \( r_0 \) and the real axis as their the bisector.

Let \( D_a \), \( 0 < a \leq 1 \), denote one of those angle-domains of angle \( \alpha \pi \) and let the function \( w = F(z) \) be analytic and univalent and such that \( F(0) = 0 \) and \( F(K_1) = D_a \). Since \( f(K_1) \subset F(K_1) \), i.e. \( f(z) = F(\omega(z)) \), where the function \( \omega(z) \) is univalent and satisfies the conditions of the Schwarz Lemma. From the principle of subordination it follows that \( r_0 = f(r_0) = F(\omega(r_0)) < F(r_0) \), i.e. \( r_0/F(r_0) < 1 \), while \( \omega(z) \neq z \).

Normalizing \( F(z) \) so as to obtain \( \Phi(z) = [r_0F(z)]/F(r_0) \), we see that the function \( \Phi(z) \) maps \( K_1 \) onto the contracted and normalized angle-domain \( \tilde{D}_a \), where \( \tilde{D}_a \) is a normalizing angle-domain of angle \( \alpha \pi \), \( 0 < a \leq 1 \).

Taking into account that for every function \( f \in \mathcal{S}_c(r_0) \) there exists a maximal set \( I_f \subseteq (0, 1] \) such that \( f(K_1) = \bigcap_{a \in I_f} D_a \), we have

\[
\bigcap_{f \in \mathcal{S}_c(r_0)} f(K_1) = \bigcap_a D_a \supseteq \bigcap_a \overline{D}_a,
\]

where \( \bigcup f = (0, 1] \).

But any \( \overline{D}_a \) is the image of \( K_1 \) by a function \( f \in \mathcal{S}_c(r_0) \), so

\[
\bigcap_a \overline{D}_a \supseteq \bigcap_{f \in \mathcal{S}_c(r_0)} f(K_1).
\]

Relations (3) and (4) allow us to write

\[
\mathcal{K} [\mathcal{S}_c(r_0)] = \bigcap_{f \in \mathcal{S}_c(r_0)} f(K_1) = \bigcap_a \overline{D}_a.
\]

Therefore, to determine the Koebe domain for \( \mathcal{S}_c(r_0) \) it is enough to establish the intersection \( \bigcap_a \overline{D}_a \) of the normalized angle-domains.

Every function \( F(z) \) which maps \( K_1 \) onto a convex angle \( \alpha \pi \) and is symmetrical with respect to the real axis has the form

\[
F(z) = a \left[ \left( \frac{1 \pm z}{1 \mp z} \right)^a - 1 \right], \quad a \in (0, 1],
\]

where \( a > 0 \).

For the function \( F(z) \) which is a majorant of a function \( f(z) \in \mathcal{S}(r_0) \) the following condition must hold

\[
r_0 < a \left[ \left( \frac{1 \pm r_0}{1 \mp r_0} \right)^a - 1 \right].
\]

By (6), every function \( \Phi(z) \) such that \( \Phi(K_1) = \overline{D}_a \), \( \Phi(0) = 0 \) and \( \Phi(r_0) = r_0 \) has the form

\[
\omega = \Phi(z) = r_0 \frac{a \left[ \left( \frac{1 \pm z}{1 \mp z} \right)^a - 1 \right]}{a \left[ \left( \frac{1 \pm r_0}{1 \mp r_0} \right)^a - 1 \right]} = r_0 \frac{\left[ \left( \frac{1 \pm z}{1 \mp z} \right)^a - 1 \right]}{\left[ \left( \frac{1 \pm r_0}{1 \mp r_0} \right)^a - 1 \right]}.
\]

Now we are going to determine \( \bigcap_a \overline{D}_a \). This is equivalent to finding the envelope of the boundary lines of all such domains. Let \( W = u + iv \).

First we find the part of the envelope of the boundary \( \overline{D}_a \) which lies in the upper half-plane, then we shall obtain the other part by the reflection of the previous part in the real axis. For the angle-domains \( \overline{D}_a \)
having the corners on the left of 0 we find the envelope of the lines \( w = \Phi(e^{i\theta}), \theta \in [0, 2\pi] \), taking the upper sing in (8) and \( \alpha \in (0, 1] \). Similarly, for the angle-domains \( \overline{D}_a \) which have corners on the right of \( r_0 \) we find the envelope of the lines \( w = \Phi(e^{i\theta}), 0 \leq \theta \leq 2\pi \), taking the lower sing in (8) and \( \alpha \in (0, 1] \).

The equations of those lines are the following:

\[
(9) \quad v = \tan \frac{a\pi}{2} \left[ u + \frac{r_0}{(1 + r_0)^{a} - 1} \right], \quad v = -\tan \frac{a\pi}{2} \left[ u + \frac{r_0}{(1 - r_0)^{a} - 1} \right].
\]

Hence, by the well-known procedure, we obtain (2).

Analogously one can prove the following theorems:

**Theorem 2.** The boundary of the set \( \mathcal{K}[\overline{S}^*_{r_0}] = \bigcap_{f \in \mathcal{K}_1} f(S_{r_0}) \) is the curve given by the equations

\[
u(a) = -\frac{1}{2}(1 + r_0)^2(1 - r_0)^{2(1 - a)}(1 - a)^{-(1 - a)} \cos a\pi, \\
v(a) = \pm \frac{1}{2}(1 + r_0)^{2a}(1 - r_0)^{2(1 - a)}(1 - a)^{a} \sin a\pi, \quad a \in [0, 1],
\]

where

\[
u(0) = -\frac{1}{2}(1 - r_0)^2, \quad v(0) = 0,
\]

and

\[
u(1) = \frac{1}{2}(1 + r_0)^2, \quad v(1) = 0.
\]

**Theorem 3.** The boundary of the set \( \mathcal{K}[\overline{S}^*_{r_0}] = \bigcap_{f \in \mathcal{K}_1} f(S_{r_0}) \) is the ellipse

\[
\frac{(u - \frac{1}{2}r_0)^2}{(\pm 1)^2} + \frac{v^2}{\left(\frac{\sqrt{1 - r_0^2}}{2}\right)^2} = 1.
\]

Putting \( r_0 \to 0 \), we obtain the Koebe domains for the class \( \overline{S}^*(0) \) and \( \overline{S}^*(0) \) (1).

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