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Solutions of the Schröder equation

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1. Let F be a locally convex space over the field of complex numbers C. We consider the Schröder equation

$$\varphi[f(z)] = s\varphi(z), \tag{1}$$

where φ of the type $\varphi\colon C^n \to F$ is the unknown function; the function f of the type $f\colon C^n \to C^n$ and the complex constant $s \neq 0$ are given. We shall investigate local analytic solutions of (1).

We assume that:

(a_1) $f(z)$ is an analytic function in a neighbourhood of $z = 0$ and $f(0) = 0$.

(a_2) $\lambda_0 \overset{\text{df}}{=} \max\limits_{i=1,\ldots,n} |\lambda_i| < 1$, where λ_i denote the characteristic roots of the matrix $f'(0)$.

It follows by (a_2) that there exists a positive number p such that

$$\frac{\lambda_0^p}{|s|} < 1. \tag{2}$$

We suppose that φ is an analytic solution of (1) in a neighbourhood of $z = 0$. We put

$$\eta_{i_1\ldots i_q} = \frac{\partial^q \varphi(z)}{\partial z_{i_1} \ldots \partial z_{i_q}}\bigg|_{z=0}, \quad i_1, \ldots, i_q = 1, \ldots, n;\ q = 1, \ldots, p, \tag{3}$$

and

$$\eta^{(q)} = \{\eta_{i_1\ldots i_q}\}_{i_1,\ldots,i_q = 1,\ldots,n}.$$

Differentiating equality (1) q times and then setting $z = 0$, we obtain

$$\eta^{(q)} = \left\{\frac{1}{s} \sum_{k=1}^{q} \sum_{j_1\ldots j_k = 1}^{n} b_{i_1\ldots i_q}^{j_1\ldots j_k} \eta_{j_1\ldots j_k}\right\}_{i_1,\ldots,i_q=1,\ldots,n}, \tag{4}$$

where $b_{i_1\ldots i_q}^{j_1\ldots j_k}$ are expressed by means of sums and products of derivatives of f at $z = 0$. Since the result of the differentiation does not depend on the order of taking the derivatives, we have

$$\eta_{i_{p_1}\dots i_{p_q}} = \eta_{i_1\dots i_q}, \quad i_1, \dots, i_q = 1, \dots, n;\ q = 1, \dots, p, \tag{5}$$

for every permutation $i_{p_1}, \dots, i_{p_q}$ of the sequence $i_1, \dots, i_q$.

We say that the system $\eta^{(1)}, \dots, \eta^{(p)}$ is allowable if it fulfils system (4) for $q = 1, \dots, p$, and conditions (5).

We shall prove the following

THEOREM 1. *If hypotheses* (a_1) *and* (a_2) *are fulfilled, then for every allowable system* $\eta^{(1)}, \dots, \eta^{(p)}$ *for equation* (1) *there exists exactly one solution* φ *of* (1) *defined and analytic in a neighbourhood of* $z = 0$ *and such that*

$$\varphi(0) = 0, \quad \varphi^{(q)}(0) = \eta^{(q)}, \quad q = 1, \dots, p. \tag{6}$$

Proof. Let $\eta^{(1)}, \dots, \eta^{(p)}$ be an arbitrary non-trivial allowable system and let

$$\omega(z) \stackrel{\mathrm{df}}{=} \sum_{q=1}^{p} \frac{1}{q!} \sum_{i_1\dots i_q=1}^{n} \eta_{i_1\dots i_q} z_{i_1} \dots z_{i_q}. \tag{7}$$

We take into consideration only those elements of the set

$$\eta_{i_1\dots i_q}, \quad i_1, \dots, i_q = 1, \dots, n;\ q = 1, \dots, p \tag{8}$$

for which $i_1 \leqslant i_2 \leqslant \dots \leqslant i_q$. We set those elements into a sequence $\eta^1, \dots, \eta^N$, $N = \binom{n+p}{n} - 1$. Similarly, we take only those equations from system (4) in which on the left-hand side $\eta_{i_1\dots i_q}$ has indices in a non-decreasing order. Thus the $\eta_{i_1\dots i_q}$, $i_1 \leqslant i_2 \leqslant \dots \leqslant i_q$, of the allowable system considered fulfil the linear homogeneous system of N equations (arising from (4))

$$\sum_{j=1}^{N} \hat{b}_j^i \eta^j = 0, \quad i = 1, \dots, N. \tag{9}$$

We can write (7) in the form

$$\omega(z) = \sum_{j=1}^{N} w_j(z) \eta^j, \tag{10}$$

where $w_j(z)$ are homogeneous polynomials of the type $w_j\colon C^n \to C$. Since $\eta^{(1)}, \dots, \eta^{(p)}$ is a non-trivial allowable system, the rank of the matrix $B = [\hat{b}_j^i]$ equals $N - l$, where $l > 0$. Thus $\eta^{l+1}, \dots, \eta^N$ are linearly dependent on $\eta^1, \dots, \eta^l$ (possibly after a change of the numbering of variables). Thus (10) may be written in the form

$$\omega(z) = \sum_{j=1}^{l} W_j(z) \eta^j,$$

where $W_j(z)$ are polynomials of the type $W_j\colon C^n \to C$.

We shall prove the convergence of the sequence $\frac{\omega[f^m(z)]}{s^m}$ to an analytic function of the type $\varphi\colon C^n \to F$ in a neighbourhood of $z = 0$ if we show the convergence of the sequences $W_j[f^m(z)]/s^m$, $j = 1, \ldots, l$, to analytic functions $\psi_j(z)$ in this neighbourhood.

We get all the allowable systems $c^{(1)}, \ldots, c^{(p)}$ for the equation

$$\psi[f(z)] = s\psi(z), \quad \psi\colon C^n \to C, \tag{11}$$

solving the system of equations

$$\sum_{j=1}^{N} \hat{b}_j^i c^j = 0, \quad i = 1, \ldots, N, \tag{12}$$

where $c^j = c_{i_1 \ldots i_p}$ if $\eta^j = \eta_{i_1 \ldots i_q}$ $(i_1 \leqslant i_2 \leqslant \ldots \leqslant i_q)$. We can take $c^1, \ldots, c^l$ arbitrarily and then calculate $c^{l+1}, \ldots, c^N$ from system (12). Then function

$$\pi(z) \overset{\mathrm{df}}{=} \sum_{q=1}^{p} \frac{1}{q!} \sum_{i_1 \ldots i_q = 1}^{n} c_{i_1 \ldots i_q} z_{i_1} \ldots z_{i_q} = \sum_{j=1}^{l} W_j(z) c^j$$

fulfils the following conditions:

$$\pi(0) = 0, \quad \pi^{(q)}(0) = c^{(q)}, \quad q = 1, \ldots, p.$$

The equalities

$$c^i = 1, \quad c^j = 0, \quad j \neq i, \; i, j = 1, \ldots, l \tag{13}$$

define exactly one allowable system $c^{(1)}, \ldots, c^{(p)}$, and for this system we have $\pi(z) = W_i(z)$. It is known (see [3]) that the functional sequence $\frac{\pi[f^m(z)]}{s^m} = \frac{W_i[f^m(z)]}{s^m}$ converges to an analytic function $\psi_i(z)$ in a neighbourhood of $z = 0$. This function fulfils equation (11) and the conditions

$$\left.\frac{\partial^q \psi_i(z)}{\partial z_{i_1} \ldots \partial z_{i_q}}\right|_{z=0} = c^j \quad (c^j = c_{i_1 \ldots i_q}), \quad i = 1, \ldots, l, \tag{14}$$

where c^j for $1 \leqslant j \leqslant l$ fulfil (13). The function

$$\varphi(z) \overset{\mathrm{df}}{=} \sum_{i=1}^{l} \psi_i(z) \eta^i \tag{15}$$

is an analytic solution of (1) in a neighbourhood of $z = 0$.

We shall prove that φ fulfils conditions (6).

We differentiate both sides of the equality $\varphi[f(z)] = s\varphi(z)$, q times, $q = 1, \ldots, p$, with respect to the variables $z_{i_1}, \ldots, z_{i_q}$, $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_q$. Let $\delta^j \overset{\mathrm{df}}{=} \left.\dfrac{\partial^q \varphi(z)}{\partial z_{i_1} \ldots \partial z_{i_q}}\right|_{z=0}$ if $\eta^j = \eta_{i_1 \ldots i_q}$. Then

$$\sum_{j=1}^{H} \hat{b}_j^i \delta^j = 0, \quad i = 1, \ldots, N. \tag{16}$$

It follows by (15), (14) and (13) that

$$\left.\frac{\partial^q \varphi(z)}{\partial z_{i_1} \ldots \partial z_{j_q}}\right|_{z=0} = \delta^j = \sum_{i=1}^{l} \left.\frac{\partial^q \psi_i(z)}{\partial z_{i_1} \ldots \partial z_{i_q}}\right|_{z=0} \eta^i = \eta^j, \quad j = 1, \ldots, l.$$

Moreover, we have by (9) and (16)

$$\delta^j = \eta^j \quad \text{for } j = l+1, \ldots, N.$$

Thus φ fulfils (6).

Now, we prove that there exists at most one solution of (1) fulfilling (6).

We suppose that φ_1 and φ_2 are analytic functions in a neighbourhood U of $z = 0$ fulfilling equation (1) and conditions (6). Let $u \in F'$, where F' denotes the vector space of all continuous linear forms $u\colon F \to C$. The functions $u \circ \varphi_1$ and $u \circ \varphi_2$ are analytic functions in U (cf. [1], Theorem 3.2) such that

$$u \circ \varphi_k[f(z)] = su \circ \varphi_k(z),$$

$$u \circ \varphi_k(0) = 0, \quad \left.\frac{\partial^q}{\partial z_{i_1} \ldots \partial z_{i_q}} u \circ \varphi_k(z)\right|_{z=0} = u(\eta_{i_1 \ldots i_q}), \quad k = 1, 2;$$

$$i_1, \ldots, i_g = 1, \ldots, n; \quad q = 1, \ldots, p.$$

It follows by the form of system (4) and conditions (5) that the system $c^{(1)}, \ldots, c^{(p)}$, $c^{(q)} \overset{\mathrm{df}}{=} \{u(\eta_{i_1 \ldots i_q})\}_{i_1, \ldots, i_q = 1, \ldots, n}$, $q = 1, \ldots, p$, is an allowable system for (11). In virtue of [3] Theorem 1 we have $u \circ \varphi_1(z) = u \circ \varphi_2(z)$ for $z \in U$. Since $u \in F'$ was arbitrary, we have $\varphi_1 = \varphi_2$.

If $\eta^{(1)}, \ldots, \eta^{(p)}$ is a trivial system, then $\varphi = 0$ is the unique solution of (1) fulfilling (6). Thus the proof is completed.

2. Let F be a locally convex space over the field of real numbers R. We consider the Schröder equation

$$\varphi[f(x)] = s\varphi(x), \tag{17}$$

where f and φ are of the types $f\colon R^n \to R^n$ and $\varphi\colon R^n \to F$, respectively, and the real constant s is different from zero.

We assume that

(b_1) $f(x)$ is defined and of class C^p $(p > 0)$ in a neighbourhood of $x = 0$ and $f(0) = 0$.

(b_2) $\dfrac{\lambda_0^p}{|s|} < 1$ for a certain positive integer p, where $\lambda_0 = \max\limits_{i=1,\ldots,n} |\lambda_i|$ and λ_i denote the characteristic roots of the matrix $f'(0)$.

We apply Theorem 6.2 from paper [2] and by an argument quite similar to that used in Theorem 1 we may prove the following

THEOREM 2. *If hypotheses* (b_1) *and* (b_2) *are fulfilled, then for every allowable system* $\eta^{(1)}, \ldots, \eta^{(p)}$ *for equation* (17) *there exists exactly one solution* $\varphi(x)$ *of* (17) *defined and of weak class* C^p (1) *in a neighbourhood of* $x = 0$ *and such that*

$$(18)\quad u\circ\varphi(0) = 0, \qquad \frac{\partial^q}{\partial x_{i_1}\ldots\partial x_{i_q}} u\circ\varphi(x)\Big|_{x=0} = u(\eta_{i_1\ldots i_q}), \quad q = 1, \ldots, p,$$

for every $u \in F'$. *This solution is of class* C^p.

Let $f(x)$ be defined in a domain $G \subset R^n$ and let $\Gamma_r \stackrel{\mathrm{df}}{=} G \cap \{x\colon |x| = r\}$, $R \stackrel{\mathrm{df}}{=} \{r > 0\colon \Gamma_r \neq 0\}$, $h(r) \stackrel{\mathrm{df}}{=} \frac{1}{r}\sup\limits_{\Gamma_r} |f(x)|$ for $r \in R$ (2). We assume also that

(b_3) $h(r)$ is continuous in R and $0 < h(r) < 1$ for $r \in R$.

Theorem 2 of this paper and Theorem 6.3 of [2] imply the following theorem of a global character:

THEOREM 3. *Let* $f(x)$ *be a* C^p*-morphism of* G *into* G *and let hypotheses* (b_2) *and* (b_3) *be fulfilled. Then for every allowable system* $\eta^{(1)}, \ldots, \eta^{(p)}$ *for equation* (17) *there exists exactly one solution* $\varphi(x)$ *of* (17) *defined and of weak class* C^p *in* G *fulfilling conditions* (18). *This solution is of class* C^p.

(1) The function $\varphi\colon R^n \to F$ is of weak class C^p in an open set U if for every linear and continuous form $u\colon F \to R$ the function $u\circ\varphi$ is of class C^p in U.

(2) Here $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$.

References

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[2] M. Kuczma, *On the Schröder equation,* Rozprawy Mat. 34 (1963).

[3] W. Smajdor, *Local analytic solutions of the functional equation* $\varphi(z) = h(z, \varphi[f(z)])$ *in multidimensional spaces,* Aequationes Math. 1 (1968), p. 20 - 36.

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