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Integral transformations and closed form expressions for sums of double infinite series

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1. Introduction. In this paper the author has established some relations between integral transformations of two variables and double infinite series, with the help of the Mellin transform of two variables. The results give a new technique for finding a closed form expression for a double infinite series.

The following well-known integral transformations of two variables will be used.

1. *Mellin transform* ([5], p. 565)

$$F(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} f(x, y)\, dx\, dy;$$

2. *Laplace transform* ([2], p. 657)

$$F(p, q) = \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy, \qquad R(p, q) > 0;$$

3. *Hankel transformation of zero orders* ([3], p. 47)

$$F(p, q) = \int_0^\infty \int_0^\infty \sqrt{pqxy}\; J_0(px) J_0(qy) f(x, y)\, dx\, dy;$$

4. *Fourier cosine transform*

$$F(p, q) = \int_0^\infty \int_0^\infty \cos px \cos qy\, f(x, y)\, dx\, dy.$$

5. *Stieltjes transform* [4]

$$F(p, q) = \int_0^\infty \int_0^\infty (x+p)^{-1} (y+q)^{-1} f(x, y)\, dx\, dy.$$

2. In this section we have established the main result which gives the relation between double infinite series and integral transformations given in section 1.

THEOREM. *If*

$$F(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} f(x, y)\, dx\, dy,$$

then

$$(2.1) \qquad \sum_{m,n=0}^{\infty} a_m b_n F(m+1, n+1) = \int_0^\infty \int_0^\infty \Big[\sum_{m,n=0}^{\infty} a_m b_n x^m y^n \Big] f(x, y)\, dx dy,$$

provided the Mellin transform of $|f(x, y)|$ exists the series involved are absolutely convergent and the change in the order of integration and summation in (2.1) is permissible.

Proof. Replacing r and s in (1.1) by $m+1$ and $n+1$ respectively; multiplying by $a_m b_n$ and summing up m and n from 0 to ∞, we get

$$\sum_{m,n=0}^{\infty} a_m b_n F(m+1, n+1) = \sum_{m,n=0}^{\infty} \int_0^\infty \int_0^\infty [a_m b_n x^m y^n] f(x, y)\, dx\, dy$$
$$= \int_0^\infty \int_0^\infty \Big[\sum_{m,n=0}^{\infty} a_m b_n x^m y^n \Big] f(x, y)\, dx\, dy,$$

since the change in the order of summation and integration is permissible.

Thus the result (2.1) is proved.

With certain specific choices for a_m and b_n, we can reduce the integral on the right-hand side of (2.1) to standard transformation integrals (sometimes a simple change of variables is also required) given in section 1.

For example:

(a) When $a_m = (-p)^m/m!$, $b_n = (-q)^n/n!$, we have

$$(2.2) \qquad \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} p^m q^n F(m+1, n+1)}{m!\, n!} = \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\, dx\, dy.$$

(b) Taking $a_m = (-p^2)^m/(2m)!$, $b_n = (-q^2)^n/(2n)!$, we get

$$(2.3) \qquad \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} p^{2m} q^{2n} F(m+1, n+1)}{(2m)!\, (2n)!}$$
$$= 4 \int_0^\infty \int_0^\infty \cos pu \cos qv \{uv f(u^2, v^2)\}\, du\, dv.$$

(c) Putting $a_m = (-p^2)^m/2^{2m}(m!)^2$, $b_n = (-q^2)^n/2^{2n}(n!)^2$, we get

$$(2.4) \qquad \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} p^{2m} q^{2n} F(m+1, n+1)}{2^{2m+2n}(m!)^2(n!)^2}$$
$$= \frac{4}{\sqrt{pq}} \int_0^\infty \int_0^\infty \sqrt{pqst}\, J_0(ps) J_0(qt) \{\sqrt{st} f(s^2, t^2)\}\, ds\, dt.$$

(d) $a_m = (-1)^m/m!$, $b_n = (-1)^n/n!$, we have

$$(2.5)\quad \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}F(m+1, n+1)}{p^m q^n} = pq \int_0^{\infty}\int_0^{\infty} (x+p)^{-1}(y+q)^{-1} f(x, y)\,dx dy.$$

In each case from (2.1) to (2.5), F-function is the double Mellin transform of the f-function.

3. Application. If the terms of the desired double series can be expressed as a product of the coefficient of a known double power series and of the Mellin transform, the series can be expressed as a definite integral, which on evaluation will lead to a closed form expression.

We shall use (2.2) for finding the closed form expression for a given double infinite series. The technique will be illustrated by the following example.

Consider the sum

$$\sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} 2^{m+n} \Gamma\left(\frac{m+\nu_1+1}{2}\right) \Gamma\left(\frac{n-\nu+1}{2}\right) \Gamma\left(\frac{n+\nu+1}{2}\right) p^m q^n}{m!\, n! \Gamma\left(\frac{\nu_1-m+1}{2}\right)}$$

$$= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} p^m q^n}{m!\, n!} F(m+1, n+1).$$

Therefore

$$F(m, n) = \frac{2^{m+n-2}\Gamma\left(\frac{m+\nu_1}{2}\right)\Gamma\left(\frac{n-\nu}{2}\right)\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{\nu_1-m+2}{2}\right)}$$

and from ([1], p. 326 (1); p. 331, (26))

$$f(x, y) = 2J_{\nu_1}(x)K_\nu(y).$$

Hence from (2.2), using ([1], p. 182, (1); p. 197, (24)) the value of the sum is given by

$$\pi \csc(\pi\nu)(rs)^{-1}(S^\nu - S^{-\nu})e^{-\nu_1 \sinh^{-1}(p)},$$

where

$$r = (1+p^2)^{\frac{1}{2}}, \quad s = (q^2-1)^{\frac{1}{2}} \quad \text{and} \quad S = q+s,$$

provided

$$\mathrm{Re}(\nu_1) > -\mathrm{Re}(m) > -\tfrac{3}{2}, \quad \mathrm{Re}(\nu) < \mathrm{Re}(n).$$

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References

[1] Bateman M. Project, *Tables of integral transforms*, Vol. I, New York 1954.
[2] P. Humbert, *Le calcul symbolique a deux variables*, Compt. Rend. Acad. Sci. (Paris), 199 (1934), p. 657-660.
[3] N. C. Jain, *On Laplace transform of two variables and self-reciprocal functions*, J. Math. Pures Appl. 47, Fasc. I (1968), p. 47-55.
[4] — *A relation between Laplace and Stieltjes transforms of two variables*, Ann. Polon. Math. 22 (1969), p. 313-315.
[5] I. S. Reed, *The Mellin type of double integral*, Duke Math. J. 11 (1944), p. 565-572.

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