ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

Boundedness and oscillatoriness of solutions of the equation $y''+a(x)g(y,y')+b(x)f(y)h(y') = 0$

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Abstract. The present paper presents a theorem which is a generalization of the boundedness results of [1], [2] and [3] in that it is concerned with solutions of the equation

(1) $$y''+a(x)g(y,y')+b(x)f(y)h(y') = 0.$$

Besides this theorem the paper contains theorem on the unboundedness of solutions of (1). The last two theorems deal with conditions under which a solution $y(x)$ of (1) is oscillatory or $\lim_{x\to\infty} y(x) = 0$

Papers [1] and [3] deal with sufficient conditions for the boundedness and oscillatoriness of all solutions of the equation

$$y''+b(x)f(y)h(y') = 0$$

if, among other conditions, $\int_0^y f(s)\,ds \to +\infty$ for $|y|\to\infty$.

The present paper presents a theorem which is a generalization of the boundedness results of [1] and [3] in that it is concerned with solutions of the equation

(1) $$y''+a(x)g(y,y')+b(x)f(y)h(y') = 0$$

and, moreover, $\int_0^y f(s)\,ds \to F \leqslant +\infty$ for $|y|\to\infty$. Besides this theorem the paper contains theorem on the unboundedness of solutions of (1). The last two theorems deal with conditions under which a solution $y(x)$ of (1) is oscillatory or $\lim_{x\to\infty} y(x) = 0$.

In the sequel we shall assume that $a(x)\in C\langle x_0,\infty)$, $b(x)\in C\langle x_0,\infty)$ $f(y)\in C(R_1)$, $h(z)\in C(R_1)$, $g(y,z)\in C(R_2)$, $\operatorname{sgn} f(y) = \operatorname{sgn} y$, $h(z)>0$, with $x_0\in(-\infty,\infty)$, $R_1 = (-\infty,\infty)$, $R_2 = R_1\times R_1$.

We introduce the following notation:

$$F(y) = \int_0^y f(s)\,ds, \qquad H(z) = \int_0^z \frac{s}{h(s)}\,ds,$$

$$\{c(t)\}_+ = \begin{cases} c(t) & \text{for } c(t)>0, \\ 0 & \text{for } c(t)\leqslant 0. \end{cases}$$

We have

THEOREM 1. *Let* $b(x) \in C^1 \langle x_0, \infty)$; *for all* $x \in \langle x_0, \infty)$ *and* $(y, z) \in R_2$ *let*

1. $a(x) \geqslant 0,\ b(x) \geqslant k > 0,\ g(y, z)z \geqslant 0$;
2. $\int\limits_{x_0}^{\infty} \{b'(t)\}_+ \, dt = K < +\infty$;
3. $\lim\limits_{|y| \to \infty} F(y) = F \leqslant +\infty$.

Then any solution $y(x)$ *of* (1) *such that*

$$\frac{1}{k} K_0 \exp\left(\frac{K}{k}\right) = \frac{1}{k}\left[H\big(y'(x_0)\big) + b(x_0) F\big(y(x_0)\big)\right] \exp\left(\frac{K}{k}\right) < F, \tag{2}$$

is bounded on its domain of existence $\langle x_0, \bar{x})$.

If in addition $\lim\limits_{|z| \to \infty} H(z) = +\infty$, *then any solution* $y(x)$ *satisfying* (2) *is bounded together with its first derivative on* $\langle x_0, \infty)$ *and the first derivative of any solution is bounded for* $x \in \langle x_0, \infty)$.

Proof. Let $y(x)$ be defined on $\langle x_0, \bar{x})$ and satisfy (2). Then from (1) we have:

$$\frac{d}{dx} H\big(y'(x)\big) + b(x) \frac{d}{dx} F\big(y(x)\big) = -\frac{a(x) g(y, y') y'(x)}{h\big(y'(x)\big)};$$

hence

$$H\big(y'(x)\big) + b(x) F\big(y(x)\big) \leqslant K_0 + \int\limits_{x_0}^{x} \{b'(t)\}_+ F\big(y(t)\big)\, dt, \tag{3}$$

and, since $H(z) > 0$ for $z \neq 0$,

$$F\big(y(x)\big) \leqslant \frac{K_0}{k} \exp\left(\frac{K}{k}\right) \tag{4}$$

for all $x \in \langle x_0, \bar{x})$. If $\limsup\limits_{x \to \bar{x}} |y(x)| = +\infty$, then a sequence $\{x_k\}_{k=1}^{\infty}$, $x_k \to \bar{x}$ exists such that $\lim\limits_{k \to \infty} F\big(y(x_k)\big) = F$, and thus the last relation leads to a contradiction. Therefore necessarily $\limsup\limits_{x \to \bar{x}} |y(x)| < \infty$.

Now let $\lim\limits_{|z| \to \infty} H(z) = +\infty$; suppose that $y(x)$ is an arbitrary solution of (1). Then from (3) and (4)

$$H\big(y'(x)\big) \leqslant K_0 + \frac{K_0}{k} K \exp\left(\frac{K}{k}\right);$$

thus $y'(x)$ is bounded for $x \in \langle x_0, \bar{x})$. If $\bar{x} < \infty$, then $y(x)$ is also bounded on $\langle x_0, \bar{x})$ so that $\bar{x} = +\infty$. This completes the proof.

Remark. Evidently, if $F = +\infty$, then under the hypotheses of Theorem 1 any solution of (1) is bounded for $x \geqslant x_0$. If $a(x) \equiv 1$, $b(x) \equiv 1$,

$h(z) \equiv 1$, $g(y, z) \equiv g_1(y) \cdot g_2(z)$ and $F = +\infty$, then as a special case of Theorem 1 we get Theorem 2 or 3 of [2].

THEOREM 2. *Let* $b(x) \in C^1 \langle x_0, \infty)$ *and suppose that for all* $x \in \langle x_0, \infty)$ *and* $(y, z) \in R_2$

$$a(x) \leqslant 0, \quad b(x) \leqslant 0, \quad b'(x) \geqslant 0, \quad g(y, z)z \geqslant 0.$$

Then any solution $y(x)$ *of* (1) *defined on* $\langle x_0, \infty)$ *such that* $K_0 > 0$ *is unbounded for* $x \to \infty$.

Proof. Clearly for all $x \geqslant x_0$

$$H\big(y'(x)\big) \geqslant K_0 > 0. \tag{5}$$

Since $H(z) \in C(R_1)$ there exists a number y_0', $\operatorname{sgn} y_0' = \operatorname{sgn} y'(x_0)$, such that $H(y_0') = K_0$. By the definition of $H(z)$, relation (5) implies that $y'(x)$ is non-zero and does not change its sign for any $x \geqslant x_0$. Let $y_0' < 0$. Then also $y'(x) < 0$ for $x \in \langle x_0, \infty)$ and

$$\frac{d}{dz} H(z) = \frac{z}{h(z)} < 0$$

for all $z = y'(x)$, so that $H(z)$ is a decreasing function for $z < 0$. Hence $H\big(y'(x)\big) \geqslant H(y_0')$ and therefore for all $x \geqslant x_0$ $y'(x) \leqslant y_0'$. This means that $y(x) \to -\infty$ for $x \to \infty$. Now let $y_0' > 0$. Then the same holds for $y'(x)$, $x \geqslant x_0$ and thus, since $H\big(y'(x)\big) \geqslant H(y_0')$, $y'(x) \geqslant y_0'$, so that $y(x) \to +\infty$ for $x \to \infty$. This completes the proof.

THEOREM 3. *In addition to the hypotheses of Theorem 2 suppose that*

$$b(x) \leqslant k < \infty, \quad k > 0.$$

If for $z \in R_1$

$$H(z) \leqslant K_1 < \infty,$$

then a solution $y(x)$ *of* (1) *for which*

$$K_1 < H\big(y'(x_0)\big) + b(x_0) F\big(y(x_0)\big) = K_0$$

has no zero to the right of x_0.

Proof. From the hypotheses we have

$$F\big(y(x)\big) \geqslant \frac{1}{k}(K_0 - K_1) + \frac{1}{k}\int_{x_0}^{x} b'(t) F\big(y(t)\big)\,dt$$

for $x \in \langle x_0, \bar{x})$, $\bar{x} \leqslant +\infty$, where $\langle x_0, \bar{x})$ is the domain of $y(x)$. This yields

$$F\big(y(x)\big) \geqslant \frac{1}{k}(K_0 - K_1) > 0,$$

so that for all $x \in \langle x_0, \bar{x})$ is $y(x) \neq 0$.

THEOREM 4. *Let* $a(x) \in C^1\langle x_0, \infty)$, $b(x) \in C^1\langle x_0, \infty)$ *and suppose that the following conditions hold for all* $x \in \langle x_0, \infty)$, $y \in R_1$:

1. $a(x) \geqslant 0$, $a'(x) \leqslant 0$, $b(x) > 0$, $b'(x) \leqslant 0$, $f'(y) \geqslant \varepsilon > 0$;

2. $\int\limits_{x_0}^{\infty} a(t)\,dt \leqslant A < \infty$, $\int\limits_{x_0}^{\infty} b(t)\,dt = +\infty$.

If $g(y, z) = z$ *and* $\lim\limits_{|z|\to\infty} H(z) = H \leqslant +\infty$, *then any solution* $y(x)$ *of* (1) *such that*

$$K_0 = H\big(y'(x_0)\big) + b(x_0)F\big(y(x_0)\big) < H \tag{6}$$

is oscillatory or $\lim\limits_{x\to\infty} y(x) = 0$.

Proof. We start by proving that if (6) holds for $y(x)$, then $y'(x)$ is bounded and therefore $y(x)$ exists on $\langle x_0, \infty)$. Using (3) we have

$$H\big(y'(x)\big) + b(x)F\big(y(x)\big) \leqslant H\big(y'(x_0)\big) + b(x_0)F\big(y(x_0)\big);$$

hence for all $x \in \langle x_0, \bar{x})$

$$H\big(y'(x)\big) \leqslant K_0. \tag{7}$$

Let $y'(x)$ be unbounded for $x \to \bar{x}$; this means that there must exist a sequence $\{x_k\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} |y'(x_k)| = +\infty$, and in that case (7) means that $H \leqslant K_0$, which contradicts (6). If $\bar{x} < \infty$, then $y(x)$ is also bounded on $\langle x_0, \bar{x})$; therefore $\bar{x} = +\infty$.

Now suppose that $y(x)$ satisfies (6) and is not oscillatory; this means that there exists an $x_1 \geqslant x_0$ such that for $x \in \langle x_1, \infty)$, e. g. $y(x) > 0$. Then

$$\frac{y''(x)}{f\big(y(x)\big)} + \frac{a(x)y'(x)}{f\big(y(x)\big)} = -b(x)h\big(y'(x)\big).$$

Integrating this from x_1 to $x \geqslant x_1$, we get

$$\frac{y'(x)}{f\big(y(x)\big)} + \int\limits_{x_1}^{x} \left[\frac{y'(t)}{f\big(y(t)\big)}\right]^2 f'\big(y(t)\big)\,dt + \int\limits_{x_1}^{x} \frac{a(t)y'(t)}{f\big(y(t)\big)}\,dt = \frac{y'(x_1)}{f\big(y(x_1)\big)} - \int\limits_{x_1}^{x} b(t)h\big(y'(t)\big)\,dt.$$

Since $|a| \leqslant \frac{1}{2}(1 + a^2)$, we obtain the inequality

$$\frac{y'(x)}{f\big(y(x)\big)} + \int\limits_{x_1}^{x} \left[\frac{y'(t)}{f\big(y(t)\big)}\right]^2 \left\{f'\big(y(t)\big) - \tfrac{1}{2}a(t)\right\}dt \leqslant K_1^* - \int\limits_{x_1}^{x} b(t)h\big(y'(t)\big)\,dt, \tag{8}$$

where

$$K_1^* = \frac{y'(x_1)}{f\big(y(x_1)\big)} + \tfrac{1}{2}A \geqslant \frac{y'(x_1)}{f\big(y(x_1)\big)} + \tfrac{1}{2}\int\limits_{x_1}^{x} a(t)\,dt.$$

Since $\lim_{x\to\infty} a(x) = 0$, there exists an $x_2 \geqslant x_1$ such that for $x \geqslant x_2$, $f'\big(y(x)\big) - \frac{1}{2}a(x) \geqslant 0$. From the boundedness of $y'(x)$ and the continuity of $h(z)$ we see that there exists an a such that, for all $x \geqslant x_0$, $h\big(y'(x)\big) \geqslant h(a) > 0$. For $x \geqslant x_2$, (8) yields

$$\frac{y'(x)}{f\big(y(x)\big)} \leqslant K_2 - h(a)\int_{x_2}^{x} b(t)\,dt. \tag{9}$$

Therefore there exists an $x_3 \geqslant x_2$ such that, for all $x \geqslant x_3$, $y'(x) < 0$. For $x \in \langle x_3, \infty)$, (1) yields

$$y'(x) = y'(x_3) + a(x_3)y(x_3) - a(x)y(x) + \int_{x_3}^{x} a'(t)y(t)\,dt - \\ - \int_{x_3}^{x} b(t)f\big(y(t)\big)h\big(y'(t)\big)\,dt$$

and therefore

$$y'(x) \leqslant K_3 - h(a)\int_{x_3}^{x} b(t)f\big(y(t)\big)\,dt. \tag{10}$$

Since for $x \in \langle x_3, \infty)$, $y(x) > 0$ and $y'(x) < 0$, there exists a $\lim_{x\to\infty} y(x) = c \geqslant 0$. We shall prove that $c = 0$. Suppose that $c > 0$. Then

$$f\big(y(x)\big) \geqslant f(c) > 0$$

and, since $\int_{x_3}^{x} b(t)\,dt \to +\infty$ for $x \to \infty$, (10) leads to a contradiction. Thus, $c = 0$.

Supposing that $y(x) < 0$ for $x \in \langle x_1, \infty)$, from (9) we see that, for sufficiently large x, $y'(x) > 0$, and therefore

$$y'(x) \geqslant K_3 - h(a)\int_{x_3}^{x} b(t)f\big(y(t)\big)\,dt.$$

Analogously as in the first case we prove the impossibility of $\lim_{x\to\infty} y(x) = c < 0$.

This completes the proof.

THEOREM 5. *The hypotheses are those of Theorem 4 with the assumptions* $a'(x) \leqslant 0$ *and* $\int_{x_0}^{\infty} b(t)\,dt = +\infty$ *replaced by*

$$C_1\int_{x_0}^{x} b(t)\,dt - C_2\int_{x_0}^{x} \{a'(t)\}_+\,dt \to \infty \quad \textit{for } x \to \infty, \tag{11}$$

where $C_1 > 0$, $C_2 > 0$ *are arbitrary constants. Then every solution* $y(x)$ *of* (1) *satisfying* (6) *is oscillatory or* $\lim_{x\to\infty} y(x) = 0$.

Proof. Analogously as in the proof of Theorem 4 we show that if $y(x) > 0$ for $x \geqslant x_1$, then there exists an x_3 such that, for $x \in \langle x_3, \infty)$, $y(x) > 0$, $y'(x) < 0$. From (1) we have

$$y'(x) \leqslant y'(x_3) + a(x_3)y(x_3) - a(x)y(x) + \int_{x_3}^{x} \{a'(t)\}_+ y(t)\,dt - \\ - \int_{x_3}^{x} b(t)f\big(y(t)\big)h\big(y'(t)\big)\,dt,$$

so that

$$y'(x) \leqslant K_3 + y(x_3)\int_{x_3}^{x} \{a'(t)\}_+\,dt - h(a)f(c)\int_{x_3}^{x} b(t)\,dt.$$

Owing to (11), this means that $y(x) \to -\infty$ for $x \to \infty$, which is a contradiction.

Analogously we complete the proof if $y(x) < 0$ for $x \in \langle x_1, \infty)$.

References

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Reçu par la Rédaction le 2. 5. 1972